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Review for Test 1
Solutions

1. The three points $A(-2,0,1)$, $B(-3,4,3)$, and $C(-5,3,4)$ determine a triangle. Is this a right triangle? Is it isosceles?
Solution.
$$\vec{AB} = \langle -1,4,2\rangle, |\vec{AB}| = \sqrt{1+16+4} = \sqrt{21},$$
$$\vec{AC} = \langle -3,3,3\rangle, |\vec{AC}| = \sqrt{9+9+9} = \sqrt{27},$$
$$\vec{BC} = \langle -2,-1,1\rangle, |\vec{BC}| = \sqrt{4+1+1} = \sqrt{6}.$$
Since no sides have equal lengths, the triangle ABC cannot be isosceles. On the other hand, since
$$|AB|^2 + |BC|^2 = 21 + 6 = 27 = |AC|^2,$$
the triangle ABC is a right triangle.

2. Write an equation of the sphere with center $C(-6,-3,7)$ and radius $4\sqrt{3}$.
Solution.
$$(x+6)^2 + (y+3)^2 + (z-7)^2 = 48,$$
or, equivalently,
$$x^2 + y^2 + z^2 + 12x + 6y - 14z + 46 = 0.$$

3. A sphere has an equation $3x^2 + 3y^2 + 3z^2 - 12x + 6y - 36z = 120$. Find the center and the radius of the sphere.
Solution. We manipulate the equation. To simplify the computations, we first divide both sides by 3.
$$x^2 + y^2 + z^2 - 4x + 2y - 12z = 40.$$
Completing the square we get
$$(x-2)^2 + (y+1)^2 + (z-6)^2 - 4 - 1 - 36 = 40,$$
or, equvalently,
$$(x-2)^2 + (y+1)^2 + (z-6)^2 = 81.$$
Therefore the circle has center $C(2,-1,6)$ and radius $r = 9$.

4. * Find an equation of the sphere with center $(-3,2,6)$ that touches the xy-plane.
Solution. Let $C(-3,2,6)$ be the center of the sphere as given, and let A be the point at which the sphere touches the xy-plane. Then $|AC| = r$ is the radius of the sphere. Moreover, since the xy-plane is tangent to the sphere, $\vec{AC}$ is orthogonal to the xy-plane (and hence, parallel to the z axis). In other words, A is the orthogonal projection of C onto the xy-plane. Thus we conclude that A has coordinates $(-3,2,0)$, whence $r = |AC| = 6$. We then write the equation:
$$(x+3)^2 + (y-2)^2 + (z-6)^2 = 36,$$
or, equvalently,
$$x^2 + y^2 + z^2 + 6x - 4y - 12z + 13 = 0.$$

5. For the given vectors $\mathbf{a}$, $\mathbf{b}$, compute $|\mathbf{a}|$, $|\mathbf{b}|$, $2\mathbf{a}-3\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$, $\text{comp}_{\mathbf{a}}\mathbf{b}$, and $\text{proj}_{\mathbf{a}}\mathbf{b}$.

(a) $\mathbf{a}=\langle 3,1,0\rangle, \mathbf{b}=\langle 1,-1,-1\rangle$.

(b) $\mathbf{a}=\langle 2,0,-1\rangle, \mathbf{b}=\langle -1,-2,3\rangle$.

(c) $\mathbf{a}=\langle 1,1,1\rangle, \mathbf{b}=\langle -1,0,1\rangle$.

(d) $\mathbf{a}=\langle 1,-1,1\rangle, \mathbf{b}=\langle 2,-2,2\rangle$.

Solution.

(a)

$$
\begin{aligned}
|\mathbf{a}| = \sqrt{9+1+0} = \sqrt{10}, \qquad & |\mathbf{b}| = \sqrt{1+1+1} = \sqrt{3}, \\
2\mathbf{a}-3\mathbf{b} \quad &= \quad \langle 6-3, 2+3, 0+3\rangle = \langle 3,5,3\rangle, \\
\mathbf{a}\cdot\mathbf{b} \quad &= \quad 3-1+0 = 2, \\
\mathbf{a}\times\mathbf{b} \quad &= \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = \langle -1,3,-4\rangle, \\
\text{comp}_{\mathbf{a}}\mathbf{b} \quad &= \quad \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{2}{\sqrt{10}} \\
\text{proj}_{\mathbf{a}}\mathbf{b} \quad &= \quad (\text{comp}_{\mathbf{a}}\mathbf{b})\frac{\mathbf{a}}{|\mathbf{a}|} = \left\langle \frac{3}{5}, \frac{1}{5}, 0\right\rangle.
\end{aligned}
$$

(b)

$$
\begin{aligned}
|\mathbf{a}| = \sqrt{5}, \qquad & |\mathbf{b}| = \sqrt{14}, \\
2\mathbf{a}-3\mathbf{b} \quad &= \quad \langle 7,6,-11\rangle, \\
\mathbf{a}\cdot\mathbf{b} \quad &= \quad -5, \\
\mathbf{a}\times\mathbf{b} \quad &= \quad \langle -2,-5,-4\rangle, \\
\text{comp}_{\mathbf{a}}\mathbf{b} \quad &= \quad -\sqrt{5} \\
\text{proj}_{\mathbf{a}}\mathbf{b} \quad &= \quad \langle -2,0,1\rangle.
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\mathbf{a}| = \sqrt{3}, \qquad & |\mathbf{b}| = \sqrt{2}, \\
2\mathbf{a}-3\mathbf{b} \quad &= \quad \langle 5, 2-1\rangle, \\
\mathbf{a}\cdot\mathbf{b} \quad &= \quad 0 \quad \text{(the vectors are orthogonal)}, \\
\mathbf{a}\times\mathbf{b} \quad &= \quad \langle 1,-2,1\rangle, \\
\text{comp}_{\mathbf{a}}\mathbf{b} \quad &= \quad 0 \\
\text{proj}_{\mathbf{a}}\mathbf{b} \quad &= \quad \langle 0,0,0\rangle.
\end{aligned}
$$

(d)

$$
\begin{aligned}
|\mathbf{a}| = \sqrt{3}, \qquad & |\mathbf{b}| = 2\sqrt{3}, \\
2\mathbf{a}-3\mathbf{b} \quad &= \quad \langle -4,4,-4\rangle, \\
\mathbf{a}\cdot\mathbf{b} \quad &= \quad 6, \\
\mathbf{a}\times\mathbf{b} \quad &= \quad \langle 0,0,0\rangle \quad \text{(the vectors are parallel)} \\
\text{comp}_{\mathbf{a}}\mathbf{b} \quad &= \quad 2\sqrt{3} \\
\text{proj}_{\mathbf{a}}\mathbf{b} \quad &= \quad \langle 2,-2,2\rangle.
\end{aligned}
$$

6. Find the cosines of the angles of the triangles with the given vertices:

   (a) $A(1,1,0), B(0,1,0), C(1,1,-1)$.

   (b) $A(19,17,35), B(43,-1,-5), C(-13,41,5)$.

   Solution.

   (a)

$$\vec{AB} = \langle -1,0,0\rangle, \vec{BC} = \langle 1,0,-1\rangle, \vec{AC} = \langle 0,0,-1\rangle.$$

$$\begin{aligned}
\cos A &= \frac{\vec{AB}\cdot\vec{AC}}{|AB|\,|AC|} = 0 \\
\cos B &= \frac{\vec{BA}\cdot\vec{BC}}{|AB|\,|BC|} = -\frac{\vec{AB}\cdot\vec{BC}}{|AB|\,|BC|} = \frac{1}{\sqrt{2}} \\
\cos C &= \frac{\vec{CA}\cdot\vec{CB}}{|AC|\,|BC|} = \frac{\vec{AC}\cdot\vec{BC}}{|AC|\,|BC|} = \frac{1}{\sqrt{2}}
\end{aligned}$$

   (b) $\cos A = 0, \cos B = \frac{1}{\sqrt{2}}, \cos C = \frac{1}{\sqrt{2}}$

7. * Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. Show that

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

   Solution.

$$\begin{aligned}
&|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 \\
&= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\
&\quad + \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\
&= 2\mathbf{a}\cdot\mathbf{a} + 2\mathbf{b}\cdot\mathbf{b} = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.
\end{aligned}$$

8. * Let $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. Find the angle between $\mathbf{a}$ and $\mathbf{b}$.
   Solution.

$$|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$$

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$$

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$$

$$\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}$$

$$\mathbf{a}\cdot\mathbf{b} = 0$$

   Therefore, the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

9. * If $|\mathbf{a}| = |\mathbf{a}-\mathbf{b}|$, show that $\mathbf{a}\cdot\mathbf{b} = (\mathbf{b}-\mathbf{a})\cdot\mathbf{b}$.
Solution.

$$\begin{aligned}
|\mathbf{a}|^2 &= |\mathbf{a}-\mathbf{b}|^2 \\
\mathbf{a}\cdot\mathbf{a} &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
\mathbf{a}\cdot\mathbf{a} &= \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
0 &= \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} \\
\mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} \\
\mathbf{a}\cdot\mathbf{b} &= (\mathbf{b}-\mathbf{a})\cdot\mathbf{b}
\end{aligned}$$

10. * If $\mathbf{c} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all non–zero vectors, show that the angle between $\mathbf{a}$ and $\mathbf{c}$ is the same as the angle between $\mathbf{b}$ and $\mathbf{c}$.
Solution. Let θ be the angle between $\mathbf{a}$ and $\mathbf{c}$ and let φ be the angle between $\mathbf{b}$ and $\mathbf{c}$. Then

$$\cos\theta = \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}|\,|\mathbf{a}|} = \frac{|\mathbf{a}|(\mathbf{b}\cdot\mathbf{a}) + |\mathbf{b}||\mathbf{a}|^2}{|\mathbf{c}|\,|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}||\mathbf{b}|}{|\mathbf{c}|}$$
$$\cos\varphi = \frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{c}|\,|\mathbf{b}|} = \frac{|\mathbf{a}||\mathbf{b}|^2 + |\mathbf{b}|(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{c}|\,|\mathbf{b}|} = \frac{|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b}}{|\mathbf{c}|},$$

and we see that the two expressions are equal.

11. Find the area of the parallelogram with vertices $A(0,1,0), B(3,0,-1), C(5,-2,0)$ and $D(2,-1,1)$.
Solution.

$$\begin{aligned}
\vec{AB} = \vec{DC} &= \langle 3,-1,-1\rangle \\
\vec{AD} = \vec{BC} &= \langle 2,-2,1\rangle
\end{aligned}$$
$$\text{area} = |\vec{AB}\times\vec{AD}| = |\langle -3,-5,-4\rangle| = \sqrt{9+25+16} = 5\sqrt{2}$$

12. Find the area of the triangle with vertices $P(1,-1,2), Q(3,2,3), R(0,1,-1)$.
Solution.

$$\vec{PQ} = \langle 2,3,1\rangle, \qquad \vec{PR} = \langle -1,2,-3\rangle.$$
$$\text{area} = \frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}|\langle -11,5,7\rangle| = \frac{1}{2}\sqrt{121+25+49} = \frac{\sqrt{195}}{2}$$

13. Given $P(0,1,2), Q(1,1,8), R(2,4,-6)$, and $S(8,-6,7)$, find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.
Solution.

$$\vec{PQ} = \langle 1,0,6\rangle, \vec{PR} = \langle 2,3,-8\rangle, \vec{PS} = \langle 8,-7,5\rangle.$$
$$\text{Volume} = \text{abs}\begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -7 & 5 \end{vmatrix} = |-269| = 269.$$

14. * Show that $|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.
Solution. Let θ be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. We have

$$\begin{aligned}
|\mathbf{u}\times\mathbf{v}|^2 &= (|\mathbf{u}|\,|\mathbf{v}|\sin\theta)^2 \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2\sin^2\theta \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2(1-\cos^2\theta) \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 - (|\mathbf{u}|\,|\mathbf{v}|\cos\theta)^2 \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2.
\end{aligned}$$

15. * Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three-dimensional vectors such that

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

and

$$\mathbf{u} \cdot \mathbf{v} = 0,$$

then

$$\mathbf{w} \times \mathbf{u} = |\mathbf{u}|^2 \mathbf{v}.$$

Solution. Recall that, for any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in 3D,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

In particular, in our case,

$$\begin{aligned} \mathbf{w} \times \mathbf{u} &= (\mathbf{u} \times \mathbf{v}) \times \mathbf{u} = -\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) \\ &= -(\mathbf{u} \cdot \mathbf{v})\mathbf{u} + (\mathbf{u} \cdot \mathbf{u})\mathbf{v} \\ &= |\mathbf{u}|^2 \mathbf{v}. \end{aligned}$$

16. * Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three-dimensional vectors such that

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = 0,$$

then

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

Solution. Cross-multiply the given equality by $\mathbf{v}$ and get

$$\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v} + \mathbf{w} \times \mathbf{v} = 0,$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w}.$$

Similarly, cross-multiplying by $\mathbf{u}$ gives $\mathbf{u} \times \mathbf{v} = \mathbf{w} \times \mathbf{u}$.

17. Given $A(0,2,0), B(2,0,2), P(1,0,0)$ and $Q(0,1,1)$, determine whether the line passing through A and B, and the line passing through P and Q are parallel, skew, or intersecting. If they intersect, find the point of intersection.

Solution. The first line passes through the point $A(0,2,0)$ and has a direction vector $\vec{AB} = \langle 2, -2, 2 \rangle$. Therefore it can be described by the following parametric equations:

$$\begin{aligned} x &= 2t \\ y &= 2 - 2t \\ z &= 2t \end{aligned}$$

Similarly, the second line passes through the point $P(1,0,0)$ and has a direction vector $\vec{PQ} = \langle -1, 1, 1 \rangle$. Therefore it can be described by the following parametric equations:

$$\begin{aligned} x &= 1 - s \\ y &= s \\ z &= s \end{aligned}$$

The two lines intersect if and only if they have a point (x, y, z) in common, i.e. if and only if the following system has a solution:

$$\begin{aligned} 2t &= 1-s \\ 2-2t &= s \\ 2t &= s \end{aligned}$$

Combining the first and third equations, we conclude that $1-s=s=\frac{1}{2}$ and hence $2t=\frac{1}{2}$. Substituting this into the second equation, we obtain the equality $\frac{3}{2}=\frac{1}{2}$, which is obviously false. Therefore, the system has no solution, and the two lines do not intersect. Further, since their direction vectors, $\vec{AB}$ and $\vec{PQ}$, are not proportional to each other, the two lines are not parallel. The conclusion is that the two lines are skew.

18. Given $A(0,1,0), B(1,0,1), P(1,0,0)$ and $Q(0,1,1)$, determine whether the line passing through A and B, and the line passing through P and Q are parallel, skew, or intersecting. If they intersect, find the point of intersection.
Solution. The first line passes through the point $A(0,1,0)$ and has a direction vector $\vec{AB}=\langle 1,-1,1\rangle$. Therefore it can be described by the following parametric equations:

$$\begin{aligned} x &= t \\ y &= 1-t \\ z &= t \end{aligned}$$

Similarly, the second line passes through the point $P(1,0,0)$ and has a direction vector $\vec{PQ}=\langle -1,1,1\rangle$. Therefore it can be described by the following parametric equations:

$$\begin{aligned} x &= 1-s \\ y &= s \\ z &= s \end{aligned}$$

The two lines intersect if and only if they have a point (x, y, z) in common, i.e. if and only if the following system has a solution:

$$\begin{aligned} t &= 1-s \\ 1-t &= s \\ t &= s \end{aligned}$$

Combining the first and third equations, we conclude that $t=1-s=s=\frac{1}{2}$. Substituting this into the second equation, we obtain the equality $\frac{1}{2}=\frac{1}{2}$, which is obviously true. Therefore, the system has a solution, and the two lines intersect. To find the point of intersection, take the value of t and plug it into the parametric equations. The intersection point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

19. Find an equation of the plane passing through the point $(-6,0,1)$ and parallel to the plane $x-y+3z=9$.
Solution. Every plane that is parallel to the plane $x-y+3z=9$ has an equation of the form $x-y+3z=d$ for some d. To find the value of d in our case, we use the fact that the point $(-6,0,1)$ is on the plane. Plugging the coordinates into the equation, we get $-6-0+3=d$, i.e., $d=-3$. The equation then is

$$x-y+3z=-3.$$

20. Find an equation of a plane passing through the point $(-1, 2, 7)$ and perpendicular to the line

$$\frac{x-3}{3} = \frac{2-y}{7} = -\frac{z}{2}.$$

Solution. Since plane must be orthogonal to the line, the direction vector of the line is a normal vector for the plane. Denoting

$$t := \frac{x-3}{3} = \frac{2-y}{7} = -\frac{z}{2},$$

the given line can be rewritten in parametric form:

$$\begin{aligned} x &= 3 + 3t \\ y &= 2 - 7t \\ z &= - 2t \end{aligned}$$

This makes it clear that the direction vector of the line is $\langle 3, -7, -2 \rangle$. Thus, we are looking for an equation of a plane with a normal vector $\mathbf{n} = \langle 3, -7, -2 \rangle$ and passing through the point $(-1, 2, 7)$. This we can readily write:

$$3(x+1) - 7(y-2) - 2(z-7) = 0,$$

or, equivalently,

$$3x - 7y - 2z + 31 = 0.$$

21. Consider the planes $3x - 4z = 12$ and $x - 2y - 2z = -2$.

(a) Find the cosine of the angle between the planes.

(b) Find symmetric equations for the line of intersection of the planes.

Solution. The first plane has a normal vector $\mathbf{a} = \langle 3, 0, -4 \rangle$, and the second plane has a normal vector $\mathbf{b} = \langle 1, -2, -2 \rangle$.

(a) The angle between the planes is equal to the angle between their normal vectors which has a cosine

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{3+8}{(5)(3)} = \frac{11}{15}.$$

(b) We need a point on the line of intersection, and a direction vector. To find a point, we have some freedom; let's choose $z = 0$ (we could use any other nuber). Having chosen z, we successively determine $x = 4$ (from the first equation) and then $y = 3$ (from the second equation). This gives us the point $(4, 3, 0)$ which lies on the line of intersection. For a direction vector, we can take just

$$\mathbf{a} \times \mathbf{b} = \langle 3, 0, -4 \rangle \times \langle 1, -2, -2 \rangle = \langle -8, 2, -6 \rangle.$$

This gives us the symmetric equations:

$$\frac{x-4}{-8} = \frac{y-3}{2} = \frac{z}{-6}.$$

22. A plane passes through the points $(0, 0, 1)$, $(0, 2, 0)$, and $(3, 0, 0)$. What is the distance between this plane and the origin?
Solution. $\vec{AB} = \langle 0, 2, -1\rangle$, $\vec{AC} = \langle 3, 0, -1\rangle$. The plane passes through the point $A(0, 0, 1)$. A normal vector is $\mathbf{n} = \vec{AB} \times \vec{AC} = \langle -2, -3, -6\rangle$. Then, an equation of the plane is

$$-2x - 3y - 6(z - 1) = 0,$$

or, equivalently

$$2x + 3y + 6z = 6.$$

By the distance formula, the distance to the origin is given by

$$\text{dist} = \frac{|0 + 0 + 0 - 6|}{\sqrt{4 + 9 + 36}} = \frac{6}{7}.$$

23. * Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x = 1$ and $2x^2+4y^2-2z^2-5y = 0$ lies in a plane.
Solution. If a point lies on the intersection of the two surfaces, it must satisfy both equations. This means that both

$$x^2 + 2y^2 - z^2 + 3x = 1$$

and

$$2x^2 + 4y^2 - 2z^2 - 5y = 0$$

must be true. Multiplying the first equation by 2 and subtracting the second, we see that

$$6x - 5y = 0$$

must be true for all points (x, y, z) of the intersection. In other words, the points of intersection of the two surfaces lie on the plane $6x - 5y = 0$.

24. Find parametric equations for the tangent line to the curve

$$\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 3t^3\mathbf{k}$$

at the point $(1, 1, -3)$.
Solution. A tangent vector to the curve at the point $\mathbf{r}(t)$ is $\mathbf{r}'(t)$. We have $\mathbf{r}'(t) = 2\mathbf{i}+2t\mathbf{j}-9t^2\mathbf{k}$. In particular, for $t = 1$, we have $\mathbf{r}(t) = (1, 1, -3)$ and $\mathbf{r}'(t) = \langle 2, 2, -9\rangle$. Thus, the tangent line passes through the point $(1, 1, -3)$ and has a direction vector $\langle 2, 2, -9\rangle$. The parametric equations of the tangent line are:

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + 2t \\ z &= -3 - 9t\,. \end{aligned}$$

25. For each of the following parametric curves, compute the unit tangent vector $\mathbf{T}(t)$, $\frac{d\mathbf{T}}{ds}$, the principal unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$. Here s denotes the natural parameter.

(a) $\mathbf{r}(t) = \langle e^t \sin(2t), e^t, e^t \cos(2t)\rangle$,

(b) $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, -4\cos t\rangle$.

Solution.

(a)

$$
\begin{aligned}
\mathbf{r}'(t) &= \langle 2e^t\cos(2t) + e^t\sin(2t), e^t, e^t\cos(2t) - 2e^t\sin(2t)\rangle \\
|\mathbf{r}'(t)| &= \sqrt{6}e^t \\
\mathbf{T} &= \frac{1}{\sqrt{6}}\langle 2\cos(2t) + \sin(2t), 1, \cos(2t) - 2\sin(2t)\rangle \\
\frac{d\mathbf{T}}{dt} &= \frac{\sqrt{6}}{3}\langle \cos(2t) - 2\sin(2t), 0, -2\cos(2t) - \sin(2t)\rangle \\
\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} &= \frac{1}{\sqrt{5}}\langle \cos(2t) - 2\sin(2t), 0, -2\cos(2t) - \sin(2t)\rangle \\
\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} &= \frac{d\mathbf{T}/dt}{|\mathbf{r}'(t)|} = \frac{1}{3}e^{-t}\langle \cos(2t) - 2\sin(2t), 0, -2\cos(2t) - \sin(2t)\rangle \\
\kappa(t) &= \left|\frac{d\mathbf{T}}{ds}\right| = \frac{\sqrt{5}}{3}e^{-t}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{r}'(t) &= \langle -3\sin t, 5\cos t, 4\sin t\rangle \\
|\mathbf{r}'(t)| &= 5 \\
\mathbf{T} &= \frac{1}{5}\langle -3\sin t, 5\cos t, 4\sin t\rangle \\
\frac{d\mathbf{T}}{dt} &= \frac{1}{5}\langle -3\cos t, -5\sin t, 4\cos t\rangle \\
\left|\frac{d\mathbf{T}}{dt}\right| &= 1 \\
\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} &= \frac{1}{5}\langle -3\cos t, -5\sin t, 4\cos t\rangle \\
\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} &= \frac{d\mathbf{T}/dt}{|\mathbf{r}'(t)|} = \frac{1}{25}\langle -3\cos t, -5\sin t, 4\cos t\rangle \\
\kappa(t) &= \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{5}
\end{aligned}
$$

26. Find the length of the curve described by:

(a) $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle, \quad 0 \le t \le \pi,$

(b) $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 2t^{3/2}\rangle, \quad 0 \le t \le 3,$

(c) $\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle, \quad 0 \le t \le 1.$

Solution.

(a)

$$\begin{aligned} \mathbf{r}'(t) &= \langle -4\sin t, 4\cos t, 3\rangle, \quad |\mathbf{r}'(t)| = 5 \\ L &= \int_0^\pi |\mathbf{r}'(t)|\, dt = \int_0^\pi 5\, dt = 5\pi \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{r}'(t) &= 3\langle \cos(3t), -\sin(3t), \sqrt{t}\rangle, \quad |\mathbf{r}'(t)| = 3\sqrt{t+1} \\ L &= \int_0^3 |\mathbf{r}'(t)|\, dt = \int_0^3 3\sqrt{t+1}\, dt = 14 \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{r}'(t) &= \langle 3\cos t, -5\sin t, 4\cos t\rangle, \quad |\mathbf{r}'(t)| = 5 \\ L &= \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 5\, dt = 5 \end{aligned}$$

27. The torsion τ of a space curve is defined by

$$\tau = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}.$$

Show that

(a) $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$

(b) $\dfrac{d\mathbf{B}}{ds} = \tau\mathbf{N},$

(c) $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B},$

Solution.

(a) Since s is the arclength parameter, we have $|\mathbf{T}'(s)| = \kappa(s)$ and

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|} = \frac{\mathbf{T}'(s)}{\kappa(s)},$$

so

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s).$$

(b) First, observe that, since $\mathbf{B}\cdot\mathbf{B} = 1$, the vector $\mathbf{B}$ is perpendicular to $d\mathbf{B}/ds$. Second, differentiating the equality $\mathbf{B}\cdot\mathbf{T} = 0$, we get

$$0 = \frac{d}{ds}(\mathbf{B}\cdot\mathbf{T}) = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} + \mathbf{B}\cdot\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} + \mathbf{B}\cdot(\kappa\mathbf{N}) = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} + 0.$$

Thus,
$$\frac{d\mathbf{B}}{ds}\cdot\mathbf{T} = 0.$$
We see that $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$. This means it must be parallel to $\mathbf{N}$, i.e
$$\frac{d\mathbf{B}}{ds} = c\mathbf{N}, \quad \text{for some } c.$$
Dot-multiplying this equality by $\mathbf{N}$ we find that
$$-\tau = \frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = c\mathbf{N}\cdot\mathbf{N} = c.$$
Therefore, $c = -\tau$ and
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

(c) For this one, we just use the product rule and the results obtained above. We have
$$\frac{d}{ds}\mathbf{N} = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = (\frac{d}{ds}\mathbf{B})\times\mathbf{T} + \mathbf{B}\times\frac{d}{ds}\mathbf{T} = -\tau\mathbf{N}\times\mathbf{T} + \kappa\mathbf{B}\times\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

28. A surface is described in spherical coordinates by the equation
$$\rho^2(\sin^2\varphi - 16\cos^2\varphi) = 1.$$
Write an equation for this surface in Cartesian coordinates and determine the type of the surface.
Solution. We manipulate the equation into consequtive equvalent forms:
$$\begin{aligned}
\rho^2(\sin^2\varphi - 16\cos^2\varphi) &= 1\\
\rho^2\sin^2\varphi - 16\rho^2\cos^2\varphi &= 1\\
\rho^2\sin^2\varphi(\cos^2\theta + \sin^2\theta) - 16\rho^2\cos^2\varphi &= 1\\
\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta - 16\rho^2\cos^2\varphi &= 1\\
x^2 + y^2 - 16z^2 &= 1
\end{aligned}$$
We see that the surface is a hyperboloid of one sheet.

29. A surface is described in Cartesian coordinates by the equation
$$y^2 + z^2 = 2y.$$
Write an equation for the same surface

(a) in cyllindrical coordinates
(b) in spherical coordinates.

Solution.

(a) In cyllindrical coordinates we have:
$$\begin{aligned}
x &= r\cos\theta\\
y &= r\sin\theta\\
z &= z
\end{aligned}$$
Plugging these into the equation, we get
$$r^2\sin^2\theta + z^2 = 2r\sin\theta.$$

(b) In spherical coordinates we have:

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}$$

Plugging these into the equation, we get

$$\rho^2 \sin^2\varphi \sin^2\theta + \rho^2 \cos^2\varphi = 2\rho \sin\varphi \sin\theta.$$

30. Find the domain and the range of the function:

(**a**) $f(x,y) = \sqrt{\ln(x-y)}$ (**b**) $f(x,y,z) = \ln(64 - 16x^2 - 9y^2 - 25z^2)$.

Solution.

(a) (x,y) is in the domain D_f if and only if $\ln(x-y)$ is defined and non-negative, i.e. when $x - y \geq 1$. Thus,

$$D_f = \{(x,y) \mid x - y \geq 1\}.$$

To obtain the range, observe that the square root can attain only non-negative values. Furthermore, for each $z \geq 0$, $(x,y) := (e^{z^2}, 0)$ gives:

$$f(x,y) = \sqrt{\ln(e^{z^2} - 0)} = \sqrt{z^2} = z.$$

Therefore the range is $[0,\infty)$.

(b)

$$D_f = \{(x,y,z) \mid 16x^2 + 9y^2 + 25z^2 < 64\}.$$

For the range, observe that, with $(x,y,z) \in D_f$, the expression $64 - 16x^2 - 9y^2 - 25z^2$ will be positive, can be as close to 0 as we want, and cannot be greater than 64. Therefore, the values of f will lie between $-\infty$ and $\ln 64$. To verify that $(-\infty, \ln 64]$ is indeed the range, we reason as following. Given $w \in (-\infty, \ln 64]$, choose

$$(x,y,z) := \left(4 - \frac{e^w}{16}, 0, 0\right).$$

Then $(x,y,z) \in D_f$ and $f(x,y,z) = w$.