Department of Mathematics
CARNEGIE MELLON UNIVERSITY
Instructor: Bogdan Doytchinov
21-259 Calulus in 3D, Spring 1999

Review for Test 4
Solutions

1. Evaluate each of the following line integrals:

   (a) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle y, -x, x^2+y^2+z^2\rangle$ and C is the curve described by $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle,\ 0 \le t \le \pi$.

   (b) $\int_C (\nabla f)\cdot d\mathbf{r}$, where $f(x,y,z) = xyz$ and C is the curve described by $\mathbf{r}(t) = \langle 1+t^4, t^3, e^t\rangle,\ 0 \le t \le 1$.

   (c) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle yz+x, xz, xy\rangle$ and C is described by $\mathbf{r}(t) = \langle t, t^2, t\rangle,\ 0 \le t \le 1$.

   Solution:

   (a) $\mathbf{r}'(t) = \langle \cos t, -\sin t, 1\rangle$.

$$\begin{aligned}&\int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^\pi \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^\pi [(\cos t)(\cos t) + (-\sin t)(-\sin t) + (\sin^2 t + \cos^2 t + t^2)(1)]\,dt\\ &= \int_0^\pi (2+t^2)\,dt = 2\pi + \frac{\pi^3}{3}\end{aligned}$$

   (b) By the Fundamental Theorem for Line Integrals,

$$\int_C (\nabla f)\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2,1,e) - f(1,0,1) = 2e.$$

   (c) This one can be solved directly. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$ and

$$\begin{aligned}&\int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^1 [(t^3+t)(1) + (t^2)(2t) + (t^3)(1)]\,dt\\ &= \int_0^1 (4t^3+t)\,dt = 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

   An alternative way is to observe that $\mathbf{F} = \nabla f$, where $f(x,y,z) = xyz + \frac{1}{2}x^2$. Then, by the Fundamental Theorem for Line Integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,1,1) - f(0,0,0) = \frac{3}{2}.$$

2. Evaluate each of the following line integrals:

   (a) $\int_C y\,dx + xy\,dy$, where C is the segment of $y = x^3$ from $(0,0)$ to $(1,1)$.

(b) $\int_C -y\,dx + (x+y^2)\,dy$, where C is the segment of $x^2+y^2=1$ from $(1,0)$ to $(0,1)$.
(c) $\int_C z\,dx - x\,dy + dz$, where C is the segment of the curve $x=t, y=t^2, z=t^3$ for $0 \le t \le 1$.

Solution: These three line integrals are straightforward (there are no tricks, just use the formulas). We give a solution for the first one and answers for the other two.

(a) With $x=t, y=t^3, 0 \le t \le 1$,

$$\int_C y\,dx + xy\,dy = \int_0^1 [(t^3) + (t^4)(3t^2)]\,dt = \frac{19}{28}.$$

(b) Ans: $\frac{\pi}{2} + \frac{1}{3}$.
(c) Ans: $\frac{7}{12}$.

3. Find a function $f(x,y)$ such that $\nabla f(x,y) = \langle 2xy + x, x^2 + y^3 \rangle$.
Solution: Since $f_x(x,y) = 2xy + x$, we have $f(x,y) = x^2y + \frac{1}{2}x^2 + g(y)$. Differentiating this with respect to y, we get:

$$x^2 + g'(y) = f_y(x,y) = x^2 + y^3,$$

so $g'(y) = y^3, g(y) = y^4/4 + C$, and

$$f(x,y) = x^2y + \frac{1}{2}x^2 + \frac{1}{4}y^4 + C.$$

4. Find a function $f(x,y,z)$ such that $\nabla f(x,y,z) = \langle 4xe^z, \cos y, 2x^2e^z \rangle$. Use this result to compute the line integral $\int_C 4xe^z\,dx + \cos y\,dy + 2x^2e^z\,dz$, where the curve C is given by $x=t, y=t^2, z=t^4, 0 \le t \le 1$.
Solution:

$$f_x(x,y,z) = 4xe^z \Rightarrow f(x,y,z) = 2x^2e^z + g(y,z)$$
$$g_y(y,z) = f_y(x,y,z) = \cos y \Rightarrow g(y,z) = \sin y + h(z) \Rightarrow f(x,y,z) = 2x^2e^z + \sin y + h(z)$$
$$2x^2e^z + h'(z) = f_z(x,y,z) = 2x^2e^z \Rightarrow h(z) = C \Rightarrow f(x,y,z) = 2x^2e^z + \sin y + C$$

Finally,

$$\int_C 4xe^z\,dx + \cos y\,dy + 2x^2e^z\,dz = f(1,1,1) - f(0,0,0) = 2e + \sin 1.$$

5. Evaluate the line integral

$$\int_C (y + y\cos x)\,dx + \sin x\,dy,$$

where C is the segment of $y = x^4$ from $(0,0)$ to $(1,1)$.
Solution: Observe that the integrand can be represented as the sum of two parts: one path-independent and one easy to integrate:

$$\int_C (y + y\cos x)\,dx + \sin x\,dy = \int_C y\,dx + \int_C (y\cos x)\,dx + \sin x\,dy.$$

The first integral is computed directly:

$$\int_C y\,dx = \int_0^1 x^4\,dx = \frac{1}{5},$$

while the second integral is path-independent, because

$$\frac{\partial}{\partial x}(\sin x) = \cos x = \frac{\partial}{\partial y}(y\cos x).$$

Hence,

$$\int_C (y\cos x)\,dx + \sin x\,dy = \int_{C'} (y\cos x)\,dx + \sin x\,dy + \int_{C''} (y\cos x)\,dx + \sin x\,dy,$$

where C' is the line segment $0 \le x \le 1, y = 0$, and C'' is the line segment $x = 1, 0 \le y \le 1$. On C', $y = 0, dx = 1$ and $dy = 0$, while on C'', $x = 1, dx = 0$ and $dy = 1$. Putting these together, we get

$$\int_C (y + y\cos x)\,dx + \sin x\,dy = \frac{1}{5} + 0 + \sin 1.$$

6. Evaluate each of the following line integrals:

   (a) $\int_C (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy$, where C is the semicircle described by $x = \cos t, y = \sin t, 0 \le t \le \pi$.

   (b) $\int_C (e^y + xy)\,dx + xe^y\,dy$, where C is the segment of $y = x^2$ from $(0,0)$ to $(1,1)$.

   (c) $\int_C (2x\cos y - x^2 y)\,dx + (x - x^2\sin y)\,dy$, where C is the segment of $y = x^4$ from $(0,0)$ to $(1,1)$.

   Solution: These problems can be solved the same way as the previous one. We will show here another method, which uses Green's theorem. Observe that Green's Theorem cannot be applied directly, because the contour of integration is not closed. In each of the following examples, we close the contour by adding a line segment.

   (a) Let C' be the line segment connecting the points $(-1,0)$ and $(1,0)$. Then $C \cup C'$ is a closed simple curve and, by Green's Theorem,

   $$\int_{C\cup C'} (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy = \iint_D [(3y^2 - 3x^2) - (6y^2)]\,dA = -3\iint_D (x^2 + y^2)\,dA,$$

   where $D = \{(x,y) : x^2 + y^2 \le 1, y \ge 0\}$. Expressing the above integral in polar coordinates, we get

   $$\int_{C\cup C'} (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy = -3\int_0^\pi \int_0^1 r^2\, r\,dr\,d\theta = -\frac{3}{4}\pi.$$

   On the other hand,

   $$\begin{aligned}&\int_{C\cup C'} (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy \\ &= \int_C (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy + \int_{C'} (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy,\end{aligned}$$

   and

   $$\int_{C'} (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy = \int_{-1}^1 (0 + x^2)\,dx + 0 = \frac{2}{3}.$$

   Putting these together, we arrive at:

   $$\int_C (2y^3 + x^2)\,dx + (3xy^2 - x^3)\,dy = -\frac{3}{4}\pi - \frac{2}{3}.$$

   (b) Let C' be the line segment from $(1,1)$ to $(0,0)$. Then $C \cup C'$ is a closed simple curve and, by Green's Theorem,

   $$\int_{C\cup C'} (e^y + xy)\,dx + xe^y\,dy = \iint_D (-x)\,dA$$

where $D = \{(x,y) : 0 \le x \le 1, x^2 \le y \le x\}$. Expressing the above integral as an iterated integral, we get

$$\int_{C \cup C'} (e^y + xy)\, dx + xe^y\, dy = \int_0^1 \int_{x^2}^x (-x)\, dy\, dx = \int_0^1 (x^3 - x^2)\, dx = -\frac{1}{12}.$$

On the other hand,

$$\int_{C \cup C'} (e^y + xy)\, dx + xe^y\, dy = \int_C (e^y + xy)\, dx + xe^y\, dy + \int_{C'} (e^y + xy)\, dx + xe^y\, dy$$

and

$$\int_{C'} (e^y + xy)\, dx + xe^y\, dy = -\int_{-C'} (e^y + xy)\, dx + xe^y\, dy = -\int_0^1 (e^x + x^2 + xe^x)\, dx = -(e + \frac{1}{3}).$$

Putting these together, we arrive at:

$$\int_C (e^y + xy)\, dx + xe^y\, dy = -\frac{1}{12} + e + \frac{1}{3} = e + \frac{1}{4}.$$

(c) Let C' be the line segment from $(1,1)$ to $(1,0)$ and letet C'' be the line segment from $(1,0)$ to $(0,0)$. Then $C \cup C' \cup C''$ is a closed simple curve and, by Green's Theorem,

$$\begin{aligned}
&\int_{C \cup C' \cup C''} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
= &\iint_D [(1 - 2x \sin y) - (-2x \sin y - x^2)]\, dA = \iint_D (1 + x^2)\, dA,
\end{aligned}$$

where $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le x^4\}$. Expressing the above integral as an iterated integral, we get

$$\int_{C \cup C' \cup C''} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy = \int_0^1 \int_0^{x^4} (1 + x^2)\, dy\, dx = \int_0^1 (x^4 + x^6)\, dx = \frac{12}{35}.$$

On the other hand,

$$\begin{aligned}
&\int_{C \cup C' \cup C''} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
= &\int_C (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
&+ \int_{C'} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
&+ \int_{C''} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_{C'} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
= &-\int_{-C'} (2x \cos y - x^2 y)\, dx + (x - x^2 \sin y)\, dy \\
= &-\int_0^1 (1 - \sin y)\, dy = -\cos 1,
\end{aligned}$$

while

$$\begin{aligned}
&\int_{C''} (2x\cos y - x^2 y)\,dx + (x - x^2 \sin y)\,dy \\
&= -\int_{-C''} (2x\cos y - x^2 y)\,dx + (x - x^2 \sin y)\,dy \\
&= -\int_0^1 2x\,dx = -1.
\end{aligned}$$

Putting these together, we arrive at:

$$\int_C (2x\cos y - x^2 y)\,dx + (x - x^2 \sin y)\,dy = \frac{12}{35} + \cos 1 + 1.$$

7. Let C be a piecewise smooth simple closed curve in the xy-plane and let D be the region enclosed by C. Given that $\iint_D dA = \pi$, $\iint_D y\,dA = 3$, $\int_C x^2\,dy = 2$, evaluate

$$\int_C (xy + 2y + 1)\,dx + (y^2 + 3xy + 5x)\,dy.$$

Solution: We use Green's Theorem for this one:

$$\begin{aligned}
&\int_C (xy + 2y + 1)\,dx + (y^2 + 3xy + 5x)\,dy \\
&= \int_C (xy + 2y + 1)\,dx + (\tfrac{1}{2}x^2 + y^2 + 3xy + 5x)\,dy - \frac{1}{2}\int_C x^2\,dy \\
&= \iint_D [(x + 3y + 5) - (x + 2)]\,dA - \frac{1}{2}\int_C x^2\,dy \\
&= 3\iint_D y\,dA + 3\iint_D dA - \frac{1}{2}\int_C x^2\,dy \\
&= 9 + 3\pi - \frac{2}{2} = 8 + 3\pi.
\end{aligned}$$

8. Use Green's Theorem to evaluate $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2 + y^2 = 1$.
Solution:

$$\int_C y^3\,dx - x^3\,dy = \iint_{x^2+y^2\le 1} (-3x^2 - 3y^2)\,dA = -3\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta = -\frac{3}{2}\pi.$$

9. Let C be a piecewise-smooth simple closed curve in the xy-plane and let D be the simply connected region enclosed by C. Assume that $u(x, y)$ and $v(x, y)$ have continuous first and second partial derivatives. Show that

$$\int_C (vu_y - uv_y)\,dx + (uv_x - vu_x)\,dy = \iint_D [u\,(v_{xx} + v_{yy}) - v\,(u_{xx} + u_{yy})]\,dA.$$

Solution: By Green's Theorem,

$$\begin{aligned}
&\int_C (vu_y - uv_y)\,dx + (uv_x - vu_x)\,dy \\
&= \iint_D [(uv_x - vu_x)_x - (vu_y - uv_y)_y]\,dA \\
&= \iint_D [(u_x v_x + uv_{xx} - v_x u_x - vu_{xx}) - (v_y u_y + vu_{yy} - u_y v_y - uv_{yy})]\,dA \\
&= \iint_D [u\,(v_{xx} + v_{yy}) - v\,(u_{xx} + u_{yy})]\,dA
\end{aligned}$$

10. Let C be a piecewise smooth simple closed curve in the xy-plane and let D be the simply connected region enclosed by C. Assume that $f(x, y)$ has continuous first and second partial derivatives and that $f_{xx} + f_{yy} = 0$. Show that

$$\begin{aligned}
\int_C (f\,\nabla f) \cdot \mathbf{n}\, ds &= \iint_D (f_x^2 + f_y^2)\, dA, \\
\int_C \nabla f \cdot \mathbf{n}\, ds &= 0,
\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal.

Solution: By Green's Theorem,

$$\begin{aligned}
&\int_C (f\,\nabla f) \cdot \mathbf{n}\, ds \\
&= \iint_D \operatorname{div}(f\,\nabla f)\, dA \\
&= \iint_D \operatorname{div}(\langle f f_x, f f_y\rangle)\, dA \\
&= \iint_D (f f_x)_x + (f f_y)_y\, dA \\
&= \iint_D (f_x f_x + f f_{xx}) + (f_y f_y + f f_{yy})\, dA \\
&= \iint_D (f_{xx} + f_{yy})\, f + (f_x^2 + f_y^2)\, dA \\
&= \iint_D (f_x^2 + f_y^2)\, dA,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_C (\nabla f) \cdot \mathbf{n}\, ds \\
&= \iint_D \operatorname{div}(\nabla f)\, dA \\
&= \iint_D \operatorname{div}(\langle f_x, f_y\rangle)\, dA \\
&= \iint_D (f_{xx} + f_{yy})\, dA = 0.
\end{aligned}$$

11. Let $f(x, y, z)$ be a function with continuous partial derivatives and let $\mathbf{F}(x, y, z)$ be a vector field whose components have continuous partial derivatives. Show that

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f) \times \mathbf{F}.$$

Solution: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\begin{aligned}
&\operatorname{curl}(f\mathbf{F}) \\
&= ((fR)_y - (fQ)_z)\mathbf{i} + ((fP)_z - (fR)_x)\mathbf{j} + ((fQ)_x - (fP)_y)\mathbf{k} \\
&= (f\,R_y - f\,Q_z)\mathbf{i} + (f\,P_z - f\,R_x)\mathbf{j} + (f\,Q_x - f\,P_y)\mathbf{k} \\
&\quad + (f_y R - f_z Q)\mathbf{i} + (f_z P - f_x R)\mathbf{j} + (f_x Q - f_y P)\mathbf{k} \\
&= f\operatorname{curl}\mathbf{F} + (\nabla f) \times \mathbf{F}.
\end{aligned}$$

12. Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.
Solution: The surface in question can be parametrized by

$$\mathbf{r}(y,z) = \langle y^2 + z^2, y, z\rangle, \qquad (y,z) \in D,$$

where D is the disk $y^2 + z^2 \leq 9$, in the $yz-$plane. Then

$$\begin{aligned} \mathbf{r}_y(y,z) &= \langle 2y, 1, 0\rangle, \\ \mathbf{r}_z(y,z) &= \langle 2z, 0, 1\rangle, \\ \mathbf{r}_y \times \mathbf{r}_z &= \langle 1, -2y, -2z\rangle, \end{aligned}$$

and

$$A(S) = \int_D |\mathbf{r}_y \times \mathbf{r}_z|\, dA = \int_D \sqrt{1 + 4y^2 + 4z^2}\, dA.$$

After changing to polar coordinates $y = r\cos\theta, z = r\sin\theta$, we get

$$A(S) = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta = 2\pi\, \frac{37\sqrt{37}-1}{12}.$$

13. Evaluate the surface integral $\iint_S xyz\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies above the cone $z = \sqrt{x^2 + y^2}$.
Solution: Clearly, the integral is zero (by symmetry). Here is an honest computation: The surface in question can be parametrized by

$$\mathbf{r}(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi\rangle, \qquad (\varphi,\theta) \in D,$$

where D is given by the inequalities $0 \leq \varphi \leq \pi/4, 0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned} \mathbf{r}_\varphi(\varphi,\theta) &= 3\langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle, \\ \mathbf{r}_\theta(\varphi,\theta) &= 3\langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle, \\ \mathbf{r}_\varphi \times \mathbf{r}_\theta &= 9\sin\varphi\langle -\sin\varphi\cos\theta, -\sin\varphi\sin\theta, \cos\varphi\rangle, \end{aligned}$$

and

$$\begin{aligned} &\iint_S xyz\, dS \\ &= \iint_D 27\sin^2\varphi\cos\varphi\sin\theta\cos\theta\, |\mathbf{r}_\varphi \times \mathbf{r}_\theta|\, dA \\ &= \iint_D 243\sin^3\varphi\cos\varphi\sin\theta\cos\theta\, dA \\ &= 243\int_0^{\pi/4}\sin^3\varphi\cos\varphi\, d\varphi \int_0^{2\pi}\sin\theta\cos\theta\, d\theta = 0, \end{aligned}$$

because

$$\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = 0.$$

14. Suppose that $f(x,y,z) = g(\sqrt{1+x^2+y^2+z^2})$, where g is a continuous function of one variable such that $g(3) = -1$. Evaluate $\iint_S f(x,y,z)\, dS$, where S is the sphere $x^2+y^2+z^2 = 8$.
Solution: Observe that on the sphere S we have $x^2+y^2+z^2 = 8$, and hence $f(x,y,z) = g(\sqrt{1+8}) = g(3) = -1$ at all points (x,y,z) of the sphere S. Thus,

$$\iint_S f(x,y,z)\, dS = \iint_S (-1)\, dS = -(4\pi)(8) = -32\pi.$$

15. Compute $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle xyz, xy, x^2yz \rangle$, and S consists of the top and four sides (but not the bottom) of the the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.
Solution: Let S' be the bottom of the given cube, that is, let S' be the the square with verices $(\pm 1, \pm 1, -1)$, oriented upward (i.e., oriented by its upward normal $\mathbf{k}$). Since S and S' have the same boundary, then, by Stokes' Theorem,

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \text{curl}\mathbf{F} \cdot \mathbf{k}\, dS = \int_0^1 \int_0^1 (y - xz)\Big|_{z=-1} dx\, dy = \int_0^1 \int_0^1 (y - (-x))\, dx\, dy = 0.$$