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Review for Test 2
Solutions

1. Determine the largest set on which the function is continuous:

$$(\mathbf{a}) f(x,y) = \begin{cases} \frac{xy^2}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases} \qquad (\mathbf{b}) f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.

(a) Clearly, f is continuous at all points $(x,y) \neq (0,0)$, as the ratio of two continuous functions, the denominator being non-zero. The question is whether f is continuous at $(0,0)$. We have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} x\frac{y^2}{x^2+2y^2}.$$

Observe that

$$0 \leq \frac{y^2}{x^2+2y^2} \leq \frac{1}{2} \qquad \text{for all } (x,y),$$

and that

$$\lim_{(x,y)\to(0,0)} x = 0.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

which makes f continuous at $(0,0)$. Therefore f is continuous on all $\mathbb{R}^2$.

(b) Again, f is continuous at all points $(x,y) \neq (0,0)$, as the ratio of two continuous functions, the denominator being non-zero. The question is whether f is continuous at $(0,0)$. In this case it will turn out that the limit at $(0,0)$ does not exist. Ideed, if the limit existed, we would get the same limit when approaching $(0,0)$ along different curves.
Along the line $y = 0$, we have $\lim_{x\to 0} f(x,0) = 1$. Along the line $x = y$, we have $\lim_{x\to 0} f(y,y) = 0$, which is different.
Since the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, f cannot be continuous at $(0,0)$, no matter how $f(0,0)$ is defined. Thus, the region of continuity is $\mathbb{R}^2 \setminus \{(0,0)\}$.

2. Find the limit, or show that the limit does not exist.

$$(\mathbf{a}) \quad \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \qquad\qquad (\mathbf{b}) \quad \lim_{(x,y)\to(0,0)} \frac{x^3}{x\,y^2+y^3}$$

Solution.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x\frac{x^2}{x^2+y^2} = 0,$$

because

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

for all $(x, y) \neq (0, 0)$ and

$$\lim_{(x,y)\to(0,0)} x = 0.$$

(b) The limit does not exist exist, as can be seen by computing limits from different directions. Along the line $x = 0$, we have $\lim_{y\to 0} f(0, y) = 0$. Along the line $x = y$, we have $\lim_{y\to 0} f(y, y) = 1/2$.

3. Let $f(x, y) = \sin(x + 3y) + x^2y^3 + 2y^2$. Compute f_x, f_y, f_{xx}, f_{xy}, and f_{yy}.
Solution.

$$\begin{aligned}
f_x(x,y) &= \cos(x+3y) + 2xy^3 \\
f_y(x,y) &= 3\cos(x+3y) + 3x^2y^2 + 4y \\
f_{xx}(x,y) &= -\sin(x+3y) + 2y^3 \\
f_{xy}(x,y) = f_{yx}(x,y) &= -3\sin(x+3y) + 6xy^2 \\
f_{yy}(x,y) &= -9\sin(x+3y) + 6x^2y + 4
\end{aligned}$$

4. * For what values of r is the function

$$f(x,y,z) = \begin{cases} \frac{|x+y+z|^r}{x^2+y^2+z^2}, & \text{if } (x,y,z) \neq (0,0,0), \\ 0, & \text{if } (x,y,z) = (0,0,0). \end{cases}$$

continuous on $\mathbb{R}^3$?
Solution. The given function f is continuous if and only if $r > 2$.

To see this, observe the following. First of all, for all r, f is continous at all points $(x, y, z) \neq (0, 0, 0)$, as the ratio of two continuous functions, the function in the denominator beeng different from zero. So the question is what happens at the origin. Consider three cases:

case $r < 2$. Then, along the x-axis, $y = z = 0$, we have

$$\lim_{x\to 0} f(x,0,0) = \lim_{x\to 0} \frac{|x|^r}{x^2} = \lim_{x\to 0} |x|^{r-2} = \infty \neq f(0,0,0).$$

case $r = 2$. Similarly, along the x-axis, $y = z = 0$, we have

$$\lim_{x\to 0} f(x,0,0) = \lim_{x\to 0} \frac{|x|^2}{x^2} = 1 \neq f(0,0,0).$$

case $r > 2$. Then

$$|x+y+z| \le |x| + |y| + |z| \le 3\max\{|x|,|y|,|z|\} \le 3\sqrt{x^2+y^2+z^2},$$

and

$$0 \le \frac{|x+y+z|^r}{x^2+y^2+z^2} \le \frac{3^r(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 3^r(x^2+y^2+z^2)^{\frac{r-2}{2}} \to 0,$$

as $(x, y, z) \to (0, 0, 0)$.

5. Find the tangent plane and normal line to the surface $z = x^2 + 4y^2$ at the point $(-1, 2, 17)$.
Solution. $z = f(x, y)$, where $f(x, y) = x^2 + 4y^2$. Further, $f(-1, 2) = 17$,

$$\begin{array}{ll} f_x(x,y) = 2x & f_x(-1,2) = -2 \\ f_y(x,y) = 8y & f_y(-1,2) = 16 \end{array}$$

The tangent plane through $(-1, 2, 17)$ has equation

$$z = f(-1, 2) + f_x(-1, 2)(x + 1) + f_y(-1, 2)(y - 2),$$

i.e.,

$$z = 17 - 2(x + 1) + 16(y - 2),$$

or, equivalently,

$$2x - 16y + z + 17 = 0.$$

The normal line is a line that passes through the point $(-1, 2, 17)$ in the direction of the normal vector $\langle 2, -16, 1\rangle$. Therefore it has parametric equations $x = -1 + 2t, y = 2 - 16t, z = 17 + t$, and symmetric equations

$$\frac{x+1}{2} = \frac{2-y}{16} = z - 17.$$

6. Suppose that $w = x^3y + y^2 - 4z^3, x = rs, y = (r^2 - s^2)^3$, and $z = 2r + s$. Use the chain rule to compute $\frac{\partial w}{\partial s}$. Express your answer in terms of r and s alone. Algebraic simplification is not necessary.
Solution.

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (3x^2y)(r) + (x^3 + 2y)((3(r^2 - s^2)^2(-2s)) + (-12z^2)(1) \\ &= 3r^2s^2r + (r^3s^3 + 2(r^2 - s^2)^3)(-6(r^2 - s^2)^2s) - 12(2r + s)^2 \end{aligned}$$

7. * For

$$f(x, y, z) = \frac{x}{x^2 + y^2} e^{\sin(x^2y)} + \sin(z - 2)\ln(x + \sqrt{1 + x^2}),$$

compute $f_x(1, 0, 2)$.
Solution. Let $\varphi(x) := f(x, 0, 2)$. Then, by the definition of partial derivative, $f_x(1, 0, 2) = \varphi'(1)$. We have:

$$\varphi(x) = f(x, 0, 2) = \frac{1}{x}, \qquad \varphi'(x) = \frac{-1}{x^2}, \qquad \varphi'(1) = -1.$$

Thus,

$$f_x(1, 0, 2) = -1.$$

8. Let f be a continuously differentiable function of one variable and define

$$w(x, y) := f\left(\frac{y}{y - x}\right).$$

Show that, for $x \neq y$:

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = 0.$$

Solution.

$$
\begin{aligned}
& x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} \\
&= x f'\left(\frac{x}{y-x}\right)\left(-\frac{-y}{(y-x)^2}\right) + y f'\left(\frac{x}{y-x}\right)\left(\frac{(y-x)-y}{(y-x)^2}\right) \\
&= \frac{xy}{(y-x)^2}\, f'\left(\frac{x}{y-x}\right) - \frac{xy}{(y-x)^2}\, f'\left(\frac{x}{y-x}\right) = 0.
\end{aligned}
$$

9. * For

$$f(x,y) = 5 + 6(x-3) + 8(y+1) + \sin(x-3)^2 + (y+1)\ln\sqrt{2+y},$$

write an equaion of the tangent plane to the graph at the point $(3,-1,5)$.
Solution. Observe that

$$f(x,y) = 5 + 6(x-3) + 8(y+1) + (x-3)\,h_1(x,y) + (y+1)\,h_2(x,y),$$

where

$$h_1(x,y) = \frac{\sin(x-3)^2}{x-3}, \qquad h_2(x,y) = \ln\sqrt{2+y},$$

and

$$\lim_{(x,y)\to(3,-1)} h_1(x,y) = \lim_{(x,y)\to(3,-1)} h_2(x,y) = 0.$$

Thus, near $(x,y) = (3,-1)$, we have

$$f(x,y) = 5 + 6(x-3) + 8(y+1) + \text{higher order error term},$$

and the tangent plane at $(-3,1,5)$ is given by

$$z = 5 + 6(x-3) + 8(y+1).$$

All done, without a single computation.

10. Assume that u and v have continuous second partial derivatives and that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Solution.

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0.
\end{aligned}
$$

11. The equation $xyz = \cos(x+y+z)$ defines implicitly z as a function of x and y. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Solution. Let $F(x, y, z) := xyz - \cos(x+y+z)$. Then

$$F_x(x,y,z) = yz + \sin(x+y+z),$$

$$F_y(x,y,z) = xz + \sin(x+y+z),$$

$$F_z(x,y,z) = xy + \sin(x+y+z),$$

and

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}.$$

12. Consider the function f defined by $f(x,y,z) = xy + y^2 z$.

(a) Find ∇f.

(b) Find the directional derivative of f at the point $(2, 3, -1)$ in the direction of the vector $\mathbf{v} = \langle 2, 4, -3 \rangle$.

Solution.

$$\nabla f(x,y,z) = \langle y, x + 2yz, y^2 \rangle, \nabla f(2,3,-1) = \langle 3, -4, 9 \rangle, |\mathbf{v}| = \sqrt{4+16+9} = \sqrt{29},$$

$$D_{\mathbf{v}} f(2,3,-1) = \nabla f(2,3,-1) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{6 - 16 - 27}{\sqrt{29}} = -\frac{37}{\sqrt{29}}.$$

13. Find the maximum rate of change of f at the given point and the direction in which it occurs.

(a) $f(x,y) = x^2 e^y - 3y$, $(1, 0)$,

(b) $f(x,y,z) = \cos(4x - 3y) + z^2$, $(\pi/16, -\pi/6, 2)$,

(c) $f(x,y) = \sqrt{5x + y^2}$, $(4, 4)$.

Solution. The maximum rate of change of a function f at a point P is $|\nabla f(P)|$ and it occurs in the direction of the gradient $\nabla f(P)$. Thus,

(a) $\nabla f(x,y) = \langle 2xe^y, x^2 e^y - 3 \rangle$,
direction of fastest increase at the point $(1,0)$ is $\nabla f(1,0) = \langle 2, -2 \rangle$,
the maximal rate of change is $|\nabla f(1,0)| = \sqrt{4+4} = 2\sqrt{2}$.

(b) $\nabla f(x,y,z) = \langle -4\sin(4x+3y), 3\sin(4x-3y), 2z \rangle$,
direction of fastest increase at the point $(\pi/16, -\pi/6, 2)$ is
$\nabla f(\pi/16, -\pi/6, 2) = \langle -2\sqrt{2}, 3/\sqrt{2}, 4 \rangle$,
the maximal rate of change is $|\nabla f(\pi/16, -\pi/6, 2)| = \sqrt{8 + 4.5 + 16} = \sqrt{57/2}$.

(c)

$$\nabla f(x,y) = \frac{\langle 5, 2y \rangle}{2\sqrt{5x + y^2}},$$

the direction of fastest increase at the point $(4,4)$ is $\nabla f(4,4) = \dfrac{\langle 5, 8 \rangle}{12}$,

the maximal rate of change is $|\nabla f(4,4)| = \sqrt{89}/12$.

14. * Let f be a smooth function with the following property: At the point (1,2,3), the function f increases most rapidly in the direction given by the vector $\langle 5,-3,0\rangle$ and the rate of change of f in this direction is 17. Find $\nabla f(1,2,3)$.
Solution. Since a function increases most rapidly in the direction of its gradient, $\nabla f(1,2,3)$ must have the same direction as the vector $\mathbf{v}=\langle 5,-3,0\rangle$. Further, the most rapid rate of increase is equal to the magnitude of the gradient, so $|\nabla f(1,2,3)|=17$. Therefore,
$$\nabla f(1,2,3)=17\frac{\mathbf{v}}{|\mathbf{v}|}=17\frac{\langle 5,-3,0\rangle}{\sqrt{25+9}}=\frac{\langle 85,-51,0\rangle}{\sqrt{34}}.$$

15. * Let f be a smooth function with the following property: At the point (0,0,0), the function f *decreases* most rapidly in the direction of $\langle 6,-1,2\rangle$ and the rate of change of f in this direction is -5. Find $\nabla f(0,0,0)$.
Solution. Since a function decreases most rapidly in the direction opposite to its gradient, $\nabla f(0,0,0)$ must be parallel to the vector $\mathbf{v}=\langle 6,-1,2\rangle$, but with opposite direction. Further, the absolute value of the most rapid rate of change is equal to the magnitude of the gradient, so $|\nabla f(0,0,0)|=5$. Therefore,
$$\nabla f(0,0,0)=5\left(-\frac{\mathbf{v}}{|\mathbf{v}|}\right)=-5\frac{\langle 6,-1,2\rangle}{\sqrt{36+1+4}}=\frac{\langle -30,5,-10\rangle}{\sqrt{41}}.$$

16. * Let $\mathbf{u}=\frac{1}{\sqrt{2}}\langle 1,1\rangle$ and $\mathbf{v}=\frac{1}{\sqrt{2}}\langle 1,-1\rangle$. Assume that f_x and f_y are continuous. If $D_{\mathbf{u}}f(2,1)=2\sqrt{2}$ and $D_{\mathbf{v}}f(2,1)=-8\sqrt{2}$, find $\nabla f(2,1)$.
Solution. Since the two vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $\nabla f(2,1)=a\mathbf{u}+b\mathbf{v}$ for some real numbers, a and b. To find these numbers, we dot-multiply the equality by $\mathbf{u}$ and $\mathbf{v}$:
$$\begin{aligned}2\sqrt{2}&=D_{\mathbf{u}}f(2,1)=\nabla f(2,1)\cdot\mathbf{u}=(a)(1)+(b)(0)=a,\\-8\sqrt{2}&=D_{\mathbf{v}}f(2,1)=\nabla f(2,1)\cdot\mathbf{v}=(a)(0)+(b)(1)=b.\end{aligned}$$
Thus,
$$\nabla f(2,1)=a\mathbf{u}+b\mathbf{v}=2\sqrt{2}\mathbf{u}-8\sqrt{2}\mathbf{v}=\langle 2,2\rangle+\langle -8,8\rangle=\langle -6,10\rangle.$$

17. Find the tangent plane and normal line to the surface $z=x^2+4y^2$ at the point $(-1,2,17)$.
Solution. The surface under consideration is described by the equation $F(x,y,z)=0$, where $F(x,y,z)=x^2+4y^2-z$. Further,
$$\begin{aligned}F_x(x,y,z)&=2x & F_x(-1,2,17)&=-2\\F_y(x,y,z)&=8y & F_y(-1,2,17)&=16\\F_z(x,y,z)&=-1 & F_z(-1,2,17)&=-1\end{aligned}$$
The tangent plane through $(-1,2,17)$ has equation
$$F_x(-1,2,17)(x+1)+F_y(-1,2,17)(y-2)+F_z(-1,2,17)(z-17)=0,$$
i.e.,
$$-2(x+1)+16(y-2)-(z-17),$$
or, equivalently,
$$-2x+16y-z-17=0.$$
The normal line is a line that passes through the point $(-1,2,17)$ in the direction of the normal vector $\langle -2,16,-1\rangle$. Therefore it has parametric equations $x=-1-2t, y=2+16t, z=17-t$, and symmetric equations
$$\frac{x+1}{-2}=\frac{2-y}{16}=17-z.$$

18. Find all the points on the surface $x^2 + 4y^2 - z^2 = 4$ at which the tangent plane is parallel to the plane $x + 2y + z = 6$.
Solution. Let $F(x, y, z) = x^2 + 4y^2 - z^2 - 4$. Then $F_x(x, y, z) = 2x, F_y(x, y, z) = 8y, F_z(x, y, z) = -2z$. The tangent plane at a point (a, b, c) on the surface $F(x, y, z) = 0$ is the given by the equation

$$2a(x - a) + 8b(y - b) - 2c(z - c) = 0,$$

or, equivalently,

$$a(x - a) + 4b(y - b) - c(z - c) = 0,$$

and has a normal vector $\langle a, 4b, -c\rangle$. The tangent plane will be parallel to the plane $x + 2y + z = 6$, if and only if the normal vectors of the two planes are equal, i.e., $\langle a, 4b, -c\rangle = \lambda\langle 1, 2, 1\rangle$ for some number λ. This gives us the system

$$\begin{cases} a &= \lambda \\ 4b &= 2\lambda \\ -c &= \lambda \\ a^2 + 4b^2 - c^2 &= 4 \end{cases}$$

Substituting a, b, c in the last equation, we get

$$\lambda^2 + 4(\lambda/2)^2 - \lambda^2 = 4,$$

whence $\lambda = \pm 2$. This gives us two points on the surface: $(a, b, c) = (2, 1, -2)$ and $(a, b, c) = (-2, -1, 2)$.

19. Find all the critical points of the given function, and determine which of them are local minima, which are local maxima, and which are saddlepoints.

(a) $f(x, y) = x^2 + xy + y^2 + x - y + 2$,
(b) $f(x, y) = 3x^2 - 3xy + y^3 + 3y^2$.

Solution.

(a) First, find f_x and f_y and set them equal to 0:

$$\begin{aligned} f_x(x, y) &= 2x + y + 1 = 0, \\ f_y(x, y) &= x + 2y - 1 = 0, \end{aligned}$$

which gives $x = -1, y = 1$. Thus, we have only one critical point, $(-1, 1)$. To see whether it is a minimum, a maximum, or a saddlepoint, we use the second derivative test.

$$f_{xx}(x, y) = 2, f_{xy}(x, y) = f_{yx}(x, y) = 1, f_{yy}(x, y) = 2$$

and therefore

$$f_{xx}(-1, 1) = 2 > 0, D(-1, 1) = 4 - 1 = 3 > 0,$$

which means that $(-1, 1)$ is a minimum point.

(b) We follow the same procedure as in part (a):

$$\begin{aligned} f_x(x, y) &= 6x - 3y = 0, \\ f_y(x, y) &= -3x + 3y^2 + 6y = 0. \end{aligned}$$

Expressing $y = 2x$ from the first equation, and substituting it into the second, we get

$$-3x + 12x^2 + 12x = 0 \Rightarrow 4x^2 + 3x = 0 \Rightarrow x = 0 \text{ or } x = -3/4.$$

which gives two critical points, $(0, 0)$ and $(-3/4, -3/2)$. Further, we again use the second derivative test.

$$\begin{aligned} f_{xx}(x, y) &= 6, \\ f_{xy}(x, y) = f_{yx}(x, y) &= -3, \\ f_{yy}(x, y) &= 6y + 6, \\ D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 &= 36y + 27 \end{aligned}$$

and therefore

$$f_{xx}(0, 0) = 6 > 0, D(0, 0) = 27 > 0,$$
$$D(-3/4, -3/2) = -27 < 0,$$

which means that $(0, 0)$ is a minimum point, and $(-3/4, -3/2)$ is a saddlepoint.

20. Let R be the region described by $0 \le y \le \sqrt{16 - x^2}$, $0 \le x \le 4$. Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 + y^2 - 4x - 2y + 1$ on R.
Solution. First we look for local extrema inside the region. We find the gradient and set it equal to 0:

$$\begin{aligned} f_x(x, y) &= 2x - 4 = 0, \\ f_y(x, y) &= 2y - 2 = 0. \end{aligned}$$

This gives us one critical point, $(2, 1)$, and it is inside R. Then we look at the boundary of R. The boundary consists of three parts: the line segment connecting $(0, 0)$ and $(0, 4)$, the line segment connecting $(0, 0)$ and $(4, 0)$, and the arc of the circle $x^2 + y^2 = 16$ connecting $(4, 0)$ and $(0, 4)$. We look at these three parts one by one.

On the segment $x = 0, 0 \le y \le 4$, we have $f(0, y) = y^2 - 2y + 1$. This expression has a local minimum for $y = 1$, and there are also the two endpoints $y = 0$ and $y = 4$ to consider. This gives us three more points: $(0, 0), (0, 1), (0, 4)$.

On the segment $y = 0, 0 \le x \le 4$, we have $f(x, 0) = x^2 - 4x + 1$. This expression has a local minimum for $x = 2$, and there are also the two endpoints $x = 0$ and $x = 4$ to consider. This gives us the points: $(0, 0), (2, 0), (4, 0)$.

Finally, the arc under consideration can be parametrized as $x = 4\cos\theta, y = 4\sin\theta, 0 \le \theta \le \pi/2$. Then the function on the arc is

$$\begin{aligned} f(4\cos\theta, 4\sin\theta) &= 16\cos^2\theta + 16\sin^2\theta - 16\cos\theta - 8\sin\theta + 1 \\ &= 17 - 16\cos\theta - 8\sin\theta. \end{aligned}$$

We then find the critical points of this expression, by differentiating it with respect to θ and setting the derivative equal to zero:

$$16\sin\theta - 8\cos\theta = 0,$$

or, equivalently,

$$4y - 2x = 0.$$

Expressing $x = 2y$ and substituting into $x^2 + y^2 = 16$, we find one more suspicious point, $(8/\sqrt{5}, 4/\sqrt{5})$. Of course, there are also the endpoints of the arc, $(4, 0)$ and $(0, 4)$. Now we list all the suspicious points and evaluate the function at them:

$$\begin{array}{cc}
(2,1) & f(2,1) = -4 \\
(0,0) & f(0,0) = 1 \\
(0,1) & f(0,1) = 0 \\
(0,4) & f(0,4) = 9 \\
(2,0) & f(2,0) = -3 \\
(4,0) & f(4,0) = 1 \\
(8/\sqrt{5}, 4/\sqrt{5}) & f(8/\sqrt{5}, 4/\sqrt{5}) = 17 - 8\sqrt{5}
\end{array}$$

Comparing the values at these points, we see that the maximum value is 9, attained at (0,4), and the minimal value is -4, attained at (2,1).