46-679 Multi-Period Pricing Models
Homework Set 1
Due July 20, 1999
Solutions

Exercise 2. Suppose in the situation of Example 1.1.1 that the option sells for 1.20 at time zero. Consider an agent who begins with wealth $X_0 = 0$ and at time zero buys Δ_0 shares of stock and Γ_0 options. The numbers Δ_0 and Γ_0 can be either positive or negative or zero. This leaves the agent with a cash position of $-4\Delta_0 - 1.20\Gamma_0$. If this is positive, it is invested in the money market; if it is negative, it represents money borrowed from the money market. At time one, the value of the agent's portfolio of stock, option and money market is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)^+ - \frac{5}{4}\left(4\Delta_0 + 1.20\Gamma_0\right).$$

Show that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. In other words, one cannot find an arbitrage when the time-zero price of the option is 1.20.

Solution. Considering the cases of a head and of a tail at the first tossing, and utilizing the concrete numbers given, we can write:

$$\begin{aligned} X_1(H) &= 8\Delta_0 + 3 \cdot \Gamma_0 - \frac{5}{4}(4\Delta_0 + 1.20 \cdot \Gamma_0) \\ X_1(T) &= 2\Delta_0 + 0 \cdot \Gamma_0 - \frac{5}{4}(4\Delta_0 + 1.20 \cdot \Gamma_0) \end{aligned}$$

Adding these, we get:

$$X_1(H) + X_1(T) = 10\Delta_0 + 3\Gamma_0 - 10\Delta_0 - 3\Gamma_0 = 0,$$

or, equivalently,

$$X_1(H) = -X_1(T).$$

In other words, either $X_1(H)$ and $X_1(T)$ are both zeros, or they have opposite signs. Taking into account that both $p > 0$ and $q > 0$, we conclude that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative.

Exercise 3. In the one-period binomial model of Section 1.1, suppose we want to determine the price at time zero of the derivative security $V_1 = S_1$, i.e., the derivative security pays off the stock price. (This can be regarded as a European call with strike price $K = 0$). What is the time-zero price V_0 given by the risk-neutral pricing formula (1.1.10)?

Solution. We have

$$\begin{aligned}
V_0 &= \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \\
&= \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] \\
&= \frac{1}{1+r}[\tilde{p}uS_0 + \tilde{q}dS_0] \\
&= S_0\frac{1}{1+r}[u\tilde{p} + d\tilde{q}d] \\
&= S_0\frac{1}{1+r}\frac{u((1+r)-d) + d(u-(1+r))}{u-d} \\
&= S_0.
\end{aligned}$$

Exercise 5. In Example 1.2.4, we considered an agent who sold the lookback option for $V_0 = 1.376$ and bought $\Delta_0 = 0.1733$ shares of stock at time zero. At time one if the stock goes up she has a portfolio valued at $V_1(H) = 2.24$. Assume that she now takes a position of $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(HT)}$ in the stock. Show that at time two, if the stock goes up again, she will have a portfolio valued at $V_2(HH) = 3.20$, whereas if the stock goes down, her portfolio will be worth $V_2(HT) = 2.40$. Finally, under the assumption that the stock goes up in the first period and down in the second period, assume the agent takes a position of $\Delta_2(HT) = \frac{V_3(HTH)-V_3(HTT)}{S_3(HTH)-S_3(HTT)}$ in the stock. Show that at time three, if the stock goes up in the third period, she will have a portfolio valued at $V_3(HTH) = 0$, whereas if the stock goes down, her portfolio will be worth $V_3(HTT) = 6$. In other words, she has hedged her short position in the option.

Solution. We have:

$$\begin{aligned}
\Delta_1(H) &= \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.20 - 2.40}{16 - 4} = .0667 \\
\Delta_2(HT) &= \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0 - 6}{8 - 2} = -1
\end{aligned}$$

Assuming the market goes first up, at time 1 we have $X_1(H) = 2.24 = V_1(H)$. Redistribute this amount, by taking a position $\Delta_1(H) = .0667$ in stock:

$$2.24 = (.0667)(8) + (1.7064)$$

At time 2:

$$\begin{aligned} X_2(HH) &= (.0667)(16) + (1.25)(1.7064) = 3.20 = V_2(HH) \\ X_2(HT) &= (.0667)(4) + (1.25)(1.7064) = 2.40 = V_2(HT) \end{aligned}$$

Assuming that the market on the first two steps goes first up and then down, redistribute the amount 2.40, by taking a position $\Delta_2(HT) = -1$ in stock:

$$2.40 = (-1)(4) + 6.40$$

At time 3:

$$\begin{aligned} X_3(HTH) &= (-1)(8) + (1.25)(6.40) = 0 = V_3(HTH) \\ X_3(HTT) &= (-1)(2) + (1.25)(6.40) = 6 = V_3(HTT) \end{aligned}$$

Exercise 7. Consider the three-period model of Example 1.2.1, with $S_0 = 4$, $u = 2$, $d = 1/2$, and take the interest rate $r = \frac{1}{4}$, so that $\tilde{p} = \tilde{q} = 1/2$. For $n = 0, 1, 2, 3$, define $Y_n = \sum_{k=0}^{n} S_k$ to be the sum of the stock prices between times zero and n. Consider an *Asian option* whose payoff at time three is $\left(\frac{1}{4}Y_3 - 4\right)^+$. This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_3(s, y) = \left(\frac{1}{4}y - 4\right)^+$.

(i) Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.

(ii) Apply the algorithm developed in (i) to compute $v_0(4, 4)$, the price of the Asian option at time zero.

(iii) Provide a formula for $\delta_n(s, y)$, the number of shares of stock which should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.

Solution. Assume that at time n, $S_n = s$ and $Y_n = y$. Then, at time $n+1$, if the maket gous up, we have

$$S_{n+1} = us, \quad Y_{n+1} = Y_n + S_{n+1} = y + us,$$

and, similarly, if the maket gous down, we have

$$S_{n+1} = ds, \quad Y_{n+1} = Y_n + S_{n+1} = y + ds.$$

Therefore, formulas (1.2.13) and (1.2.14) take the form:

$$\begin{aligned}
v_n(s,y) &= \frac{1}{1+r}[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)] \\
\delta_n(s,y) &= \frac{v_{n+1}(us, y+us) - v_{n+1}(ds, y+ds)}{us - ds}.
\end{aligned}$$

Applying this algorithm to the numerical example at hand, we have:

$$\begin{aligned}
v_0(4,4) &= \frac{4}{5}\left[\frac{1}{2}v_1(8,12) + \frac{1}{2}v_1(2,6)\right] \\
v_1(8,12) &= \frac{4}{5}\left[\frac{1}{2}v_2(16,28) + \frac{1}{2}v_2(4,16)\right] \\
v_1(2,6) &= \frac{4}{5}\left[\frac{1}{2}v_2(4,10) + \frac{1}{2}v_2(1,7)\right] \\
v_2(16,28) &= \frac{4}{5}\left[\frac{1}{2}v_3(32,60) + \frac{1}{2}v_3(8,36)\right] \\
v_2(4,16) &= \frac{4}{5}\left[\frac{1}{2}v_3(8,24) + \frac{1}{2}v_3(2,18)\right] \\
v_2(4,10) &= \frac{4}{5}\left[\frac{1}{2}v_3(8,18) + \frac{1}{2}v_3(2,12)\right] \\
v_2(1,7) &= \frac{4}{5}\left[\frac{1}{2}v_3(2,9) + \frac{1}{2}v_3(.50,7.50)\right]
\end{aligned}$$

And for v_3 we have

$$\begin{aligned}
v_3(32,60) &= (60/4-4)^+ = 11 \\
v_3(8,36) &= (36/4-4)^+ = 5 \\
v_3(8,24) &= (24/4-4)^+ = 2 \\
v_3(2,18) &= (18/4-4)^+ = .50
\end{aligned}$$

$$\begin{aligned}
v_3(8,18) &= (18/4-4)^+ = .50 \\
v_3(2,12) &= (12/4-4)^+ = 0 \\
v_3(2,9) &= (9/4-4)^+ = 0 \\
v_3(.50,7.50) &= (7.50/4-4)^+ = 0
\end{aligned}$$

Substituting these in the previous equations, we obtain successively:

$$v_2(16,28) = 6.40, v_2(4,16) = 1, v_2(4,10) = .20, v_2(1,7) = 0,$$

$$v_1(8,12) = 2.96, v_1(2,6) = 0.08,$$
$$v_0(4,4) = 1.216$$