Department of Mathematics
CARNEGIE MELLON UNIVERSITY
Professor B. Doytchinov
21-259, Calculus in 3D, Spring 1999

Review for Test 1

For Test 1 you must be familiar with the following topics and be able to solve problems related to them.

- coordinates in 3D;
- vectors; addition, subtraction and scalar multiplication of vectors;
- dot product and angles between vectors; projections;
- cross product of vectors in 3D;
- equations of lines and planes;
- equations of spheres;
- collinear and coplanar vectors;
- using cross products to find areas and volumes;
- distance from a plane to a point and distance between planes;
- quadric surfaces;
- parametric curves, arclength and curvature;
- polar, cyllindrical, and spherical coordinates;
- functions of several variables, graphs, level curves and level surfaces.

The test will have 7 problems. Six of the problems will be worth 15 points each and will be pretty similar to the homework problems and the in-class examples. The last problem will be worth 10 points and will be a "surprize" problem, i.e., it will require some creativity for its solution.

Below are some sample problems similar to the ones to appear on the test. The problems marked with a * give you an idea of what the "surprize" problems are like.

Also, be sure to review all homework problems. Any type of problem that appears on the homework, could appear on the test, too.

Review Problems

1. The three points $A(-2,0,1)$, $B(-3,4,3)$, and $C(-5,3,4)$ determine a triangle. Is this a right triangle? Is it isosceles?
2. Write an equation of the sphere with center $C(-6,-3,7)$ and radius $4\sqrt{3}$.
3. A sphere has an equation $3x^2+3y^2+3z^2-12x+6y-36z=120$. Find the center and the radius of the sphere.
4. * Find an equation of the sphere with center $(-3,2,6)$ that touches the xy-plane.

5. For the given vectors $\mathbf{a}$, $\mathbf{b}$, compute $|\mathbf{a}|$, $|\mathbf{b}|$, $2\mathbf{a} - 3\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, $\text{comp}_{\mathbf{a}}\mathbf{b}$, and $\text{proj}_{\mathbf{a}}\mathbf{b}$.

   (a) $\mathbf{a} = \langle 3, 1, 0\rangle, \mathbf{b} = \langle 1, -1, -1\rangle$.

   (b) $\mathbf{a} = \langle 2, 0, -1\rangle, \mathbf{b} = \langle -1, -2, 3\rangle$.

   (c) $\mathbf{a} = \langle 1, 1, 1\rangle, \mathbf{b} = \langle -1, 0, 1\rangle$.

   (d) $\mathbf{a} = \langle 1, -1, 1\rangle, \mathbf{b} = \langle 2, -2, 2\rangle$.

6. Find the cosines of the angles of the triangles with the given vertices:

   (a) $A(1, 1, 0), B(0, 1, 0), C(1, 1, -1)$.

   (b) $A(19, 17, 35), B(43, -1, -5), C(-13, 41, 5)$.

7. * Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. Show that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

8. * Let $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$. Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

9. * If $|\mathbf{a}| = |\mathbf{a} - \mathbf{b}|$, show that $\mathbf{a} \cdot \mathbf{b} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{b}$.

10. * If $\mathbf{c} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all non–zero vectors, show that the angle between $\mathbf{a}$ and $\mathbf{c}$ is the same as the angle between $\mathbf{b}$ and $\mathbf{c}$.

11. Find the area of the parallelogram with vertices $A(0, 1, 0), B(3, 0, -1), C(5, -2, 0)$ and $D(2, -1, 1)$.

12. Find the area of the triangle with vertices $P(1, -1, 2), Q(3, 2, 3), R(0, 1, -1)$.

13. Given $P(0, 1, 2), Q(1, 1, 8), R(2, 4, -6)$, and $S(8, -6, 7)$, find the volume of the parallellepiped with adjacent edges PQ, PR, and PS.

14. * Show that $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

15. * Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three-dimensional vectors such that

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

   and

$$\mathbf{u} \cdot \mathbf{v} = 0,$$

   then

$$\mathbf{w} \times \mathbf{u} = |\mathbf{u}|^2\mathbf{v}.$$

16. * Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three-dimensional vectors such that

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = 0,$$

   then

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

17. Given $A(0, 2, 0), B(2, 0, 2), P(1, 0, 0)$ and $Q(0, 1, 1)$, determine whether the line passing through A and B, and the line passing through P and Q are parallel, skew, or intersecting. If they intersect, find the point of intersection.

18. Given $A(0,1,0), B(1,0,1), P(1,0,0)$ and $Q(0,1,1)$, determine whether the line passing through A and B, and the line passing through P and Q are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. Find an equation of the plane passing through the point $(-6,0,1)$ and parallel to the plane $x-y+3z=9$.

20. Find an equation of a plane passing through the point $(-1,2,7)$ and perpendicular to the line

$$\frac{x-3}{3} = \frac{2-y}{7} = -\frac{z}{2}.$$

21. Consider the planes $3x-4z=12$ and $x-2y-2z=-2$.

    (a) Find the cosine of the angle between the planes.

    (b) Find symmetric equations for the line of intersection of the planes.

22. A plane passes through the points $(0,0,1)$, $(0,2,0)$, and $(3,0,0)$. What is the distance between this plane and the origin?

23. * Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x=1$ and $2x^2+4y^2-2z^2-5y=0$ lies in a plane.

24. Find parametric equations for the tangent line to the curve

$$\mathbf{r}(t) = (2t-1)\mathbf{i} + t^2\mathbf{j} - 3t^3\mathbf{k}$$

at the point $(1,1,-3)$.

25. For each of the following parametric curves, compute the unit tangent vector $\mathbf{T}(t)$, $\frac{d\mathbf{T}}{ds}$, the principal unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$. Here s denotes the natural parameter.

    (a) $\mathbf{r}(t) = \langle e^t\sin(2t), e^t, e^t\cos(2t)\rangle$,

    (b) $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, -4\cos t\rangle$.

26. Find the length of the curve described by:

    (a) $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle, \quad 0 \le t \le \pi$,

    (b) $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 2t^{3/2}\rangle, \quad 0 \le t \le 3$,

    (c) $\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle, \quad 0 \le t \le 1$.

27. The torsion τ of a space curve is defined by

$$\tau = -\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}.$$

Show that

   (a) $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$,

   (b) $\frac{d\mathbf{B}}{ds} = \tau\mathbf{N}$,

   (c) $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$,

28. A surface is described in spherical coordinates by the equation

$$\rho^2(\sin^2\varphi - 16\cos^2\varphi) = 1.$$

Write an equation for this surface in Cartesian coordinates and determine the type of the surface.

29. A surface is described in Cartesian coordinates by the equation

$$y^2 + z^2 = 2y.$$

Write an equation for the same surface

(a) in cyllindrical coordinates

(b) in spherical coordinates.

30. Find the domain and the range of the function:

(a) $f(x, y) = \sqrt{\ln(x - y)}$,

(b) $f(x, y, z) = \ln(64 - 16x^2 - 9y^2 - 25z^2)$.