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21-256 Multivariate Analysis and Approximation, Spring 1998

Review for Test 2
Solutions

1. Find the domain and the range of the function:

   (**a**) $f(x,y) = \sqrt{\ln(x-y)}$ (**b**) $f(x,y,z) = \ln(64 - 16x^2 - 9y^2 - 25z^2)$.

   Solution.

   (a) (x,y) is in the domain D_f if and only if $\ln(x-y)$ is defined and non-negative, i.e. when $x - y \geq 1$. Thus,
   $$D_f = \{(x,y) \mid x - y \geq 1\}.$$
   To obtain the range, observe that the square root can attain only non-negative values. Furthermore, for each $z \geq 0$, $(x,y) := (e^{z^2}, 0)$ gives:
   $$f(x,y) = \sqrt{\ln(e^{z^2} - 0)} = \sqrt{z^2} = z.$$
   Therefore the range is $[0, \infty)$.

   (b)
   $$D_f = \{(x,y,z) \mid 16x^2 + 9y^2 + 25z^2 < 64\}.$$
   For the range, observe that, with $(x,y,z) \in D_f$, the expression $64 - 16x^2 - 9y^2 - 25z^2$ will be positive, can be as close to 0 as we want, and cannot be greater than 64. Therefore, the values of f will lie between $-\infty$ and $\ln 64$. To verify that $(-\infty, \ln 64]$ is indeed the range, we reason as following. Given $w \in (-\infty, \ln 64]$, choose
   $$(x,y,z) := \left(4 - \frac{e^w}{16}, 0, 0\right).$$
   Then $(x,y,z) \in D_f$ and $f(x,y,z) = w$.

2. Determine the largest set on which the function is continuous:

   $$(\mathbf{a}) f(x,y) = \begin{cases} \frac{xy^2}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases} \qquad (\mathbf{b}) f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Solution.

   (a) Clearly, f is continuous at all points $(x,y) \neq (0,0)$, as the ratio of two continuous functions, the denominator being non-zero. The question is whether f is continuous at $(0,0)$. We have
   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} x\frac{y^2}{x^2+2y^2}.$$
   Observe that
   $$0 \leq \frac{y^2}{x^2+2y^2} \leq \frac{1}{2} \qquad \text{for all } (x,y),$$

and that

$$\lim_{(x,y)\to(0,0)} x = 0.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

which makes f continuous at $(0,0)$. Therefore f is continuous on all $\mathbb{R}^2$.

(b) Again, f is continuous at all points $(x,y) \neq (0,0)$, as the ratio of two continuous functions, the denominator being non-zero. The question is whether f is continuous at $(0,0)$. In this case it will turn out that the limit at $(0,0)$ does not exist. Ideed, if the limit existed, we would get the same limit when approaching $(0,0)$ along different curves.

Along the line $y = 0$, we have $\lim_{x\to 0} f(x,0) = 1$. Along the line $x = y$, we have $\lim_{x\to 0} f(y,y) = 0$, which is different.

Since the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, f cannot be continuous at $(0,0)$, no matter how $f(0,0)$ is defined. Thus, the region of continuity is $\mathbb{R}^2 \setminus \{(0,0)\}$.

3. Find the limit, or show that the limit does not exist.

$$\textbf{(a)} \quad \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \qquad\qquad \textbf{(b)} \quad \lim_{(x,y)\to(0,0)} \frac{x^3}{x\,y^2+y^3}$$

Solution.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x\frac{x^2}{x^2+y^2} = 0,$$

because

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

for all $(x,y) \neq (0,0)$ and

$$\lim_{(x,y)\to(0,0)} x = 0.$$

(b) The limit does not exist exist, as can be seen by computing limits from different directions. Along the line $x = 0$, we have $\lim_{y\to 0} f(0,y) = 0$. Along the line $x = y$, we have $\lim_{y\to 0} f(y,y) = 1/2$.

4. Let $f(x,y) = \sin(x+3y) + x^2y^3 + 2y^2$. Compute f_x, f_y, f_{xx}, f_{xy}, and f_{yy}.
Solution.

$$\begin{aligned}
f_x(x,y) &= \cos(x+3y) + 2xy^3 \\
f_y(x,y) &= 3\cos(x+3y) + 3x^2y^2 + 4y \\
f_{xx}(x,y) &= -\sin(x+3y) + 2y^3 \\
f_{xy}(x,y) = f_{yx}(x,y) &= -3\sin(x+3y) + 6xy^2 \\
f_{yy}(x,y) &= -9\sin(x+3y) + 6x^2y + 4
\end{aligned}$$

5. * For what values of r is the function

$$f(x,y,z)=\begin{cases}\frac{|x+y+z|^r}{x^2+y^2+z^2}, & \text{if } (x,y,z)\neq(0,0,0),\\ 0, & \text{if } (x,y,z)=(0,0,0).\end{cases}$$

continuous on $\mathbb{R}^3$?
Solution. The given function f is continuous if and only if $r>2$.

To see this, observe the following. First of all, for all r, f is continous at all points $(x,y,z)\neq(0,0,0)$, as the ratio of two continuous functions, the function in the denominator beeng different from zero. So the question is what happens at the origin. Consider three cases:

case $r<2$. Then, along the x-axis, $y=z=0$, we have

$$\lim_{x\to0} f(x,0,0)=\lim_{x\to0}\frac{|x|^r}{x^2}=\lim_{x\to0}|x|^{r-2}=\infty\neq f(0,0,0).$$

case $r=2$. Similarly, along the x-axis, $y=z=0$, we have

$$\lim_{x\to0} f(x,0,0)=\lim_{x\to0}\frac{|x|^2}{x^2}=1\neq f(0,0,0).$$

case $r>2$. Then

$$|x+y+z|\leq|x|+|y|+|z|\leq 3\max\{|x|,|y|,|z|\}\leq 3\sqrt{x^2+y^2+z^2},$$

and

$$0\leq\frac{|x+y+z|^r}{x^2+y^2+z^2}\leq\frac{3^r(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2}=3^r(x^2+y^2+z^2)^{\frac{r-2}{2}}\to0,$$

as $(x,y,z)\to(0,0,0)$.

6. Find the tangent plane and normal line to the surface $z=x^2+4y^2$ at the point $(-1,2,17)$.
Solution. $z=f(x,y)$, where $f(x,y)=x^2+4y^2$. Further, $f(-1,2)=17$,

$$\begin{array}{ll} f_x(x,y)=2x & f_x(-1,2)=-2\\ f_y(x,y)=8y & f_y(-1,2)=16\end{array}$$

The tangent plane through $(-1,2,17)$ has equation

$$z=f(-1,2)+f_x(-1,2)(x+1)+f_y(-1,2)(y-2),$$

i.e.,

$$z=17-2(x+1)+16(y-2),$$

or, equivalently,

$$2x-16y+z+17=0.$$

The normal line is a line that passes through the point $(-1,2,17)$ in the direction of the normal vector $\langle 2,-16,1\rangle$. Therefore it has parametric equations $x=-1+2t, y=2-16t, z=17+t$, and symmetric equations

$$\frac{x+1}{2}=\frac{2-y}{16}=z-17.$$

7. Suppose that $w = x^3y + y^2 - 4z^3, x = rs, y = (r^2 - s^2)^3$, and $z = 2r + s$. Use the chain rule to compute $\frac{\partial w}{\partial s}$. Express your answer in terms o r and s alone. Algebraic simplification is not necessary.
Solution.

$$
\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (3x^2y)(r) + (x^3 + 2y)((3(r^2 - s^2)^2(-2s)) + (-12z^2)(1) \\
&= 3r^2s^2r + (r^3s^3 + 2(r^2 - s^2)^3)(-6(r^2 - s^2)^2s) - 12(2r + s)^2
\end{aligned}
$$

8. * For

$$f(x, y, z) = \frac{x}{x^2 + y^2} e^{\sin(x^2y)} + \sin(z - 2)\ln(x + \sqrt{1 + x^2}),$$

compute $f_x(1, 0, 2)$.
Solution. Let $\varphi(x) := f(x, 0, 2)$. Then, by the definition of partial derivative, $f_x(1, 0, 2) = \varphi'(1)$. We have:

$$\varphi(x) = f(x, 0, 2) = \frac{1}{x}, \qquad \varphi'(x) = \frac{-1}{x^2}, \qquad \varphi'(1) = -1.$$

Thus,

$$f_x(1, 0, 2) = -1.$$

9. Let f be a continuously differentiable function of one variable and define

$$w(x, y) := f\left(\frac{y}{y - x}\right).$$

Show that, for $x \neq y$:

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = 0.$$

Solution.

$$
\begin{aligned}
& x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} \\
&= xf'\left(\frac{x}{y - x}\right)\left(-\frac{-y}{(y - x)^2}\right) + yf'\left(\frac{x}{y - x}\right)\left(\frac{(y - x) - y}{(y - x)^2}\right) \\
&= \frac{xy}{(y - x)^2}\, f'\left(\frac{x}{y - x}\right) - \frac{xy}{(y - x)^2}\, f'\left(\frac{x}{y - x}\right) = 0.
\end{aligned}
$$

10. * For

$$f(x, y) = 5 + 6(x - 3) + 8(y + 1) + \sin(x - 3)^2 + (y + 1)\ln\sqrt{2 + y},$$

write an equaion of the tangent plane to the graph at the point $(3, -1, 5)$.
Solution. Observe that

$$f(x, y) = 5 + 6(x - 3) + 8(y + 1) + (x - 3)\,h_1(x, y) + (y + 1)\,h_2(x, y),$$

where

$$h_1(x, y) = \frac{\sin(x - 3)^2}{x - 3}, \qquad h_2(x, y) = \ln\sqrt{2 + y},$$

and

$$\lim_{(x,y)\to(-3,1)} h_1(x, y) = \lim_{(x,y)\to(-3,1)} h_2(x, y) = 0.$$

Thus, near $(x, y) = (-3, 1)$, we have

$$f(x, y) = 5 + 6(x - 3) + 8(y + 1) + \text{small error term},$$

and the tangent plane at $(-3, 1, 5)$ is given by

$$z = 5 + 6(x - 3) + 8(y + 1).$$

All done, without a single computation.

11. Assume that u and v have continuous second partial derivatives and that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Solution.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0.
\end{aligned}$$

12. The equation $xyz = \cos(x + y + z)$ defines implicitly z as a function of x and y. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Solution. Let $F(x, y, z) := xyz - \cos(x + y + z)$. Then

$$F_x(x, y, z) = yz + \sin(x + y + z),$$

$$F_y(x, y, z) = xz + \sin(x + y + z),$$

$$F_z(x, y, z) = xy + \sin(x + y + z),$$

and

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}, \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}.
\end{aligned}$$