46-679 Multi-Period Pricing Models
Homework Set 2
Due July 27, 1999
Solutions

Exercise 1. Using Definition 1.1, show the following.

(i) If A is an event and A^c denotes its complement, then $\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$.

(ii) If $A_1, A_2, \ldots, A_N$ is a finite set of events, then

$$\mathbf{P}\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mathbf{P}(A_n).$$

If the events $A_1, A_2, \ldots, A_N$ are disjoint, then the above expession becomes an equality.

Solution.

(i) If A is an event and A^c is its complement, then

$$\mathbf{P}(A) + \mathbf{P}(A^c) = \sum_{\omega\in A} \mathbf{P}(\omega) + \sum_{\omega\in A^c} \mathbf{P}(\omega) = \sum_{\omega\in A\cup A^c} \mathbf{P}(\omega) = \sum_{\omega\in\Omega} \mathbf{P}(\omega) = 1,$$

or, equivalently,

$$\mathbf{P}(A^c) = 1 - \mathbf{P}(A).$$

(ii) We will consider first the case when the events $A_1, A_2, \ldots, A_N$ are disjoint. We have:

$$\mathbf{P}(A_1) + \mathbf{P}(A_2) = \sum_{\omega\in A_1} \mathbf{P}(\omega) + \sum_{\omega\in A_2} \mathbf{P}(\omega) = \sum_{\omega\in A_1\cup A_2} \mathbf{P}(\omega) = \mathbf{P}(A_1 \cup A_2).$$

Thus, the equality is true for two sets. Using this fact, we can generalise it for three or more sets:

$$\begin{aligned} \mathbf{P}(A_1 \cup A_2 \cup A_3) &= \mathbf{P}((A_1 \cup A_2) \cup A_3) = \mathbf{P}(A_1 \cup A_2) + \mathbf{P}(A_3) \\ &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3). \end{aligned}$$

We can go on in the same way, to prove the equality for any finite number of disjoint events (the proof is done by induction).

Now let's look at the case when the events $A_1, A_2, \ldots, A_N$ are not necessarily disjoint. We define successively:

$$B_1 = A_1, B_2 = A_2 \setminus A_1,$$

$$B_3 = A_3 \setminus (A_1 \cup A_2), \ldots, B_N = A_N \setminus (A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{N-1}).$$

Then the events $B_1, B_2, \ldots, B_N$ are disjoint and $\bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} A_n$. We have

$$\mathbf{P}\left(\bigcup_{n=1}^{N} A_n\right) = \mathbf{P}\left(\bigcup_{n=1}^{N} B_n\right) = \sum_{n=1}^{N} \mathbf{P}(B_n) \leq \sum_{n=1}^{N} \mathbf{P}(A_n).$$

Exercise 2. Consider the stock price S_3 in Figure 3.1 on page 34.

(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$.

(ii) Compute $\widetilde{\mathbf{E}}S_1$, $\widetilde{\mathbf{E}}S_2$, $\widetilde{\mathbf{E}}S_3$. What is the average rate of growth of the stock price under $\widetilde{\mathbf{P}}$?

(iii) Answer (i) and (ii) again under the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$.

Solution.

(i) The distribution of S_3 under the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ is

32	8	2	.50
$\tilde{p}^3$	$3\tilde{p}^2\tilde{q}$	$3\tilde{p}\tilde{q}^2$	$\tilde{q}^3$

With $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$, this becomes

32	8	2	.50
.125	.375	.375	.125

(ii) By Theorem 4.4,

$$\widetilde{\mathbf{E}}\frac{S_3}{(1+r)^3} = \widetilde{\mathbf{E}}\frac{S_2}{(1+r)^2} = \widetilde{\mathbf{E}}\frac{S_1}{(1+r)} = \widetilde{\mathbf{E}}S_0 = S_0 = 4.$$

Therefore,

$$\begin{aligned}
\widetilde{\mathbf{E}}S_1 &= (1+r)S_0 = (1.25)(4) = 5 \\
\widetilde{\mathbf{E}}S_2 &= (1+r)^2S_0 = (1.25)^2(4) = 6.25 \\
\widetilde{\mathbf{E}}S_3 &= (1+r)^3S_0 = (1.25)^3(4) = 7.8125
\end{aligned}$$

In particular, we see that

$$\widetilde{\mathbf{E}}S_3 = 1.25\widetilde{\mathbf{E}}S_2, \quad \widetilde{\mathbf{E}}S_2 = 1.25\widetilde{\mathbf{E}}S_1, \quad \widetilde{\mathbf{E}}S_1 = 1.25S_0.$$

Thus, the average rate of growth of the stock price under $\widetilde{\mathbf{P}}$ is the same as the interest rate of the money market.

(iii) The distribution of S_3 under the probabilities p and q is

32	8	2	.50
p^3	$3p^2q$	$3pq^2$	q^3

With $p = \frac{2}{3}$, $q = \frac{1}{3}$, this becomes

32	8	2	.50
.2963	.4444	.2222	.0371

To compute the average rate of growth, we reason as follows:

$$\mathbf{E}_nS_{n+1} = \mathbf{E}_n\left(S_n\frac{S_{n+1}}{S_n}\right) = S_n\mathbf{E}_n\left(\frac{S_{n+1}}{S_n}\right) = (pu+qd)S_n.$$

In our case,

$$pu + qd = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 12 = 1.5,$$

i.e., the average rate of growth of the stock price under the actual probabilities is 50%.

Filally, taking expectations, we have

$$\mathbf{E}S_{n+1} = \mathbf{E}\mathbf{E}_nS_{n+1} = 1.5\mathbf{E}S_n,$$

so that

$$\begin{aligned}
ES_1 &= (1.5)ES_0 = 6 \\
ES_2 &= (1.5)ES_1 = 9 \\
ES_3 &= (1.5)ES_2 = 13.50
\end{aligned}$$

Exercise 3. Using the definition of conditional expectation (formula (3.6) on page 34) prove of Theorem 3.1:
Let N be a positive integer and let X and Y be random variables depending on the first N coin tosses. Let $0 \leq n \leq N$ be given. The following properties hold.

(i) Linearity of conditional expectations. For all constants c_1 and c_2, we have

$$\mathbf{E}_n[c_1 X + c_2 Y] = c_1 \mathbf{E}_n[X] + c_2 \mathbf{E}_n[Y].$$

(ii) Taking out what is known. If X actually depends only on the first n coin tosses, then

$$\mathbf{E}_n[XY] = X \cdot \mathbf{E}_n[Y].$$

(iii) Iterated conditioning. If $0 \leq n \leq m \leq N$, then

$$\mathbf{E}_n\Big[\mathbf{E}_m[X]\Big] = E_n[X].$$

In particular, $\mathbf{E}\Big[E_m[X]\Big] = \mathbf{E}X$.

(iv) Independence. If X depends only on tosses $n+1$ through N, then

$$\mathbf{E}_n[X] = \mathbf{E}X.$$

(v) Conditional Jensen's inquality. If $\varphi(x)$ is a convex function of the dummy variable x, then

$$\mathbf{E}_n[\varphi(X)] \geq \varphi(E_n[X]).$$

Solution. We start by recalling the definition of conditional expectation:

$$\begin{aligned} &\mathbf{E}_n[X](\omega_1 \ldots \omega_n) \\ &= \sum_{\omega_{n+1} \ldots \omega_N} p^{\#H(\omega_{n+1} \ldots \omega_N)} q^{\#T(\omega_{n+1} \ldots \omega_N)} X(\omega_1 \ldots \omega_n \omega_{n+1} \ldots \omega_N). \end{aligned}$$

(i)

$$
\begin{aligned}
&\mathbf{E}_n[c_1X + c_2Y](\omega_1 \dots \omega_n)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} [c_1X(\omega_1 \dots \omega_N) + c_2Y(\omega_1 \dots \omega_N)]\\
&\quad = \ c_1 \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} X(\omega_1 \dots \omega_N)\\
&\qquad +c_2 \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} Y(\omega_1 \dots \omega_N)\\
&\quad = \ c_1\mathbf{E}_n[X](\omega_1 \dots \omega_n) + c_2\mathbf{E}_n[Y](\omega_1 \dots \omega_n)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\mathbf{E}_n[XY](\omega_1 \dots \omega_n)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} X(\omega_1 \dots \omega_n) Y(\omega_1 \dots \omega_n\omega_{n+1} \dots \omega_N)\\
&\quad = X(\omega_1 \dots \omega_n) \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} Y(\omega_1 \dots \omega_n\omega_{n+1} \dots \omega_N)\\
&\quad = \ X(\omega_1 \dots \omega_n)\mathbf{E}_n[Y](\omega_1 \dots \omega_n).
\end{aligned}
$$

(iii) Denote $Z = E_m[X]$. Then Z actually depends on $\omega_1\omega_2 \dots \omega_m$ only and

$$
\begin{aligned}
&\mathbf{E}_n[\mathbf{E}_m[X]](\omega_1 \dots \omega_n) = \mathbf{E}_n[Z](\omega_1 \dots \omega_n)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} Z(\omega_1 \dots \omega_n\omega_{n+1} \dots \omega_m)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_m} p^{\#H(\omega_{n+1}\dots\omega_m)} q^{\#T(\omega_{n+1}\dots\omega_m)} Z(\omega_1 \dots \omega_m) \sum_{\omega_{m+1}\dots\omega_N} p^{\#H(\omega_{m+1}\dots\omega_N)} q^{\#T(\omega_{m+1}\dots\omega_N)}\\
&\quad = \sum_{\omega_{n+1}\dots\omega_m} p^{\#H(\omega_{n+1}\dots\omega_m)} q^{\#T(\omega_{n+1}\dots\omega_m)} Z(\omega_1 \dots \omega_m)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_m} p^{\#H(\omega_{n+1}\dots\omega_m)} q^{\#T(\omega_{n+1}\dots\omega_m)} \sum_{\omega_{m+1}\dots\omega_N} p^{\#H(\omega_{m+1}\dots\omega_N)} q^{\#T(\omega_{m+1}\dots\omega_N)} X(\omega_1 \dots \omega_N)\\
&\quad = \sum_{\omega_{n+1}\dots\omega_N} p^{\#H(\omega_{n+1}\dots\omega_N)} q^{\#T(\omega_{n+1}\dots\omega_N)} X(\omega_1 \dots \omega_N)\\
&\quad = \ \mathbf{E}_n[X](\omega_1 \dots \omega_n).
\end{aligned}
$$

(iv)

$$
\mathbf{E}_n[X](\omega_1 \dots \omega_n)
$$

$$
\begin{aligned}
&= \sum_{\omega_{n+1}\ldots\omega_N} p^{\#H(\omega_{n+1}\ldots\omega_N)} q^{\#T(\omega_{n+1}\ldots\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
&= \sum_{\omega_1\ldots\omega_n} p^{\#H(\omega_1\ldots\omega_n)} q^{\#T(\omega_1\ldots\omega_n)} \sum_{\omega_{n+1}\ldots\omega_N} p^{\#H(\omega_{n+1}\ldots\omega_N)} q^{\#T(\omega_{n+1}\ldots\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
&= \sum_{\omega_1\ldots\omega_N} p^{\#H(\omega_1\ldots\omega_N)} q^{\#T(\omega_1\ldots\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
&= \mathbf{E}X
\end{aligned}
$$

(v) Let φ be a convex function, and denote by $\mathcal{L}$ the class of all linear functions l which lie below φ, i.e., such that $l(y) \leq \varphi(y)$ for all y. Then, as shown in the proof of Theorem 2.5,

$$\varphi(x) = \max_{l \in \mathcal{L}} l(x)$$

for all x.

For a random variable x, and for all $l \in \mathcal{L}$, we have

$$\varphi(X) \geq l(X),$$

and consequently,

$$\mathbf{E}_n[\varphi(X)] \geq \mathbf{E}_n[l(X)].$$

On the other hand, by property (i) (linearity),

$$\mathbf{E}_n[l(X)] = l(\mathbf{E}_n[X])$$

so that

$$\mathbf{E}_n[\varphi(X)] \geq l(\mathbf{E}_n[X])$$

for all $l \in \mathcal{L}$. Therefore

$$\mathbf{E}_n[\varphi(X)] \geq \max_{l \in \mathcal{L}} l(\mathbf{E}_n[X]) = \varphi(\mathbf{E}_n[X]).$$

Exercise 4. Show that a convex function of a martingale is a submartingale. In other words, let $M_0, M_1, \ldots, M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.
Solution. Let an arbitrary n with $0 \leq n \leq N-1$ be given. By the Martingale Property, we have

$$\mathbf{E}_n M_{n+1} = M_n,$$

and hence

$$\varphi(\mathbf{E}_n M_{n+1}) = \varphi(M_n).$$

On the other hand, by the Conditional Jensen's Inequality,

$$\mathbf{E}_n \varphi(M_{n+1}) \geq \varphi(\mathbf{E}_n M_{n+1}).$$

Combining these two, we get

$$\mathbf{E}_n \varphi(M_{n+1}) \geq \varphi(M_n),$$

and since n was arbitrary, this implies that the sequence of random variables $\varphi(M_0)$, $\varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.