Department of Mathematics
CARNEGIE MELLON UNIVERSITY
Professor Doytchinov
21-259, Calculus in 3D, Spring 1999

Review for Test 2

For Test 2 you must be familiar with the following topics and be able to solve problems related to them.

- Limits of functions of several variables;
- continuity of functions of several variables;
- partial derivatives;
- Clairaut's Theorem;
- equation of a tangent plane to the surface $z = f(x, y)$;
- differentiability of functions of several variables;
- differential;
- the chain rule;
- implicit differentiation;
- directional derivatives;
- gradient — definition, meaning, geometric interpretation;
- equations of a tangent plane and normal line to the surface $F(x, y, z) = k$;
- minimal and maximal values of functions of one variable, applications;
- local and absolute extrema of functions of several variables, critical points, saddlepoints;
- second dreivative test for functions of two variables.

The test will have 7 problems. Six of the problems will be worth 15 points each and will be pretty similar to the homework problems and the in-class examples. The sixth problem will be worth 10 points and will be a "surprize" problem, i.e., it will require some creativity for its solution.

Below are some sample problems similar to the ones to appear on the test. The problems marked with a * give you an idea of what the "surprize" problems are like. Also, be sure to review all homework problems.

Review Problems

1. Determine the largest set on which the function is continuous:

$$(\mathbf{a}) f(x,y) = \begin{cases} \frac{xy^2}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases} \qquad (\mathbf{b}) f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

2. Find the limit, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$ **(b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x\,y^2+y^3}$

3. Let $f(x,y)=\sin(x+3y)+x^2y^3+2y^2$. Compute f_x, f_y, f_{xx}, f_{xy}, and f_{yy}.

4. * For what values of r is the function

$$f(x,y,z)=\begin{cases} \frac{(x+y+z)^r}{x^2+y^2+z^2}, & \text{if } (x,y,z)\neq(0,0,0), \\ 0, & \text{if } (x,y,z)=(0,0,0). \end{cases}$$

continuous on $\mathbb{R}^3$?

5. Find the tangent plane and normal line to the surface $z=x^2+4y^2$ at the point $(-1,2,17)$.

6. Suppose that $w=x^3y+y^2-4z^3, x=rs, y=(r^2-s^2)^3$, and $z=2r+s$. Use the chain rule to compute $\frac{\partial w}{\partial s}$. Express your answer in terms o r and s alone. Algebraic simplification is not necessary.

7. * For

$$f(x,y,z)=\frac{x}{x^2+y^2}e^{\sin(x^2y)}+\sin(z-2)\ln(x+\sqrt{1+x^2}),$$

compute $f_x(1,0,2)$.

8. Let f be a continuously differentiable function of one variable and define

$$w(x,y):=f\left(\frac{y}{y-x}\right).$$

Show that, for $x\neq y$:

$$x\,\frac{\partial w}{\partial x}+y\,\frac{\partial w}{\partial y}=0.$$

9. * For

$$f(x,y)=5+6(x-3)+8(y+1)+\sin(x-3)^2+(y+1)\ln\sqrt{2+y},$$

write an equaion of the tangent plane to the graph at the point $(3,-1,5)$.

10. Assume that u and v have continuous second partial derivatives and that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

11. The equation $xyz=\cos(x+y+z)$ defines implicitly z as a function of x and y. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

12. Consider the function f defined by $f(x,y,z)=xy+y^2z$.

   (a) Find ∇f.

   (b) Find the directional derivative of f at the point $(2,3,-1)$ in the direction of the vector $\mathbf{v}=\langle 2,4,-3\rangle$.

13. Find the maximum rate of change of f at the given point and the direction in which it occurs.

    (a) $f(x, y) = x^2 e^y - 3y, \quad (1, 0),$

    (b) $f(x, y, z) = \cos(4x - 3y) + z^2, \quad (\pi/16, -\pi/6, 2),$

    (c) $f(x, y) = \sqrt{5x + y^2}, \quad (4, 4).$

14. * Let f be a smooth function with the following property: At the point (1,2,3), the function f increases most rapidly in the direction given by the vector $\langle 5, -3, 0\rangle$ and the rate of change of f in this direction is 17. Find $\nabla f(1, 2, 3)$.

15. * Let f be a smooth function with the following property: At the point (0,0,0), the function f *decreases* most rapidly in the direction of $\langle 6, -1, 2\rangle$ and the rate of change of f in this direction is -5. Find $\nabla f(0, 0, 0)$.

16. * Let $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$ and $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle$. Assume that f_x and f_y are continuous. If $D_{\mathbf{u}}f(2, 1) = 2\sqrt{2}$ and $D_{\mathbf{v}}f(2, 1) = -8\sqrt{2}$, find $\nabla f(2, 1)$.

17. Find the tangent plane and normal line to the surface $z = x^2 + 4y^2$ at the point $(-1, 2, 17)$.

18. Find all the points on the surface $x^2 + 4y^2 - z^2 = 4$ at which the tangent plane is parallel to the plane $x + 2y + z = 6$.

19. Find all the critical points of the given function, and determine which of them are local minima, which are local maxima, and which are saddlepoints.

    (a) $f(x, y) = x^2 + xy + y^2 + x - y + 2,$

    (b) $f(x, y) = 3x^2 - 3xy + y^3 + 3y^2.$

20. Let R be the region described by

$$0 \leq y \leq \sqrt{16 - x^2}, \; 0 \leq x \leq 4.$$

Find the absolute maximum and absolute minimum values of

$$f(x, y) = x^2 + y^2 - 4x - 2y + 1$$

on R.