Pricing and Modeling Credit Derivatives

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August 24, 2006

Abstract

The market involving credit derivatives has become increasingly popular and extremely liquid in the most recent years. The pricing of such instruments offers a myriad of new challenges to the research community as the dimension of credit risk should be explicitly taken into account by a quantitative model. In this paper, we describe a doubly stochastic model with the purpose of pricing and hedging derivatives on securities subject to default risk. The default event is modeled by the first jump of a counting process N_t, doubly stochastic with respect to the Brownian filtration which drives the uncertainty of the level of the underlying state process conditional on no-default event. Assuming absence of arbitrage, we provide all the possible equivalent martingale measures under this setting. In order to illustrate the method, two simple examples are presented: the pricing of defaultable stocks, and a framework to price multi-name credit derivatives like basket defaults.

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1 Introduction

The market for credit derivatives has become increasingly popular and extremely liquid in the most recent years. Credit risk is basically everywhere, in swaps, corporate bonds, Collateralized Debt Obligation (CDO), basket default instruments, sovereign bonds, etc. The necessity of explicitly considering credit risk by making use of quantitative modeling techniques is evident. For instance, Duffie and Singleton (1999)[5] proposed a reduced form model for term structures of defaultable bonds, which is extended by Collin Dufresne at al. (2004)[3], while Hull and White (2004)[8] propose an implementation of Merton's (1974)[13] seminal credit risk model estimated from the implied volatilities of options on the companies equities, which outperform the original model. Hundreds of papers on credit risk modeling, which can be found on websites like "defaultrisk.com" and "gloriamundi.org", suggest this topic as an important issue under consideration.
Credit risk models are usually obtained by one of two concurrent techniques: structural models or reduced form models. In structural models, whose first representant is Merton model (1974)[13], a default is triggered when the process representing the assets of the firm falls bellow a certain barrier value. In reduced form models, whose first formal description appears in Duffie and Singleton (1999)[5][1], the default event is modeled by the first jump of a counting process N_t. Duffie and Singleton propose that N_t should be a doubly stochastic process, which means, a conditionally Poison process, with the uncertainty driving the intensity of the process coming from a sigma-algebra which does not contain information regarding the jumps of this process. In their paper however, all the calculations are obtained under the risk neutral measure, with no allusion to the physical measure.
In this paper, we describe a doubly stochastic model with the purpose of pricing and hedging derivatives on securities subject to default risk. The default event is modeled by the first jump of a counting process N_t, doubly stochastic with respect to a brownian filtration which drives the uncertainty of the volatility process, and of the security price conditional on no-default event. Assuming absence of arbitrage, we provide all the possible equivalent martingale measures under this setting. This allows the implementation of dynamic credit risk models where investors can charge risk premia which takes into account the probability that default events happen. Our main contribution is a clear description of the dynamics of credit default securities under both measures.
We also provide two examples of applications using our methodology: The pricing of defaultable equities (see Bielecki et al. (2004)[1] or Das and Sundaram (2004)[4]), and the pricing of multi-name products, like basket defaults and CDOs (see Duffie and Singleton (2003)[6]). For the multi-name example we propose modeling the intensity of the counting process that drives default as a hybrid between a mean reverting state vector capturing specific firm risks and a function of a market index that would capture common factors risk. We show that the inclusion of this common factor improves the ability of the model in capturing correlation between default times of different companies.
The paper is organized as follows. Section 2 describes the basic framework, provides the theoretical results on changes of measures under our model, and proposes a first simple example

[1]Others have used special cases of this approach before. For instance, Pye (1974)[14] proposes a discrete time precursor version of this model where interest rates, and default intensities are deterministic.

on a defaultable stock. Section 3 considers how well the model is able to capture correlation of default times when dealing with multi-name securities. Section 4 concludes the article. The Appendix present Girsanov's Theorem for counting processes and an extension of the model that includes stochastic volatility for the underlying asset.

2 Pricing Defaultable Securities

Let us fix a Probability space $(\Omega, \mathcal{F}, P)$ and the $\sigma-$ algebra $\mathcal{F}_t = \sigma(W_t^S, Z_t^\lambda)$ where W_t^S, Z_t^λ are independent standard Brownian Motions. Let's also introduce the $\sigma-$ algebra $\mathcal{G}_t = \sigma(\mathcal{F}_t \vee N_t)$ where N_t is a nonexplosive doubly stochastic (with respect to $\mathcal{F}_t$) counting process with intensity λ_t, i.e.

i. λ_t is $\mathcal{F}_t$ predictable and $\int_0^t \lambda_s ds < \infty$ a.s.

ii. $N_t - \int_0^t \lambda_s ds$ is a $\mathcal{G}_t$ local martingale

iii. $P\{N_s - N_t = k|\mathcal{G}_t \vee \mathcal{F}_s\} = \frac{e^{-\int_t^s \lambda_u du}(\int_t^s \lambda_u du)^k}{k!}$

In this section we will price derivatives on a defaultable security, where the price of the security is modeled as a Geometric Brownian Motion with stochastic volatility, and the default event is modeled by the stopping time τ, the first jump of the counting process N_t. We assume that the short term interest rate process is constant and equal to r.[2]. The default intensity process itself is modeled by an OU process. We also introduce correlation between the security price and the intensity brownian motions allowing that changes in prices influence the likelihood of default.

$$\begin{cases} dS_t &= \mu S_t dt + \sigma S_t dW_t^S \\ d\lambda_t &= a(b - \lambda_t)dt + \beta_\lambda dW_t^\lambda \qquad dW_t^\lambda = \rho_\lambda dW_t^S + \sqrt{1-\rho_\lambda^2} dZ_t^\lambda \end{cases} \tag{1}$$

where S is the stock price, σ is the volatility, λ is the instantaneous probability of default of the underlying.
Clearly, the way the problem is set up gives rise to an incomplete market model in the sense that there exist derivatives that can not be hedged by a portfolio of the basic securities. Assumption of no arbitrage guarantees the existence of a set of equivalent martingale measures.[3] In this setting, an EMM P^* is a probability measure equivalent to P, under which the discounted price of the defaultable security[4], $e^{-rt}S_t 1_{\{\tau > t\}}$ is a $\mathcal{G}_t$-martingale. At this point, we look for all possible EMM's P^* that allow us to write the price of a defaultable object as an expectation in terms of the intensity of the counting process N_t under P^*.[5] Let us call the set of all such measures to be $\mathcal{S}$.
In order to characterize all EMM's in the set $\mathcal{S}$, we make use of the two versions of the

[2]This implies that the money market account, the usual instrument adopted for deflation, will be $B_t = e^{rt}$.
[3]non-empty, non-unitary.
[4]Defaultable equity derivatives have been also studied in Bielecki et al. (2004) and Das and Sundaram (2004).
[5]More general versions, useful for instance in markets with multiple defaults, would allow the intensity to depend not only on a Brownian filtration but also on default events.

Girsanov's theorem, where one is for changes in the Brownian Filtration and one for the changes in the intensity process λ_t. In order to construct our argument we state the standard version of Girsanov's theorem for a d-dimensional Brownian filtration[6] and also the Girsanov's theorem version for counting processes.[7]

2.0.1 Girsanov's Theorem (G1):

Given $\theta \in (\mathcal{L}^2)^d$, assume that $\xi_t^\theta = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s \cdot \theta_s ds}$ is a martingale (Novikov's condition is sufficient.) Then a Standard Brownian Motion W^θ is defined by

$$W_t^\theta = W_t + \int_0^t \theta_s ds, \qquad 0 \leq t \leq T$$

Moreover, W^θ has the martingale representation theorem under the new measure P^* where $\frac{dP^*}{dP} = \xi_T^\theta$. Hence, any P^* martingale can be represented as

$$M_t = M_0 + \int_0^t \phi_s dW_s^\theta, \qquad t \leq T$$

for some $\phi \in (\mathcal{L}^2)^d$

2.0.2 Girsanov's Theorem for Counting Processes (G2):

Suppose N_t is a nonexplosive counting process with intensity λ_t, and ϕ is a strictly positive predictable process such that, for some fixed T, $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s)\lambda_s ds} \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t. In addition, if ξ_t^ϕ is a martingale (bounded ϕ suffices), then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t \phi_t$.

Now, suppose the counting process N_t is doubly stochastic with respect to $\mathcal{F}_t$ under measure P^*, say with an intensity λ^*. Then one can show that:

$$E^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\} = E^*\{1_{\{\tau>t\}}1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} \tag{2}$$
$$= 1_{\{\tau>t\}}E^*\{1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} \tag{3}$$
$$= 1_{\{\tau>t\}}e^{-\int_t^s \lambda_u^* du} \tag{4}$$

If we go from P^* to a measure P^{**} making use of G1, then one can prove that there is an equivalence between the discounted defaultable price being a $\mathcal{G}_t$ $P^{**}-$martingale and the

[6]For the proof see Karatzas and Shreve (1991)[11].
[7]For the proof see the Appendix.

process $e^{\int_0^t (r+\lambda_u^*)du} S_t$ being a $\mathcal{G}_t$ P^{**}-martingale as follows

$$\begin{aligned}
E^{**}\{e^{-rs}S_s 1_{\{\tau>s\}}|\mathcal{G}_t\} &= E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s 1_{\{\tau>s\}}|\mathcal{G}_t\} \\
&= E^*\{E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s 1_{\{\tau>s\}}|\mathcal{G}_t\}|\mathcal{G}_t \vee \mathcal{F}_s\} \\
&= E^*\{E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s 1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\} \\
&= E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s E^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\} \\
&= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^*\{\frac{\xi_s}{\xi_t}e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\} \\
&= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^{**}\{e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\}
\end{aligned}$$

where we used Bayes rule and Equation (2,3,4).
How do we characterize the elements in $\mathcal{S}$? The idea is to realize a two step change of measure, where we first change the intensity of the counting process going from P to P^*, using a particular case of G2 where N_t counts up to 1. Then, we apply G1 changing the measure from P^* to P^{**}. The Radon-Nikodym derivative from P to P^{**} is, by construction, the product of the two Radon-Nikodym derivatives from P to P^* and from P^* to P^{**}. By theorems G1 and G2

$$\frac{dP^*}{dP} = e^{\int_0^t (1-\phi_s)\lambda_s ds}(1_{\{\tau>t\}} + \phi(\tau)1_{\{\tau\le t\}}) \quad \text{and} \quad \frac{dP^{**}}{dP^*} = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s\cdot\theta_s ds}$$

In general, ϕ and θ are free parameters, but for simplicity we assume that ϕ is deterministic and to guarantee that the process $e^{-rt}1_{\{\tau>t\}}S_t$ is a $\mathcal{G}_t$ martingale, we choose the θ as follows:

$$\theta_t = \begin{bmatrix} \frac{\mu-r-\lambda_t^*}{\sigma} \\ \delta_t \end{bmatrix}$$

where the parameter δ is free. The Radon-Nikodym Derivative of the new measure becomes

$$\frac{dP^{**}}{dP}|_{F_t} \quad = e^{-\int_0^t \theta_1(u)dW_u^S - \frac{1}{2}\int_0^t \theta_1^2(u)du} \quad e^{-\int_0^t \theta_2(u)dZ_u^\lambda - \frac{1}{2}\int_0^t \theta_2^2(u)du}e^{\int_0^t (1-\phi_u)\lambda_u du}(1_{\{\tau>t\}} + \phi(\tau)1_{\{\tau\le t\}})$$

where
$\theta_1 = \frac{\mu-r-\lambda_t^*}{f(Y_t)}$
$\theta_2 = \delta_t$
and System (1) becomes:

$$\left\{\begin{array}{lcl} dS_t &=& (r+\lambda_t^*)S_t dt + \sigma S_t dW_t^{*S} \\ d\lambda_t^* &=& [(a-\frac{\phi_t'}{\phi_t})(\frac{ab\phi_t}{a-\frac{\phi_t'}{\phi_t}} - \lambda_t^*) - \beta_\lambda\phi_t(\rho_\lambda\frac{\mu-r-\lambda_t^*}{f(Y_t)} + \delta_t\sqrt{1-\rho_\lambda^2})]dt + \beta_\lambda\phi_t dW_t^{*\lambda} \end{array}\right. \tag{5}$$

Let's introduce the 2 dimensional vector process X_t as follows.

$X_t = \begin{bmatrix} S_t \\ \lambda_t^* \end{bmatrix}$. Then the process followed by X_t can be written as:

$$dX_t = \begin{bmatrix} r + \lambda_t^* \\ (a - \frac{\phi_t'}{\phi_t})(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^*) - \beta_\lambda \phi_t(\rho_\lambda \frac{\mu - r - \lambda_t^*}{\sigma} + \delta_t \sqrt{1 - \rho_\lambda^2}) \end{bmatrix} dt$$
$$+ \begin{bmatrix} \sigma S_t & 0 \\ \phi_t \beta_\lambda \rho_\lambda & \phi_t \sqrt{1 - \rho_\lambda^2} \beta_\lambda \end{bmatrix} \begin{bmatrix} dW_t^{*S} \\ dZ_t^{*\lambda} \end{bmatrix}$$

directly implying that the Feynman-Kac PDE for the function $P(x,t) = E\{e^{-\int_0^t (r+\lambda_u^*)du} g(X_t)\}$,where $g(X_t) = h(S_t)$, is

$$P_t + (r + \lambda_t^*)P_S + [(a - \frac{\phi_t'}{\phi_t})(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^*) - \beta_\lambda \phi_t(\rho_\lambda \frac{\mu - r - \lambda_t^*}{\sigma} + \delta_t \sqrt{1 - \rho_\lambda^2})]P_{\lambda^*}$$
$$\frac{1}{2}\sigma_t^2 S_t^2 P_{SS} + \frac{1}{2}\phi_t^2 \beta_\lambda^2 P_{\lambda^* \lambda^*} + \sigma S_t \phi_t \rho_\lambda \beta_\lambda P_{S\lambda^*}$$

with the boundary condition $P(X_T, T) = h(S_T)$

2.1 First Example: Poisson process with constant intensity

Let S_t be a defaultable security which follows a Geometric Brownian Motion with constant mean and volatility parameters. The counting process modeling default events has a constant intensity λ, i.e.

i. $dS_t = \mu S_t dt + \sigma S_t dW_t^S$

ii. $N_t \sim$ Poisson(λt)

Define $\mathcal{F}_t = \sigma\{W_t^S\}$ and $\mathcal{G}_t = \sigma\{W_t^S, N_t\}$. We are interested in pricing a derivative based on S_t with a maturity date T. We would like to find an equivalent measure P^{**} under which the process $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a $\mathcal{G}_t$ martingale. In the spirit of the argument for the general change of measure which appears in the previous section, we can change the measure in two steps. First, Girsanov's theorem for counting processes guarantees that if we take ϕ constant and realize the change of measure $\frac{dP^*}{dP} = e^{\int_0^T (1-\phi)\lambda du}(1_{\{\tau > T\}} + \phi 1_{\{\tau \leq T\}})$, N_t is a Poisson process under P^* with intensity $\lambda^* = \phi\lambda$.
Now, we want to make a second change of measure from P^* to P^{**} just changing the brownian filtration in a way that $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a $\mathcal{G}_t$ martingale under P^{**}. We know that this change will not affect the characteristics of the process N_t which will consequently be Poisson under P^{**} with intensity $\lambda^* = \phi\lambda$. By the previous section, the general change of measure is

$$\frac{dP^{**}}{dP} = e^{-\int_0^T \frac{\mu - r - \lambda_t^*}{\sigma} dW_u^S - \frac{1}{2}\int_0^T (\frac{\mu - r - \lambda_t^*}{\sigma})^2 du} e^{\int_0^T (1-\phi)\lambda du}(1_{\{\tau > T\}} + \phi 1_{\{\tau \leq T\}})$$

Finally, it is interesting to note that, under this simple model, the problem of pricing a derivative on the defaultable stock boils down to pricing a derivative on the non-defaultable stock with default adjusted parameters that represent a spread in interest rates:

$$dS_t = (r + \lambda_t^*)S_t dt + \sigma S_t dW_t^{*S}$$

2.2 Second Example: Poisson process with random intensity but constant risk premium

Define the following $\sigma-$algebras $\mathcal{F}_t = \sigma\{W_t^S, Z_t^\lambda\}$ and $\mathcal{G}_t = \sigma\{W_t^S, Z_t^\lambda, N_t\}$, where W_t^S and Z_t^λ are two independent Brownian Motions, and N_t is a counting process, doubly stochastic with respect to $\mathcal{F}_t$. Let S_t be a defaultable stock which follows a Geometric Brownian Motion with constant mean and volatility parameters. The default process is modeled by the first jump of the counting process N_t. The stochastic intensity λ follows an OU process driven by a Brownian Motion that is correlated with the Brownian Motion which drives the dynamics of the stock price. The model is described by the following equations:

i. $dS_t = \mu S_t dt + \sigma S_t dW_t^S$

ii. $d\lambda_t = a(b - \lambda_t)dt + \beta_\lambda dW_t^\lambda, \quad dW_t^\lambda = \rho_\lambda dW_t^S + \sqrt{1-\rho_\lambda^2}dZ_t^\lambda$

iii. $N_s - N_t|\mathcal{G}_t \vee \mathcal{F}_s \sim \text{Poisson}(\int_t^s \lambda_u du)$

We would like to find a general measure P^{**} equivalent to P, under which the process $Y_t \equiv e^{-rt}S_t 1_{\{\tau>t\}}$ is a $\mathcal{G}_t$ martingale. Using the same idea of the previous example, we first apply Girsanov's theorem for counting processes taking ϕ constant and realize the change of measure $\frac{dP^*}{dP} = e^{\int_0^T (1-\phi)\lambda du}(1_{\{\tau \geq T\}} + \phi 1_{\{\tau \leq T\}})$. Again, we want to make a second change of measure from P^* to P^{**} just changing the brownian filtration in a way that $Y_t \equiv e^{-rt}S_t 1_{\{\tau>t\}}$ is a $\mathcal{G}_t$ martingale under P^{**}.
Then, the general change of measure is

$$\frac{dP^{**}}{dP} = e^{-\int_0^T \frac{\mu-r-\lambda_u^*}{\sigma}dW_u^S - \frac{1}{2}\int_0^T (\frac{\mu-r-\lambda_u^*}{\sigma})^2 du} e^{-\int_0^T \delta_u dW_u^\lambda - \frac{1}{2}\int_0^T \delta_u^2 du} e^{\int_0^T (1-\phi)\lambda_u^* du}(1_{\{\tau>T\}} + \phi 1_{\{\tau\leq T\}})$$

The equations for the security price and intensity processes under P^{**} are given by:

$$\begin{cases} dS_t &= (r + \lambda_t^*)S_t dt + \sigma_t S_t dW_t^{**S} \\ d\lambda_t^* &= [a(\phi b - \lambda_t^*) - \phi\beta_\lambda(\rho_\lambda \frac{(\mu-r-\lambda_t^*)}{\sigma} + \sqrt{1-\rho_\lambda^2}\delta_t)]dt + \phi\beta_\lambda dW_t^{**\lambda} \end{cases} \tag{6}$$

Let's assume, for instance, that $\rho_\lambda = 0$ and that δ is an affine function of λ_t^*. In particular, $\delta_t = m\lambda_t^*$. Then the equation for the intensity process takes the form of

$$d\lambda_t^* = (a + \beta_\lambda m)(\frac{\phi ba}{a + \beta_\lambda m} - \lambda_t^*)dt + \phi\beta_\lambda dW_t^{*\lambda}$$

Observe that, ϕ and δ have different effects on the intensity process. While ϕ only effects the long term mean of the process, δ also effects the speed of mean reversion.

3 Modeling the Default Correlation in Multi-Name Products

In today's financial markets there are lots of multi-name products whose pricing is critically dependent on the correlation of defaults of these different names. Basket default swaps,

CDO's, CBO's are such examples. In this section we try to develop a model to price this kind of products paying particular attention to the default correlations. We try to combine the capital structure models and reduced form models by modeling the default intensities of different names as both a function of the overall market and a function of its individual structure. Modeling the effect of overall market is done through a proxy like a big common index, e.g. S&P 500 and the effect of individual structure is like a surprise default.
Fair amount of this section is devoted to explain the ability of the model in capturing the high levels of correlation between defaults of different names. Since there are various definitions for default correlation in the literature, we would like to clarify which definition we will use throughout the section. The default correlation of two names, say S_1 and S_2 is defined with respect to their exact default times. say τ_1 and τ_2 as follows:

$$\begin{cases} \rho & = \frac{cov(\tau_1,\tau_2)}{\sqrt{var(\tau_1)var(\tau_2)}} \\ & = \frac{E\{\tau_1\tau_2\}-E\{\tau_1\}E\{\tau_2\}}{\sqrt{var(\tau_1)var(\tau_2)}} \end{cases} \tag{7}$$

Hereafter we simply call this definition to be the default correlation. This kind of default correlation is a much more general concept than that of the discrete default correlation based on a one period, i.e. the correlation between defaults of two different names occurred or not in a certain period. Certainly, if the joint distributions of default times are known we could calculate quantities like $E\{1_{\tau_1<T}1_{\tau_2<T}\}$ or $E\{1_{\tau_1<T}\}$ and therefore calculate the discrete default correlation using the methodology described in Lucas(1995)[12]. However, even if we know the discrete correlation we cannot calculate the default correlation in the above sense.
The straight forward intuition behind the setting is the following: When the overall market is not doing well, the default probability of each name tend to go up together, not necessarily with the same rate. Or there could be something happening not in the whole market but in a specific sector which would bump up all the default probabilities of names in that sector. In addition to that, there could be also something happening within a firm which would only effect that particular firm but not the others. So it is natural to assume the default probabilities (intensities) have two different components, one for the overall market effect and one for the individual firm effect.
In the typical setting of the model, the proxy used to capture the overall market impact is modeled as a Geometric Brownian Motion. All the default intensities are modeled as product of a state process, which is an OU process with appropriate parameters and a positive function of the index level above. The Brownian Motions that drive the dynamics of all the state processes that effect the intensities are correlated with each other. One can, in general, introduce the correlation between the Brownian Motions of the index level and state processes but we rather capture that effect in the specific form that we choose for the intensity processes.
Now suppose we try to price a product that depends on N different names. Then the SDE's that describe the event look as follows:

$$\begin{cases} dS_t & = \mu S_t dt + \sigma_t S dW_t^S \\ \lambda_t^i & = X_t^i g_i(S_t, K^i) \qquad \text{for} \quad i = 1,2,...,N \\ dX_t^i & = a^i(b^i - X_t^i)dt + \beta_X^i dW_t^i \qquad E\{dW_t^i dW_t^j\} = \rho_{ij}^X dt \end{cases} \tag{8}$$

where S is the common index level, σ is the volatility of the index level, λ^i is the instantaneous probability of default of name i. X is the OU state process whose each component affects each specific intensity process (capturing the firm specific risk), and the function g is some power function which blows up at a certain fixed boundary level K^i. Although one can keep the function g general for the rest of the section we will assume that $g(S_t, K) = (\frac{S_t - K}{S_0 - K})^n$ where $S_0 - K$ is a normalization factor and S_0 is the initial value of the index in the period of interest.

Although there seems to be a lot of parameters in the general setting, as far as the correlation of default times are concerned there are just a few key parameters. The most important one is the actual exponent in the function g. Clearly a positive power corresponds to a positive correlation and negative power corresponds to a negative correlation between the market and the default intensity of the individual name. If we assume same type of g, i.e. the same power and the same boundary level for two different names and keeping the other variables fixed, we observe that the correlation is almost a linear function of the square root of this power parameter (See Figure 1)[8]. The difference between the two pictures is the level of the volatility of the common index process. So the first one corresponds to a market with high volatility and the second one with a low volatility. As we can see from the figures above

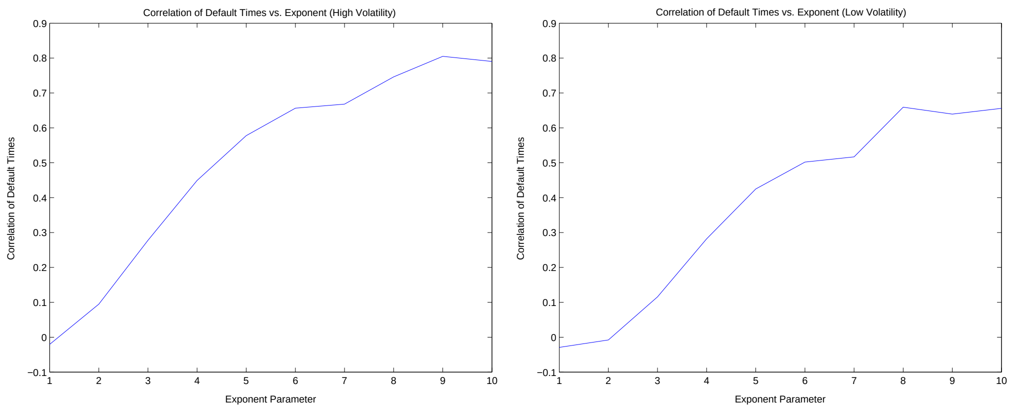

Figure 1: Correlation effect of the parameter in the exponent of the function g, under low volatile and high volatile environments

through this model we get correlation between default times up to 90%. Here the correlation is defined in the classical sense. In the market, it is also of interest the correlation between consecutive defaults, i.e. the defaults that happened within the same year. However, one gets just similar results for that definition of correlation too.

At this point we see that in the above pair of figures although each of them are almost straight lines, the slopes of those lines are different. This means that the level of correlation introduced by the specific form of the function g creates different effects in different regimes in the market. Note that, besides the exponent and the volatility parameters, another very effective parameter is the explosion boundary K in the function g. We call it explosion

[8]At this point all our observations are based on simulated data with parameters chosen from the published literature (see Duffie and Singleton (2003)).

boundary because once the value of the common index gets close to this level, it increases all the intensities by incredible amount and we get simultaneous defaults. Also the more we are further away from this level, the smaller the intensities are, i.e. when the market is doing well all the default intensities tend to go lower. And the closeness of this level to the index level is basically the sensitivity of the individual to the overall market. But in order to create a uniform effect of this exponent under different regimes of the process S_t, we also define this explosion boundary as a function of S_0 and σ and we let $K = S_0 - L * \sigma$. Then if we generate the first two pictures with this new definition of the boundary we observe the same level of correlation effect under both regimes, as shown in Figure 2. Regarding

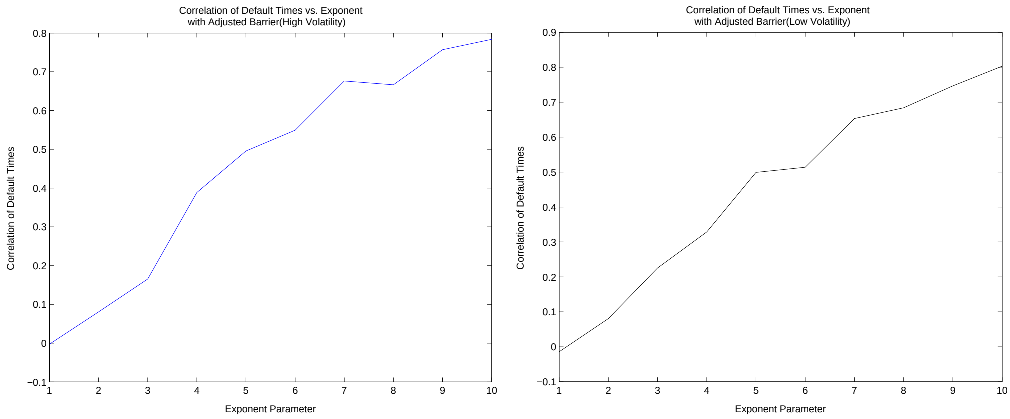

Figure 2: Correlation effect of the parameter in the exponent of the function g, without the effect of the volatility of the index process

the other set of parameters that could possibly effect the default time correlation, that is, the correlation parameters ρ_{ij}^X of the Brownian Motions driving the state process X or the mean reverting level of the state processes, we experimented the same phenomena under different sets of values of all those parameters: We observe that for all possible values of these parameters we get the same effect on the correlation of the default times. For instance, consider a first-to-default instrument. It is a contingent claim that pays off \$1 at the time where the first from a set of n names defaults. Figure 3 shows the effect of the correlation parameters between the state variables X_is on the premium of a First-to-Default insurance contract. As it is clear from the picture, there is almost no effect of the correlation parameter for Brownian motions on the price of the first-to-default instrument. This is an indication that the correlation parameter for the Brownian motions is not generating enough correlation between the different default intensity processes, and therefore is not affecting the correlation of default times, otherwise the price would be sensitive to it. Why? Intuitively we have an idea of the effect of the correlation of default intensities on the price of this simple derivative. Namely, if we have n perfectly uncorrelated default intensities then the intensity of the first-to-default event is the sum of all intensities. On the other hand if they are all perfectly correlated then having an insurance against the first-to-default or against any one of them would be the same (if we assume that each name has the same default intensity). Therefore the price of the contract should be much less in the case of the perfectly correlated case then the uncorrelated case. By a similar argument, one can convince himself that actually the

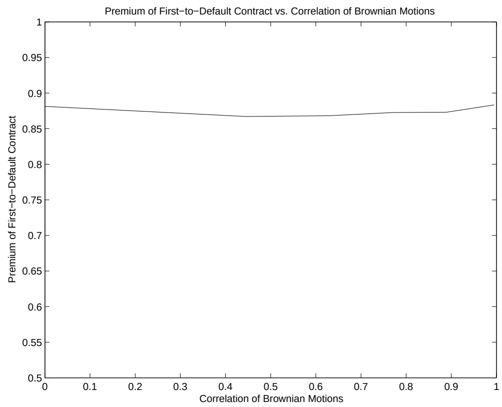

Figure 3: Correlation between Brownian Motions vs. the premium of a First-to-Default contract

price of this contract is a decreasing function of the correlation of the default times. Hence in our model a decreasing function of the square root of the exponent parameter. Figure 4 shows the impact of the exponent on the price of the first-to-default contract. Note that the price is clearly a decreasing function of the exponent, except for a small region where the exponent assumes values around 0.4. This indicates that we can generate correlation on default times using our common factor captured by the market index.

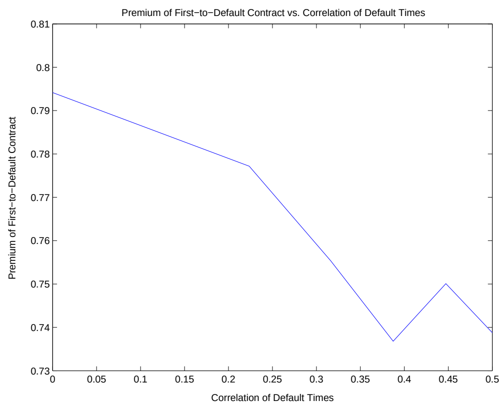

Figure 4: Correlation of Default Times vs. the premium of a First-to-Default contract

4 Conclusion

Adopting a doubly stochastic counting process to represent default events, we present theoretical results on the construction of equivalent martingale measures for defaultable securities. Examples are provided involving defaultable equities, and multi-name products. In order to price multi-name products, we propose the intensity of the counting process to be a combined function of both a mean reverting state vector representing firm-specific risk and

a function of a common market index. We show that it is possible to generate correlation between default times of the different instruments that compose the basket by controlling one specific parameter of the common factor function (the exponent).
In the general theoretical model, two sources contribute to changes in the intensity λ of the counting process from the physical to the risk neutral measure: the Brownian filtration which influences the dynamics of λ indirectly changing its behavior, and a possible deterministic shift ϕ that re-scales λ. These results might be applied on econometric studies of defaultable claims to determine the price of credit risk charged by investors, and in particular, if investors price credit risk sharply (significant ϕ) or more smoothly (significant change in the drift of Brownian motion driving λ).

5 APPENDIX

5.1 Girsanov's Theorem for Counting Processes

Claim 1: Suppose N_t is a nonexplosive counting process with intensity λ_t, and ϕ_t is a strictly positive predictable process such that, for some fixed T, $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s)\lambda_s ds} \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t.

Proof : Define

$$X_t = e^{\int_0^t (1-\phi_s)\lambda_s ds} \quad \text{and} \quad Y_t = \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)} \quad \text{and} \quad M_t = N_t - \lambda_t dt$$

Then

i. $\xi_t^\phi = X_t Y_t$

ii. M_t is a local martingale

iii. $dX_t = (1-\phi_t)\lambda_t e^{\int_0^t (1-\phi_s)\lambda_s ds} dt = (1-\phi_t)\lambda_t X_t dt$

iv. $dY_t = (\prod_{\{i:\tau(i)<t\}} \phi_{\tau(i)})(\phi_t - 1)dN_t = Y_t(\phi_t - 1)dN_t$

v. $dM_t = dN_t - \lambda_t dt$

By the above five facts and general Ito formula with jumps, ξ_t^ϕ is calculated as:

$$\begin{aligned}
d\xi_t^\phi &= d(X_t Y_t) \\
&= dX_t Y_{t-} + X_{t-} dY_t + \Delta X_t \Delta Y_t \\
&= (1-\phi_t)\lambda_t X_t Y_{t-} dt + X_{t-} Y_{t-}(\phi_t - 1)dN_t \\
&= (1-\phi_t)\lambda_t \xi_{t-}^\phi + (\phi_t - 1)\xi_{t-}^\phi dN_t \\
&= (1-\phi_t)\lambda_t \xi_{t-}^\phi + (\phi_t - 1)\xi_{t-}^\phi (dM_t + \lambda_t dt) \\
&= (\phi_t - 1)\xi_{t-}^\phi dM_t
\end{aligned}$$

In the third equation we made use of the fact that X_t is a continuous process which implies $\Delta X_t = 0$. Since M_t is a local martingale, we know that an integral against a local martingale is also a local martingale under certain regularity conditions (see Bremaud (1981)[2]) for the integrand $\diamond$

Claim 2: If ξ_t^ϕ is a martingale, then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t\phi_t$.

Proof : To say that N_t is counting process with intensity $\lambda_t\phi_t$ what we need to show is $A_t = N_t - \int_0^t \lambda_s\phi_s$ is P^* local martingale where $\frac{dP^*}{dP} = \xi_T^\phi$. Or equivalently we can show that the process $Z_t = \xi_t^\phi A_t$ is a P local martingale.
By the first claim

$$dA_t = dN_t - \lambda_t\phi_t dt \quad \text{and} \quad d\xi_t^\phi = (\phi_t - 1)\xi_{t-}dM_t$$

Then by the Ito's formula with jumps we get

$$\begin{aligned}
dZ_t &= d\xi_t^\phi A_{t-} + \xi_{t-}^\phi dA_t + \Delta\xi_t^\phi \Delta A_t \\
&= (\phi_t - 1)\xi_{t-}^\phi A_{t-}dM_t + \xi_{t-}^\phi(dN_t - \lambda_t\phi_t dt) + (\phi_t - 1)\xi_{t-}dN_t dN_t \\
&= (\phi_t - 1)\xi_{t-}^\phi A_{t-}dM_t - \xi_{t-}^\phi\lambda_t\phi_t dt + \phi_t\xi_{t-}dN_t \\
&= (\phi_t - 1)\xi_{t-}^\phi A_{t-}dM_t - \xi_{t-}^\phi\lambda_t\phi_t dt + \phi_t\xi_{t-}(dM_t + \lambda_t dt) \\
&= [(\phi_t - 1)\xi_{t-}^\phi A_{t-} + \phi_t\xi_{t-}]dM_t
\end{aligned}$$

Hence Z_t can be written as an integral against a local martingale, which would imply Z_t itself is a P local martingale. Therefore, A_t is a P^* local martingale and therefore N_t is a counting process with intensity $\lambda_t\phi_t$ under the new measure P^* $\diamond$

5.2 Extension, Index Process with Stochastic Volatility

Let us consider a more generalized version of the same problem where intensity rate process depends upon the process of the underlying, and the underlying dynamics present stochastic volatility. The stochastic volatility process is defined as a positive bounded function of a OU process as proposed in Fouque et al. (2000)[9]. Let us assume that $\lambda_t = g(X_t, S_t)$ where the function g has a certain form, but kept general for now, and X_t is some state process. Then, our SDE system for the prices will be

$$\begin{cases}
dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S \\
\sigma_t &= f(Y_t) \\
dY_t &= \alpha(m - Y_t)dt + \beta_\sigma dW_t^\sigma \qquad dW_t^\sigma = \rho_\sigma dW_t^S + \sqrt{1-\rho_\sigma^2}dZ_t^\sigma \\
dX_t &= a(b - X_t)dt + \beta_\lambda dW_t^\lambda \qquad dW_t^\lambda = \rho_\lambda dW_t^S + \sqrt{1-\rho_\lambda^2}dZ_t^\lambda \\
\lambda_t &= g(X_t, S_t)
\end{cases} \tag{9}$$

We would like to find a measure P^* under which the process $e^{-rt}S_t 1_{\{\tau>t\}}$ is a $\mathcal{G}_t$-martingale. As we showed earlier this is equivalent to have the process $e^{-\int_0^t (r+\lambda_u^*)du}S_t$ a $\mathcal{G}_t$-martingale,

where λ_t^* is the intensity process under the measure P^*.
Using this and the two step change of measure described in the notes we obtain the system under consideration under P^* using the following change in the Brownian filtration

$$W_t^* = W_t + \int_0^t \theta_u du$$

where

$$W_t = \begin{bmatrix} W_t^S \\ Z_t^\sigma \\ Z_t^\lambda \end{bmatrix} \qquad \text{and} \qquad \theta_t = \begin{bmatrix} \frac{\mu - r - \phi_t g(X_t, S_t)}{f(Y_t)} \\ \gamma_t \\ \delta_t \end{bmatrix}$$

and the parameters γ, δ and ϕ are free.
Finally the system becomes:

$$\left\{ \begin{array}{rcl} dS_t & = & (r + \phi_t g(X_t, S_t))S_t dt + \sigma_t S_t dW_t^{*S} \\ \sigma_t & = & f(Y_t) \\ dY_t & = & [\alpha(m - Y_t) - \beta_\sigma(\rho_\sigma \frac{\mu - r - \lambda_t^*}{f(Y_t)} + \gamma_t \sqrt{1 - \rho_\sigma^2})]dt + \beta_\sigma dW_t^{*\sigma} \\ dX_t & = & a(b - X_t) - \beta_\lambda(\rho_\lambda \frac{\mu_t - r - \phi_t g(X_t, S_t)}{f(Y_t)} + \delta_t \sqrt{1 - \rho_\lambda^2})dt + \beta_\lambda dW_t^{*\lambda} \end{array} \right. \tag{10}$$

6 Acknowledgements

The second author is grateful for financial support from a Post-doctorate fellowship granted by the Department of Mathematics at Stanford University, during the development of this paper.

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