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Review for Test 3
Solutions

1. Consider the function f defined by $f(x, y, z) = xy + y^2 z$.

   (a) Find ∇f.

   (b) Find the directional derivative of f at the point $(2, 3, -1)$ in the direction of the vector $\mathbf{v} = \langle 2, 4, -3\rangle$.

   Solution.
$$\nabla f(x, y, z) = \langle y, x + 2yz, y^2\rangle, \nabla f(2, 3, -1) = \langle 3, -4, 9\rangle, |\mathbf{v}| = \sqrt{4 + 16 + 9} = \sqrt{29},$$
$$D_{\mathbf{v}} f(2, 3, -1) = \nabla f(2, 3, -1) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{6 - 16 - 27}{\sqrt{29}} = -\frac{37}{\sqrt{29}}.$$

2. Find the maximum rate of change of f at the given point and the direction in which it occurs.

   (a) $f(x, y) = x^2 e^y - 3y, \ \ (1, 0)$,

   (b) $f(x, y, z) = \cos(4x - 3y) + z^2, \ \ (\pi/16, -\pi/6, 2)$,

   (c) $f(x, y) = \sqrt{5x + y^2}, \ \ (4, 4)$.

   Solution. The maximum rate of change of a function f at a point P is $|\nabla f(P)|$ and it occurs in the direction of the gradient $\nabla f(P)$. Thus,

   (a) $\nabla f(x, y) = \langle 2xe^y, x^2 e^y - 3\rangle$,
   direction of fastest increase at the point $(1, 0)$ is $\nabla f(1, 0) = \langle 2, -2\rangle$,
   the maximal rate of change is $|\nabla f(1, 0)| = \sqrt{4 + 4} = 2\sqrt{2}$.

   (b) $\nabla f(x, y, z) = \langle -4 \sin(4x + 3y), 3 \sin(4x - 3y), 2z\rangle$,
   direction of fastest increase at the point $(\pi/16, -\pi/6, 2)$ is
   $\nabla f(\pi/16, -\pi/6, 2) = \langle -2\sqrt{2}, 3/\sqrt{2}, 4\rangle$,
   the maximal rate of change is $|\nabla f(\pi/16, -\pi/6, 2)| = \sqrt{8 + 4.5 + 16} = \sqrt{57/2}$.

   (c)
$$\nabla f(x, y) = \frac{\langle 5, 2y\rangle}{2\sqrt{5x + y^2}},$$
   the direction of fastest increase at the point $(4, 4)$ is $\nabla f(4, 4) = \dfrac{\langle 5, 8\rangle}{12}$,
   the maximal rate of change is $|\nabla f(4, 4)| = \sqrt{89}/12$.

3. * Let f be a smooth function with the following property: At the point (1,2,3), the function f increases most rapidly in the direction given by the vector $\langle 5, -3, 0\rangle$ and the rate of change of f in this direction is 17. Find $\nabla f(1, 2, 3)$.
   Solution. Since a function increases most rapidly in the direction of its gradient, $\nabla f(1, 2, 3)$ must have the same direction as the vector $\mathbf{v} = \langle 5, -3, 0\rangle$. Further, the most rapid rate of increase is equal to the magnitude of the gradient, so $|\nabla f(1, 2, 3)| = 17$. Therefore,
$$\nabla f(1, 2, 3) = 17 \frac{\mathbf{v}}{|\mathbf{v}|} = 17 \frac{\langle 5, -3, 0\rangle}{\sqrt{25 + 9}} = \frac{\langle 85, -51, 0\rangle}{\sqrt{34}}.$$

4. * Let f be a smooth function with the following property: At the point (0,0,0), the function f *decreases* most rapidly in the direction of $\langle 6,-1,2\rangle$ and the rate of change of f in this direction is -5. Find $\nabla f(0,0,0)$.
Solution. Since a function decreases most rapidly in the direction opposite to its gradient, $\nabla f(0,0,0)$ must be parallel to the vector $\mathbf{v} = \langle 6,-1,2\rangle$, but with opposite direction. Further, the absolute value of the most rapid rate of change is equal to the magnitude of the gradient, so $|\nabla f(0,0,0)| = 5$. Therefore,

$$\nabla f(0,0,0) = 5\left(-\frac{\mathbf{v}}{|\mathbf{v}|}\right) = -5\frac{\langle 6,-1,2\rangle}{\sqrt{36+1+4}} = \frac{\langle -30,5,-10\rangle}{\sqrt{41}}.$$

5. * Let $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$ and $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$. Assume that f_x and f_y are continuous. If $D_{\mathbf{u}}f(2,1) = 2\sqrt{2}$ and $D_{\mathbf{v}}f(2,1) = -8\sqrt{2}$, find $\nabla f(2,1)$.
Solution. Since the two vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $\nabla f(2,1) = a\mathbf{u} + b\mathbf{v}$ for some real numbers, a and b. To find these numbers, we dot-multiply the equality by $\mathbf{u}$ and $\mathbf{v}$:

$$2\sqrt{2} = D_{\mathbf{u}}f(2,1) = \nabla f(2,1)\cdot\mathbf{u} = (a)(1) + (b)(0) = a,$$

$$-8\sqrt{2} = D_{\mathbf{v}}f(2,1) = \nabla f(2,1)\cdot\mathbf{v} = (a)(0) + (b)(1) = b.$$

Thus,

$$\nabla f(2,1) = a\mathbf{u} + b\mathbf{v} = 2\sqrt{2}\mathbf{u} - 8\sqrt{2}\mathbf{v} = \langle 2,2\rangle + \langle -8,8\rangle = \langle -6,10\rangle.$$

6. Find the tangent plane and normal line to the surface $z = x^2 + 4y^2$ at the point $(-1,2,17)$.
Solution. The surface under consideration is described by the equation $F(x,y,z) = 0$, where $F(x,y,z) = x^2 + 4y^2 - z$. Further,

$$\begin{array}{ll} F_x(x,y,z) = 2x & F_x(-1,2,17) = -2 \\ F_y(x,y,z) = 8y & F_y(-1,2,17) = 16 \\ F_z(x,y,z) = -1 & F_z(-1,2,17) = -1 \end{array}$$

The tangent plane through $(-1,2,17)$ has equation

$$F_x(-1,2,17)(x+1) + F_y(-1,2,17)(y-2) + F_z(-1,2,17)(z-17) = 0,$$

i.e.,

$$-2(x+1) + 16(y-2) - (z-17),$$

or, equivalently,

$$-2x + 16y - z - 17 = 0.$$

The normal line is a line that passes through the point $(-1,2,17)$ in the direction of the normal vector $\langle -2,16,-1\rangle$. Therefore it has parametric equations $x = -1-2t, y = 2+16t, z = 17-t$, and symmetric equations

$$\frac{x+1}{-2} = \frac{2-y}{16} = 17 - z.$$

7. Find all the points on the surface $x^2 + 4y^2 - z^2 = 4$ at which the tangent plane is parallel to the plane $x + 2y + z = 6$.
Solution. Let $F(x,y,z) = x^2 + 4y^2 - z^2 - 4$. Then $F_x(x,y,z) = 2x, F_y(x,y,z) = 8y, F_z(x,y,z) = -2z$. The tangent plane at a point (a,b,c) on the surface $F(x,y,z) = 0$ is the given by the equation

$$2a(x-a) + 8b(y-b) - 2c(z-c) = 0,$$

or, equivalently,

$$a(x-a)+4b(y-b)-c(z-c)=0,$$

and has a normal vector $\langle a, 4b, -c\rangle$. The tangent plane will be parallel to the plane $x+2y+z=6$, if and only if the normal vectors of the two planes are equal, i.e., $\langle a, 4b, -c\rangle = \lambda\langle 1,2,1\rangle$ for some number λ. This gives us the system

$$\begin{cases} a &= \lambda \\ 4b &= 2\lambda \\ -c &= \lambda \\ a^2+4b^2-c^2 &= 4 \end{cases}$$

Substituting a, b, c in the last equation, we get

$$\lambda^2 + 4(\lambda/2)^2 - \lambda^2 = 4,$$

whence $\lambda = \pm 2$. This gives us two points on the surface: $(a,b,c) = (2,1,-2)$ and $(a,b,c) = (-2,-1,2)$.

8. Which point on the parabola $y = x^2$ is nearest to $(8, 5/2)$?
Solution. All points on the parabola have coordinates (x, x^2). The square of the distance from a point like this to the point $(8, 5/2)$ is given by

$$f(x) = (x-8)^2 + \left(x^2 - \frac{5}{2}\right)^2 = x^4 - 4x^2 - 16x + \frac{281}{4}.$$

To minimize this function, we find its derivative and set it equal to 0:

$$4x^3 - 8x - 16 = 0.$$

This gives us a solution $x^* = 2$, and the coordinates of the point are $(x^*, (x^*)^2) = (2,4)$. Since the second derivative is positive, this is a minimum.

9. A landscape architect plans to enclose a 2400 square foot rectangular region in a botanical garden. She will use shrubs costing \$30 per foot along 3 sides and fencing costing \$10 per foot along the fourth side. Find the minimum total cost.
Solution. Let x denote the length of the side along which there will fencing. Then the other side of the rectangle will have length $2400/x$ and the total cost will be

$$f(x) = 10x + 30(x + 2\frac{2400}{x}) = 40x + \frac{144000}{x}.$$

To find the minimum of this function, we find the derivative and set it equal to 0:

$$40 - \frac{144000}{x^2} = 0, \qquad x^* = 60, \qquad f(x^*) = 4800.$$

Since the second derivative is positive, this is a minimum.

10. The cost of fuel to propel a boat through water (in dollars per hour) is proportional to the cube of the speed. A certain ferry boat uses \$100 worth of fuel per hour, when cruising at 10 miles per hour. Apart from fuel, the cost of running this ferry (labor, maintenance, and so on) is \$675 per hour. At what speed should it travel so as to minimize the cost *per mile* traveled?

Solution. The cost of fuel per hour is Cv^3 for some constant C. Since for $v = 10$ we have $Cv^3 = 100$, it follows that $C = 1/10$. Thus, the total cost per hour is

$$\frac{v^3}{10} + 675.$$

At speed v, it takes the ferry $1/v$ hours to travel one mile. Therefore, the total cost per mile traveled will be

$$f(v) = \frac{1}{v}\left(\frac{v^3}{10} + 675\right) = \frac{v^2}{10} + \frac{675}{v}.$$

To find the minimum of this function, we find the derivative and set it equal to 0:

$$f'(v) = \frac{v}{5} - \frac{675}{v^2},$$

and this is zero when $v^3 = (5)(675)$, which gives $v = 15$ miles per hour. The cost at this speed is \$67.5 per mile traveled. Since the second derivative is positive, this is a minimum.

11. Find all the critical points of the given function, and determine which of them are local minima, which are local maxima, and which are saddlepoints.

(a) $f(x, y) = x^2 + xy + y^2 + x - y + 2$,

(b) $f(x, y) = 3x^2 - 3xy + y^3 + 3y^2$.

Solution.

(a) First, find f_x and f_y and set them equal to 0:

$$\begin{aligned} f_x(x, y) &= 2x + y + 1 = 0, \\ f_y(x, y) &= x + 2y - 1 = 0, \end{aligned}$$

which gives $x = -1, y = 1$. Thus, we have only one critical point, $(-1, 1)$. To see whether it is a minimum, a maximum, or a saddlepoint, we use the second derivative test.

$$f_{xx}(x, y) = 2, f_{xy}(x, y) = f_{yx}(x, y) = 1, f_{yy}(x, y) = 2$$

and therefore

$$f_{xx}(-1, 1) = 2 > 0, D(-1, 1) = 4 - 1 = 3 > 0,$$

which means that $(-1, 1)$ is a minimum point.

(b) We follow the same procedure as in part (a):

$$\begin{aligned} f_x(x, y) &= 6x - 3y = 0, \\ f_y(x, y) &= -3x + 3y^2 + 6y = 0. \end{aligned}$$

Expressing $y = 2x$ from the first equation, and substituting it into the second, we get

$$-3x + 12x^2 + 12x = 0 \Rightarrow 4x^2 + 3x = 0 \Rightarrow x = 0 \text{ or } x = -3/4.$$

which gives two critical points, $(0, 0)$ and $(-3/4, -3/2)$. Further, we again use the second derivative test.

$$\begin{aligned} f_{xx}(x, y) &= 6, \\ f_{xy}(x, y) = f_{yx}(x, y) &= -3, \\ f_{yy}(x, y) &= 6y + 6, \\ D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 &= 36y + 27 \end{aligned}$$

and therefore

$$f_{xx}(0,0) = 6 > 0, D(0,0) = 27 > 0,$$
$$D(-3/4, -3/2) = -27 < 0,$$

which means that $(0,0)$ is a minimum point, and $(-3/4, -3/2)$ is a saddlepoint.

12. Let R be the region described by $0 \le y \le \sqrt{16 - x^2}$, $0 \le x \le 4$. Find the absolute maximum and absolute minimum values of $f(x,y) = x^2 + y^2 - 4x - 2y + 1$ on R.

Solution. First we look for local extrema inside the region. We find the gradient and set it equal to 0:

$$\begin{aligned} f_x(x,y) &= 2x - 4 = 0, \\ f_y(x,y) &= 2y - 2 = 0. \end{aligned}$$

This gives us one critical point, $(2,1)$, and it is inside R. Then we look at the boundary of R. The boundary consists of three parts: the line segment connecting $(0,0)$ and $(0,4)$, the line segment connecting $(0,0)$ and $(4,0)$, and the arc of the circle $x^2 + y^2 = 16$ connecting $(4,0)$ and $(0,4)$. We look at these three parts one by one.

On the segment $x = 0, 0 \le y \le 4$, we have $f(0,y) = y^2 - 2y + 1$. This expression has a local minimum for $y = 1$, and there are also the two endpoints $y = 0$ and $y = 4$ to consider. This gives us three more points: $(0,0), (0,1), (0,4)$.

On the segment $y = 0, 0 \le x \le 4$, we have $f(x,0) = x^2 - 4x + 1$. This expression has a local minimum for $x = 2$, and there are also the two endpoints $x = 0$ and $x = 4$ to consider. This gives us the points: $(0,0), (2,0), (4,0)$.

Finally, the arc under consideration can be parametrized as $x = 4\cos\theta, y = 4\sin\theta, 0 \le \theta \le \pi/2$. Then the function on the arc is

$$\begin{aligned} f(4\cos\theta, 4\sin\theta) &= 16\cos^2\theta + 16\sin^2\theta - 16\cos\theta - 8\sin\theta + 1 \\ &= 17 - 16\cos\theta - 8\sin\theta. \end{aligned}$$

We then find the critical points of this expression, by differentiating it with respect to θ and setting the derivative equal to zero:

$$16\sin\theta - 8\cos\theta = 0,$$

or, equivalently,

$$4y - 2x = 0.$$

Expressing $x = 2y$ and substituting into $x^2 + y^2 = 16$, we find one more suspicious point, $(8/\sqrt{5}, 4/\sqrt{5})$. Of course, there are also the endpoints of the arc, $(4,0)$ and $(0,4)$. Now we list all the suspicious points and evaluate the function at them:

$$\begin{array}{cc} (2,1) & f(2,1) = -4 \\ (0,0) & f(0,0) = 1 \\ (0,1) & f(0,1) = 0 \\ (0,4) & f(0,4) = 9 \\ (2,0) & f(2,0) = -3 \\ (4,0) & f(4,0) = 1 \\ (8/\sqrt{5}, 4/\sqrt{5}) & f(8/\sqrt{5}, 4/\sqrt{5}) = 17 - 8\sqrt{5} \end{array}$$

Comparing the values at these points, we see that the maximum value is 9, attained at (0,4), and the minimal value is -4, attained at (2,1).

13. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

(a) $f(x,y) = x^2 - y^2; \quad x^2 + y^2 = 1$

(b) $f(x,y) = x^2 + y^2; \quad x^4 + y^4 = 1$

(c) $f(x,y,z) = yz + xy; \quad xy = 1, y^2 + z^2 = 1$

Solution.

(a) The Lagrange function is $L(x,y,\lambda) = (x^2 - y^2) - \lambda(x^2 + y^2 - 1)$. We write the necessary conditions:

$$0 = \frac{\partial L}{\partial x} = 2x - 2\lambda x$$
$$0 = \frac{\partial L}{\partial x} = -2y - 2\lambda y$$
$$x^2 + y^2 = 1$$

Then, we solve this system of equations. If $\lambda \neq \pm 1$, then $x = y = 0$, which violates the third equation. So, λ is either 1, or -1. If $\lambda = 1$, then $y = 0$ and hence $x^2 = 1$. If $\lambda = -1$, then $x = 0$ and hence $y^2 = 1$. Thus, we get four points: $(\pm 1, 0)$ and $(0, \pm 1)$. Evaluating f at these point we get:
$f(\pm 1, 0) = 1$ — maximum,
$f(0, \pm 1) = -1$ — minimum.

(b) The Lagrange function is $L(x,y,\lambda) = (x^2 + y^2) - \lambda(x^4 + y^4 - 1)$. Necessary conditions:

$$0 = \frac{\partial L}{\partial x} = 2x - 4\lambda x^3$$
$$0 = \frac{\partial L}{\partial x} = -2y - 4\lambda y^3$$
$$x^4 + y^4 = 1$$

The first two equations can be rewritten as:

$$x(1 - 2\lambda x^2) = 0$$
$$y(1 - 2\lambda y^2) = 0$$

Since, by the third equation, $x^4 + y^4 = 1$, we have either $x = 0$ and $y^4 = 1$, or $y = 0, x^4 = 1$, or $x^2 = y^2 = 1/(2\lambda)$. This gives us the following eight points: $(0, \pm 1), (\pm 1, 0)$, and $(\pm 1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2})$. Then we evaluate f at these points:
$f(\pm 1, 0) = f(0, \pm 1) = 1$ — minimum,
$f(\pm 1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2}) = \sqrt{2}$ — maximum

(c) Lagrange function $L(x,y,z,\lambda_1,\lambda_2) = (yz + xy) - \lambda_1(xy - 1) - \lambda_2(y^2 + z^2 - 1)$. Necessary conditions:

$$\begin{aligned} (1 - \lambda_1)y &= 0 \\ z + (1 - \lambda_1)x - 2\lambda_2 y &= 0 \\ y - 2\lambda_2 z &= 0 \\ xy &= 1 \\ y^2 + z^2 &= 1 \end{aligned}$$

According to the fourth equation, $y \neq 0$, so the first equation yields $\lambda_1 = 1$. Then the second and third equations become

$$z = 2\lambda_2 y, \qquad y = 2\lambda_2 z$$

Eliminating λ_2, we get $y = \pm z$. Combining this with the last equation, we get

$$y^2 = z^2 = \frac{1}{2}.$$

Keeping in mind that $x = 1/y$, we arrive at the following four points:

$$\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Evaluating f at these points, we see that the first two provide a maximum, the second two — a minimum.

14. Use Kuhn-Tucker's Theorem to find the maximum values of the function subject to the given constraint(s).

    (a) $f(x,y) = x^2 - y^2;\quad x^2 + y^2 \leq 1$

    (b) $f(x,y) = x^2 + y^3;\quad x^2 + y \leq 1, -y \leq 1$

    (c) $f(x,y,z) = x + 2y;\quad x + y + z = 1, y^2 + z^2 \leq 8$

Solution.

(a) Lagrange function $L(x,y,\mu) = (x^2 - y^2) - \mu(x^2 + y^2 - 1)$. Necessary conditions:

$$\begin{aligned} (1-\mu)x &= 0 \\ (1+\mu)y &= 0 \\ \mu(x^2 + y^2 - 1) &= 0 \\ \mu &\geq 0 \end{aligned}$$

Since $\mu \geq 0$, the second equation implies $y = 0$. There are two cases. If $\mu = 0$, then, by the first equation, $x = 0$, and we get the point $(0,0)$. If $\mu \neq 0$, then, by the third equation $x = \pm 1$ and we get two additional points, $(\pm 1, 0)$. Evaluating f at thse points we find that $f(0,0) = 0$, and $f(\pm 1, 0) = 1$. Thus, the two points $(\pm 1, 0)$ provide a maximum.

(b) Lagrange function: $L(x,y,\mu_1,\mu_2) = (x^2 + y^3) - \mu_1(x^2 + y - 1) - \mu_2(-y - 1)$. Necessary conditions:

$$\begin{aligned} 2x - 2\mu_1 x &= 0 \\ 3y^2 - \mu_1 y + \mu_2 &= 0 \\ \mu_1(x^2 + y - 1) &= 0 \\ \mu_2(y+1) &= 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0 \end{aligned}$$

If $\mu_2 \neq 0$, then, from the fourth equation, $y = -1$, and the second equation gives $3 + \mu_1 + \mu_2 = 0$, which is impossible, because μ_1 and μ_2 must be non-negative. Thus, $\mu_2 = 0$. There are two cases now. If $\mu_1 = 0$, then, by the first two equations, $x = y = 0$, and we get the point $(0,0)$. If $\mu_1 \neq 0$, then, by the third equation, $x^2 = 1 - y$. The first equation can be rewritten as $x(1-\mu_1) = 0$. This gives us two possibilities. If $x = 0$, then $y = 1, \mu_1 = 3$, and we get the point $(0,1)$. If $\mu_1 = 1$, then we look at the scond equation and get the solutions $y = 0, x = \pm 1$ and $y = 1/3, x = \pm\sqrt{2/3}$. This gives us four more points: $(\pm 1, 0)$ and $(\pm\sqrt{2/3}, 1/3)$. We then evaluate f at these points:

$$\begin{aligned} f(0,0) &= 0 \\ f(0,1) &= 1 \qquad \text{- maximum} \\ f(\pm 1, 0) &= 1 \qquad \text{- maximum} \\ f(\pm\sqrt{2/3}, 1/3) &= 19/27 \end{aligned}$$

(c) Lagrange function: $L(x,y,z,\lambda,\mu) = (x+2y) - \lambda(x+y+z-1) - (y^2+z^2-8)$. Necessary conditions:

$$\begin{aligned} 1 - \lambda &= 0 \\ 2 - \lambda - 2\mu y &= 0 \\ -\lambda - 2\mu z &= 0 \\ x + y + z &= 1 \\ \mu(y^2 + z^2 - 8) &= 0 \\ \mu &\geq 0 \end{aligned}$$

From the first equation we get $\lambda = 1$, substituting this into the second and third, we get $\mu \neq 0$ and $y = -z = 1/(2\mu)$. The fourth equation then gives us $x = 1$, and the fifth yields $y^2 = z^2 = 4$. Thus, we get two points, $(1,2,-2)$ and $(1,-2,2)$. The first one supplies the maximum, $f(1,2,-2) = 6$.