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21-259 Calulus in 3D, Spring 1999

Review for Test 3
Solutions

1. Write an equivalent double integral with the order of integration reversed.

$$(\mathbf{a})\ \int_0^1\int_y^{\sqrt{y}} f(x,y)\,dx\,dy, \qquad (\mathbf{b})\ \int_0^2\int_1^{e^{2x}} f(x,y)\,dy\,dx, \qquad (\mathbf{c})\ \int_0^2\int_{2x}^4 f(x,y)\,dy\,dx.$$

Solution.
(a) We have

$$\int_0^1\int_y^{\sqrt{y}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA,$$

where D is the region bounded by the curves

$$x = y \text{ and } x = \sqrt{y}, \qquad 0 \le y \le 1,$$

or, equvalently,

$$y = x \text{ and } y = x^2, \qquad 0 \le x \le 1.$$

Thus,

$$\int_0^1\int_y^{\sqrt{y}} f(x,y)\,dx\,dy = \int_0^1\int_{x^2}^{x} f(x,y)\,dy\,dx.$$

(b)

$$\int_0^2\int_1^{e^{2x}} f(x,y)\,dy\,dx = \int_1^{e^4}\int_{\frac{1}{2}\ln y}^{2} f(x,y)\,dx\,dy.$$

(c)

$$\int_0^2\int_{2x}^4 f(x,y)\,dy\,dx = \int_0^4\int_0^{y/2} f(x,y)\,dx\,dy.$$

2. Evaluate each of the following

$$(\mathbf{a})\ \int_0^2\int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2}\,dy\,dx, \qquad (\mathbf{b})\ \int_0^1\int_x^1 3x\sqrt{x^2+y^2}\,dy\,dx.$$

Solution.
(a) Change to polar coordinates:

$$\int_0^2\int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2}\,dy\,dx = \int_0^2\int_{3\pi/2}^{2\pi} e^{r^2} r\,d\theta\,dr = \frac{\pi}{4}(e^4-1).$$

(b) Change the order of integration:

$$\int_0^1\int_x^1 3x\sqrt{x^2+y^2}\,dy\,dx = \int_0^1\int_0^y 3x\sqrt{x^2+y^2}\,dx\,dy$$
$$= \int_0^1 \frac{3}{2}\cdot\frac{2}{3}\cdot(x^2+y^2)^{3/2}\Big|_{x=0}^{x=y}\,dy = \int_0^1 (2y^2)^{3/2} - (y^2)^{3/2}\,dy = \frac{1}{4}(2^{3/2}-1).$$

3. Compute

$$\iint_D (x^2+y^2)(1+\sqrt{x^2+y^2})\,dx\,dy,$$

where D is the region in the first quadrant that is bounded by $y=\sqrt{1-x^2}, y=x$, and the x-axis.

Solution. In polar coordinates, the region is described by the inequalities

$$0 \le r \le 1, \qquad 0 \le \theta \le \frac{\pi}{4}.$$

Therefore,

$$\iint_D (x^2+y^2)(1+\sqrt{x^2+y^2})\,dx\,dy = \int_0^1 \int_0^{\pi/4} r^2(1+r)r\,d\theta\,dr$$
$$= \frac{\pi}{4}\int_0^1 (r^3+r^4)\,dr = \frac{\pi}{4}\left(\frac{1}{4}+\frac{1}{5}\right) = \frac{9}{80}\pi.$$

4. * Let $f(x,y)$ be defined and continuous for all $(x,y)\in\mathbb{R}^2$. For every $t>0$ consider the region

$$D_t := \{(x,y) : x^2+y^2 \le t^2\}$$

and define

$$F(t) := \iint_{D_t} f(x,y)\,dA.$$

Show that, for $t>0$, the derivative of F is given by

$$F'(t) = \int_0^{2\pi} t f(t\cos\theta, t\sin\theta)\,d\theta.$$

Solution. In polar coordinates, we have

$$F(t) = \int_0^t \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\,d\theta\,dr = \int_0^t g(r)\,dr,$$

where

$$g(r) = \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\,d\theta.$$

By the Fundamental Theorem of Calculus,

$$F'(t) = g(t) = \int_0^{2\pi} f(t\cos\theta, t\sin\theta) t\,d\theta.$$

5. Find the volume of the region enclosed by $y=x^2$, $y=1$, $z=0$, and $z=2$.

Solution. $$V = \int_{-1}^1 \int_{x^2}^1 \int_0^2 1\,dz\,dy\,dx = \int_{-1}^1 2(1-x^2)\,dx = \frac{8}{3}.$$

6. Let E be the region in the first octant that is bounded by the coordinate planes, the plane $x+y=1$, and the plane $z=3$. Evaluate

$$\iiint_E (1+2z)\,dV.$$

Solution.

$$\iiint_E (1+2z)\,dV = \int_0^1 \int_0^{1-x} \int_0^3 (1+2z)\,dz\,dy\,dx = \int_0^1 12(1-x)\,dx = 6.$$

7. Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$, but outside the cylinder $x^2 + y^2 = 1$.
Solution. A point is inside the sphere if
$$x^2 + y^2 \leq 4 - z^2,$$
and is outside the cylinder, if
$$x^2 + y^2 \geq 1.$$
Therefore, the region E is described by the inequalities:
$$1 \leq x^2 + y^2 \leq 4 - z^2.$$
In particular, we see that we must have $4 - z^2 \geq 1$, i.e., $|z| \leq \sqrt{3}$. Then, in cylindrical coordinates, the region E is given by the inequalities:
$$-\sqrt{3} \leq z \leq \sqrt{3}, \qquad 1 \leq r \leq \sqrt{4 - z^2}, \qquad 0 \leq \theta \leq 2\pi.$$
Therefore, for the volume we have:
$$\begin{aligned} V &= \int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} \int_0^{2\pi} r \, d\theta \, dr \, dz = \int_{-\sqrt{3}}^{\sqrt{3}} \pi r^2 \Big|_{r=1}^{r=\sqrt{4-z^2}} dz \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \pi(4 - z^2 - 1) \, dz = \pi(3z - \frac{z^3}{3}) \Big|_{z=-\sqrt{3}}^{z=\sqrt{3}} = 4\pi\sqrt{3} \end{aligned}$$

8. Given
$$\int_0^1 \int_{-3}^3 \int_{x^2}^9 f(x, y, z) \, dz \, dx \, dy,$$
write the equivalent triple integral in the order $dy \, dx \, dz$.
Solution. Draw a picture!
$$\int_0^1 \int_{-3}^3 \int_{x^2}^9 f(x, y, z) \, dz \, dx \, dy = \int_0^9 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^1 f(x, y, z) \, dy \, dx \, dz.$$

9. Let E be the region that is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 1$. Evaluate
$$\iiint_E z^3 \sqrt{x^2 + y^2 + z^2} \, dV.$$
Solution. In spherical coordinates, the region E is given by the inequalities:
$$0 \leq \rho \leq 1, \qquad 0 \leq \varphi \leq \frac{\pi}{4}, \qquad 0 \leq \theta \leq 2\pi.$$
Also, in polar coordinates, we have
$$z = \rho \cos\varphi, \qquad |J| = \rho^2 \sin\varphi.$$
Therefore
$$\begin{aligned} &\iiint_E z^3 \sqrt{x^2 + y^2 + z^2} \, dV \\ &= \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^3 \cos^3\varphi \, \rho \, \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho \\ &= \int_0^{2\pi} d\theta \int_0^1 \rho^6 \, d\rho \int_0^{\pi/4} \cos^3\varphi \sin\varphi \, d\varphi \\ &= 2\pi \cdot \frac{1}{7} \cdot \frac{3}{16} = \frac{3}{56}\pi. \end{aligned}$$

10. * Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0),(1,0)$, and $(0,1)$. Show that

$$\iint_R f(x+y)\,dA = \int_0^1 uf(u)\,du.$$

Solution. Make the change of variables

$$u = x+y, \qquad v = y,$$

which gives

$$x = u - v, \qquad y = v.$$

For the Jacobian we have

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

The region of integration in (x,y) coordinates is

$$x \geq 0, \qquad y \geq 0, \qquad x+y \leq 1.$$

In (u,v) coordinates, this gives

$$u - v \geq 0, \qquad v \geq 0, \qquad u \leq 1,$$

or, equivalently,

$$0 \leq u \leq 1, \qquad 0 \leq v \leq u.$$

Thus,

$$\iint_R f(x+y)\,dA = \int_0^1 \int_0^u f(u)\,dv\,du = \int_0^1 uf(u)\,du.$$

11. * Compute the integral

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\,dA,$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.
Solution. Make the change of variables

$$u = x+y, \qquad v = x-y.$$

Then

$$x = \frac{1}{2}(u+v), \qquad y = \frac{1}{2}(u-v), \qquad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

The region of integration in (x,y) coordinates is

$$x \geq 0, \qquad y \geq 0, \qquad 1 \leq x+y \leq 2.$$

In (u,v) coordinates, this gives

$$u+v \geq 0, \qquad u-v \geq 0, \qquad 1 \leq u \leq 2,$$

or, equivalently,

$$1 \leq u \leq 2, \qquad -u \leq v \leq u.$$

Thus,

$$\begin{aligned}
&\iint_R \cos\left(\frac{y-x}{y+x}\right)\,dA \\
&= \int_1^2 \int_{-u}^u \cos\left(\frac{v}{u}\right)\frac{1}{2}\,dv\,du = \frac{1}{2}\int_1^2 u\sin\left(\frac{v}{u}\right)\Big|_{v=-u}^{v=u}\,du \\
&= \frac{1}{2}\int_1^2 (u\sin 1 - u\sin(-1))\,du = \sin(1)\int_1^2 u\,du = \frac{3}{2}\sin 1.
\end{aligned}$$