Department of Mathematics
CARNEGIE MELLON UNIVERSITY
Professor Doytchinov
21-256 Multivariate Analysis and Approximation, Fall 1998

Review for Test 4
Solutions

1. * Compute the sum
$$1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\cdots+\frac{n}{3^{n-1}}.$$

Solution. Let $r \neq 1$. Then,

$$\begin{aligned}
S_n(r) &:= 1+2r+3r^2+4r^3+\cdots+nr^{n-1}\\
&= 1+r+r^2+r^3+\cdots+r^{n-1}\\
&\quad +r+r^2+r^3+\cdots+r^{n-1}\\
&\quad +r^2+r^3+\cdots+r^{n-1}\\
&\quad +r^3+\cdots+r^{n-1}\\
&\quad +\cdots\\
&\quad +r^{n-1}\\
&= \frac{1-r^n}{1-r}+r\frac{1-r^{n-1}}{1-r}+r^2\frac{1-r^{n-2}}{1-r}+r^3\frac{1-r^{n-3}}{1-r}+\cdots+r^{n-1}\frac{1-r}{1-r}\\
&= \frac{1-r^n}{1-r}+\frac{r-r^n}{1-r}+\frac{r^2-r^n}{1-r}+\frac{r^3-r^n}{1-r}+\cdots+\frac{r^{n-1}-r^n}{1-r}\\
&= \frac{1}{1-r}(1+r+r^2+r^3+\cdots+r^{n-1}-nr^n)\\
&= \frac{1}{1-r}\left(\frac{1-r^n}{1-r}-nr^n\right)=\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}.
\end{aligned}$$

In particular, for $r=1/3$, we have
$$S_n(1/3)=\left(\frac{3}{2}\right)^2\left(1-(n+1)\left(\frac{1}{3}\right)^n+n\left(\frac{1}{3}\right)^{n+1}\right).$$

2. * Let $0<r<1$. Find the value of
$$S=1+2r+3r^2+\cdots+nr^{n-1}+\cdots$$
as a function of r.

Solution. As we showed in problem 1,
$$S_n(r):=1+2r+3r^2+4r^3+\cdots+nr^{n-1}=\frac{1}{1-r}\left(\frac{1-r^n}{1-r}-nr^n\right)=\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}.$$

Now, for $0<r<1$, we have
$$S(r)=\lim_{n\to\infty}S_n(r)=\frac{1}{(1-r)^2}.$$

Remark. For another approach, see Problem 13(b).

3. * Consider the following periodic decimal fractions:

$$\begin{aligned} a &= 8.04848484848\ldots \\ b &= 0.00404040404\ldots \end{aligned}$$

Compute ab and a/b.
Solution. Observe that

$$\begin{aligned} a &= 8.04848484848\ldots = 8 + 10^{-3}(48) + 10^{-5}(48) + 10^{-7}(48) + \cdots \\ &= 8 + 10^{-3}(48)\frac{1}{1-10^{-2}} = 8 + \frac{48}{990} = \frac{7968}{990} \end{aligned}$$

and similarly,

$$\begin{aligned} b &= 0.00404040404\ldots = 10^{-3}(4) + 10^{-5}(4) + 10^{-7}(4) + \cdots \\ &= 10^{-3}(4)\frac{1}{1-10^{-2}} = \frac{4}{990} \end{aligned}$$

Therefore,

$$ab = \left(\frac{7968}{990}\right)\left(\frac{4}{990}\right), \text{ and } a/b = \frac{7968}{4} = 1992.$$

4. * Let $a = \underbrace{1111\cdots111}_{1998 \text{ ones}}$, and $b = \underbrace{2222\cdots222}_{999 \text{ twos}}$. Compute $\sqrt{a-b}$.
Solution. Observe that

$$a = 1 + 10 + 10^2 + 10^3 + \cdots + 10^{1997} = \frac{10^{1998}-1}{10-1} = \frac{10^{1998}-1}{9},$$

and

$$b = 2(1 + 10 + 10^2 + 10^3 + \cdots + 10^{998}) = 2\frac{10^{999}-1}{10-1} = \frac{2(10^{999})-2}{9}.$$

Therefore,

$$a - b = \frac{10^{1998}-1}{9} - \frac{2(10^{999})-2}{9} = \frac{10^{1998}-2(10^{999})+1}{9} = \left(\frac{10^{999}-1}{3}\right)^2,$$

so that

$$\sqrt{a-b} = \frac{10^{999}-1}{3} = \underbrace{3333\cdots333}_{999 \text{ threes}}.$$

5. The population of a city is 20000, and increases by 7% each year. After how many years will the population double?
Solution. Let P_n denote the population at the end of the n^{th} year. Then $P_0 = 20000$ and

$$P_n = P_0(1.07)^n.$$

The population will double when $P_n = 2P_0$, i.e. when $(1.07)^n = 2$. Solving this for n gives

$$n = \frac{\ln 2}{\ln 1.07} \text{ years.}$$

6. In a far-off country, there exist 400 animals of a certain unique endangered species. It is estimated, that each year the the existing animals give offspring that is numerically equal to 25% of the previous year's population, while 5% of the previous year's population dies from natural causes. It is also estimated, that each year 100 animals are killed by illegal local hunters for their valuable fur. After how many years will the species become extinct?
Solution. Let P_n denote the number of animals at the end of the n^{th} year. Then $P_0 = 400$, and

$$P_{n+1} = P_n + .25P_n - .05P_n - 100 = 1.2P_n - 100.$$

The above formula is true as long as $1.2P_n \geq 100$. As soon as $1.2P_n \leq 100$, we have $P_{n+1} = 0$, which means the species is extinct. We have

$$P_n = (1.2)^n P_0 - 100(1.2^{n-1} + 1.2^{n-2} + \cdots + 1.2 + 1) = 1.2^n 400 - 100\frac{1.2^n - 1}{.2}.$$

Setting $P_n = 0$, we get

$$\begin{aligned} 1.2^n 400 &= 100\frac{1.2^n - 1}{.2} \\ 400 &= 500(1 - 1.2^{-n}) \\ 1 - 1.2^{-n} &= .8 \\ 1.2^{-n} &= .2 \\ n &= \frac{-\ln .2}{\ln 1.2} \end{aligned}$$

7. A certain savings account pays 8%, compounded annually. At the beginning, \$5000 are deposited. Further, deposits are made at the end of each year, increasing the amount of the deposit by 2%. Give a general formula for the amount accumulated at the end of the n^{th} year.
Solution. We have

$$P_n = 5000(1.08)^n + 5000(1.02)(1.08)^{n-1} + 5000(1.02)^2(1.08)^{n-2} + \cdots + 5000(1.02)^n.$$

Observe that this is the sum of a geometric sequence: each term is obtained from the previous by multiplying it by $1.02/1.08$. Therefore,

$$P_n = 5000(1.08)^n \frac{1 - (1.02/1.08)^{n+1}}{1 - (1.02/1.08)} = 5000\frac{1.08^{n+1} - 1.02^{n+1}}{1.08 - 1.02}.$$

8. In order to generate income for 15 years, \$300000 are deposited into a savings account paying 7% compounded annually. If a constant amount of money is to be withdrawn at the end of each of the 15 years, how large can this amount be?
Soluton. Let R be the amount of dollars that we withdraw at the end of each year, and denote by P_n the amount of dollars left in the account at the end of the n^{th} year. Then $P_0 = 300000$ and

$$P_{n+1} = 1.07P_n - R,$$

so that

$$P_n = 1.07^n P_0 - R(1.07^{n-1} + 1.07^{n-2} + \cdots + 1.07 + 1) = 1.07^n P_0 - R\frac{1.07^n - 1}{.07}.$$

Setting $n = 15$ and $P_{15} = 0$, we get

$$\begin{aligned} 1.07^{15} P_0 &= R\frac{1.07^{15} - 1}{.07} \\ R &= P_0\frac{.07}{1 - 1.07^{-15}} = 300000\frac{.07}{1 - 1.07^{-15}} \end{aligned}$$

9. Determine the monthly payment on a 30-year mortgage with a loan amount of \$240000 and annual percentage rate 12%, compounded monthly.
Solution. In a similar way, denote by P_n the unpaid amount after the n^{th} month, and by R the monthly payment. Then
$$P_n = 1.01^n P_0 - R\frac{1.01^n - 1}{0.01}.$$
Setting $n = 360$ and $P_{360} = 0$, we get
$$\begin{aligned} 1.01^{360} P_0 &= 100R(1.01^{360} - 1) \\ R &= \frac{P_0}{100(1 - 1.01^{-360})} = \frac{2400}{1 - 1.01^{-360}}. \end{aligned}$$

10. Use Taylor's formula to find the Taylor series for $f(x)$ at the given value of a.
   (a) $f(x) = \cos x, \quad a = \pi/4$,
   (b) $f(x) = \sin x, \quad a = -\pi/4$,
   (c) $f(x) = \sqrt{x}, \quad a = 9$,
   (d) $f(x) = \ln x, \quad a = 2$.

   Solution.

   (a) $a = \pi/4$
$$\begin{aligned} f(x) &= \cos x, & f(a) &= \sqrt{2}/2 \\ f'(x) &= -\sin x, & f'(a) &= -\sqrt{2}/2 \\ f''(x) &= -\cos x, & f''(a) &= -\sqrt{2}/2 \\ f'''(x) &= \sin x, & f'''(a) &= \sqrt{2}/2 \\ f^{\text{iv}}(x) &= \cos x, & f^{\text{iv}}(a) &= \sqrt{2}/2 \\ &\cdots \quad \cdots & &\cdots \end{aligned}$$
   and
$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{2}\left(1 - \frac{(x-\pi/4)}{1!} - \frac{(x-\pi/4)^2}{2!} + \frac{(x-\pi/4)^3}{3!}\right. \\ &\quad \left. + \frac{(x-\pi/4)^4}{4!} - \frac{(x-\pi/4)^5}{5!} - \frac{(x-\pi/4)^6}{6!} + \cdots\right). \end{aligned}$$

   (b) This is similar to part (a). $a = -\pi/4$
$$\begin{aligned} f(x) &= \sin x, & f(a) &= -\sqrt{2}/2 \\ f'(x) &= \cos x, & f'(a) &= \sqrt{2}/2 \\ f''(x) &= -\sin x, & f''(a) &= \sqrt{2}/2 \\ f'''(x) &= -\cos x, & f'''(a) &= -\sqrt{2}/2 \\ f^{\text{iv}}(x) &= \sin x, & f^{\text{iv}}(a) &= -\sqrt{2}/2 \\ &\cdots \quad \cdots & &\cdots \end{aligned}$$
   and
$$\begin{aligned} f(x) &= -\frac{\sqrt{2}}{2}\left(1 - \frac{(x+\pi/4)}{1!} - \frac{(x+\pi/4)^2}{2!} + \frac{(x+\pi/4)^3}{3!}\right. \\ &\quad \left. + \frac{(x+\pi/4)^4}{4!} - \frac{(x+\pi/4)^5}{5!} - \frac{(x+\pi/4)^6}{6!} + \cdots\right). \end{aligned}$$

(c) $a = 9$.

$$\begin{aligned} f(x) &= x^{1/2}, & f(a) &= 3 \\ f'(x) &= \frac{1}{2}x^{-1/2}, & f'(a) &= \frac{1}{2}\cdot 3^{-1} \\ f''(x) &= \frac{1}{2}(-\frac{1}{2})x^{-3/2}, & f''(a) &= -\frac{1}{4}\cdot 3^{-3} \\ f'''(x) &= \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{-5/2}, & f'''(a) &= \frac{3}{8}\cdot 3^{-5} \\ & \cdots & \cdots & \cdots \end{aligned}$$

$$f^{(n)}(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots\left(-\frac{2n-3}{2}\right)x^{-(2n-1)/2},$$

$$f^{(n)}(a) = (-1)^{n-1}\frac{(1)(3)\ldots(2n-3)}{2^n}\cdot 3^{-(2n-1)}$$

and

$$\begin{aligned} \sqrt{x} &= 3 + \frac{1}{2}\frac{1}{3}\frac{(x-9)}{1!} - \frac{1}{4}\frac{1}{3^3}\frac{(x-9)^2}{2!} + \frac{3}{8}\frac{1}{3^5}\frac{(x-9)^3}{3!} \\ &\quad + \cdots + (-1)^{n-1}\frac{(1)(3)\ldots(2n-3)}{2^n}\frac{1}{3^{2n-1}}\frac{(x-9)^n}{n!} + \cdots \end{aligned}$$

(d) $a = 2$.

$$\begin{aligned} f(x) &= \ln x, & f(a) &= \ln 2 \\ f'(x) &= x^{-1}, & f'(a) &= 1/2 \\ f''(x) &= -x^{-2}, & f''(a) &= -1/4 \\ f'''(x) &= (-1)(-2)x^{-3}, & f'''(a) &= 2/8 \\ & \cdots & \cdots & \cdots \end{aligned}$$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n+1)x^{-n} = (-1)^{n-1}x^{-n}, \qquad f^{(n)}(a) = (-1)^{n-1}(n-1)!2^{-n}$$

and

$$\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2}\cdot\frac{1}{2^2}(x-2)^2 + \cdots + (-1)^{n-1}\frac{1}{n\,2^n}(x-2)^n + \cdots$$

11. Obtain the Maclaurin series for the given function by manipulating standard Maclaurin series decompositions given on the formula sheet.

(a) $f(x) = e^{2x}$,

(b) $f(x) = e^{x+2}$,

(c) $f(x) = x^2\cos(x^3)$,

(d) $f(x) = \ln\sqrt{\frac{1+x}{1-x}}$.

Solution.

(a)

$$f(x) = e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

(b)

$$f(x) = e^{x+2} = e^2\,e^x = e^2\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{e^2}{n!}x^n$$

(c)

$$f(x) = x^2\cos(x^3) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(2n)!}$$

(d)

$$\begin{aligned} f(x) &= \ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}(\ln(1+x)-\ln(1-x)) \\ &= \frac{1}{2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\cdots\right) \\ &\quad -\frac{1}{2}\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right) \\ &= x+\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \end{aligned}$$

12. Evaluate the indefinite integral as a power series:

(a) $\int e^{x^3}\,dx$,
(b) $\int \frac{\sin x}{x}\,dx$,
(c) $\int \frac{\arctan x}{x}\,dx$,
(d) $\int \frac{x}{1-x^7}\,dx$.

Solution.

(a)

$$\int e^{x^3}dx = \int\sum_{n=0}^{\infty}\frac{(x^3)^n}{n!}dx = \sum_{n=0}^{\infty}\int\frac{x^{3n}}{n!}dx = \sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!(3n+1)}+C$$

(b)

$$\int\frac{\sin x}{x}dx = \int\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\int\frac{(-1)^n x^{2n}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}+C$$

(c)

$$\int\frac{\arctan x}{x}dx = \int\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}dx = \sum_{n=0}^{\infty}(-1)^n\int\frac{x^{2n}}{(2n+1)}dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2}+C$$

(d)

$$\int\frac{x}{1-x^7}dx = \int x\sum_{n=0}^{\infty}x^{7n}\,dx = \sum_{n=0}^{\infty}\int x^{7n+1}\,dx = \sum_{n=0}^{\infty}\frac{x^{7n+2}}{(7n+2)}+C$$

13. Compute the following sums as functions of x:

(a) $1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$,
(b) $1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots$,
(c) $1-x^4+\frac{x^8}{2!}-\frac{x^{12}}{3!}+\cdots+(-1)^n\frac{x^{4n}}{n!}+\cdots$.

Solution.

(a)

$$\begin{aligned}
&1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \\
&\quad = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n + \cdots \\
&\quad = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}
\end{aligned}$$

(b)

$$\begin{aligned}
&1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots \\
&\quad = (1)' + (x)' + (x^2)' + (x^3)' + (x^4)' + \cdots + (x^{n+1})' + \cdots \\
&\quad = (1 + x + x^2 + x^3 + x^4 + \cdots + x^{n+1} + \cdots)' \\
&\quad = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}
\end{aligned}$$

(c)

$$\begin{aligned}
&1 - x^4 + \frac{x^8}{2!} - \frac{x^{12}}{3!} + \cdots + (-1)^n \frac{x^{4n}}{n!} + \cdots \\
&\quad = 1 + \frac{(-x^4)}{1!} + \frac{(-x^4)^2}{2!} + \frac{(-x^4)^3}{3!} + \cdots + \frac{(-x^4)^n}{n!} + \cdots \\
&\quad = e^{-x^4}
\end{aligned}$$