A Density Result for GSBD and its Application to the Approximation of Fracture Energies

Flaviana Iurlano - iurlano@sissa.it
SISSA - International School for Advanced Studies, Trieste, Italy

SCUOLA INTERNAZIONALE SUPERIORE di STUDI AVANZATI
International School for Advanced Studies

Why GSBD?

The classical mathematical formulation of many problems arising from Fracture Mechanics requires the minimization in the space $SBD(\Omega)$ of functionals whose main energy term is given by

$$\int_\Omega Q(e(u))dx + \mathcal{H}^{n-1}(J_u), \tag{1}$$

where Q is a positive definite quadratic form. Through the Compactness Theorem in SBD by Bellettini, Coscia, and Dal Maso, we are able to guarantee the existence of solutions to such minimum problems **assuming an *a priori* L^∞-bound** for minimizing sequences (see [2]). Regrettably, this requirement in general is not satisfied in spite of the presence of L^∞-boundary data, since the maximum principle does not hold for these vector valued problems. For the same reason we can approximate (1) by integral functionals depending on Sobolev functions only if we assume an *a priori* bound on the L^∞-norm of the functions, as proved by Chambolle in [3, 4].

The results we present in this poster show that, rather than $SBD(\Omega)$, the proper space where to formulate, solve, and approximate minimum problems related to (1) is the space $GSBD(\Omega)$ of ***Generalised Special functions of Bounded Deformation***, introduced by Dal Maso in [6].

Few preliminaries: Definition of GBD

Definition. *The space $GBD(\Omega)$ is the space of all $\mathcal{L}^n$-measurable functions $u\colon \Omega \to \mathbb{R}^n$ with the following property:*

there exists $\lambda \in \mathcal{M}_b^+(\Omega)$ such that for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the slice $\hat{u}_y^\xi := u_y^\xi \cdot \xi$ belongs to $BV_{loc}(\Omega_y^\xi)$ and

$$\int_{\Pi^\xi} \Big(|D\hat{u}_y^\xi|(B_y^\xi \setminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi}) \Big) d\mathcal{H}^{n-1}(y) \le \lambda(B)$$

holds for every Borel set $B \subset \Omega$, where we have set
$$J^1_{\hat{u}_y^\xi} := \{t \in J_{\hat{u}_y^\xi} : |[\hat{u}_y^\xi]|(t) \ge 1\}.$$

The space $GSBD(\Omega)$ is the space of all functions $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}_y^\xi := u_y^\xi \cdot \xi$ belongs to $SBV_{loc}(\Omega_y^\xi)$.

Few preliminaries: Fine properties of GBD

Structure Theorem. *Let $u \in GBD(\Omega)$, let $\xi \in \mathbb{S}^{n-1}$, and let $J_u^\xi := \{x \in J_u : [u](x) \cdot \xi \neq 0\}$. Then*

1) *the approximate jump set J_u is $(\mathcal{H}^{n-1}, n-1)$-rectifiable and*
$$(J_u^\xi)_y^\xi = J_{\hat{u}_y^\xi}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi;$$
2) *the approximate limits u^+ and u^- from the two sides of J_u are $\mathcal{H}^{n-1}$-measurable functions defined $\mathcal{H}^{n-1}$-a.e. on J_u; moreover for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and for every $t \in (J_u^\xi)_y^\xi$, we have*
$$u^\pm(y + t\xi) \cdot \xi = (\hat{u}_y^\xi)^\pm(t),$$
where the normals to J_u and $J_{\hat{u}_y^\xi}$ are oriented so that $\xi \cdot \nu_u \ge 0$ and $\nu_{\hat{u}_y^\xi} = 1$;
3) *there exists the approximate symmetric gradient $e(u) \in L^1(\Omega, \mathbb{M}^{n\times n}_{sym})$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have*
$$e(u)_y^\xi \xi \cdot \xi = \nabla \hat{u}_y^\xi, \quad \mathcal{L}^1\text{-a.e. in } \Omega_y^\xi.$$

Few preliminaries: A compactness theorem

Compactness Theorem. *Let $(u_k) \subset GSBD(\Omega)$. Suppose that there exist $M < +\infty$ and two increasing continuous functions $\psi_0, \psi_1\colon \mathbb{R}^+ \to \mathbb{R}^+$, with $\psi_0(s) \to +\infty$ and $\psi_1(s)/s \to +\infty$ as $s \to +\infty$, such that*

$$\int_\Omega \psi_0(|u_k|)\,dx + \int_\Omega \psi_1(|e(u_k)|)\,dx + \mathcal{H}^{n-1}(J_{u_k}) \le M$$

for every k. Then there exist a subsequence, still denoted by u_k, and a function $u \in GSBD(\Omega)$, such that

$$u_k \to u \quad \text{pointwise } \mathcal{L}^n\text{-a.e. on } \Omega,$$
$$e(u_k) \rightharpoonup e(u) \quad \text{weakly in } L^1(\Omega, \mathbb{M}^{n\times n}_{sym}),$$
$$\mathcal{H}^{n-1}(J_u) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}).$$

The main theorem: a density result for GSBD

The main result we present is the following theorem about the approximation of $GSBD \cap L^2$ functions whose approximate symmetric gradient is square integrable and whose approximate jump set has finite $\mathcal{H}^{n-1}$-measure. The proof, for which we refer to [10], follows the lines of [3, 4]. Briefly, for every function u as above, we first provide a **unified estimate** for the bulk and the surface energy (**with a bad constant**) using an interpolation argument. After that we are able to **correct the coefficient** in such estimate through the Besicovitch covering theorem. To **separate the estimates** eventually we use the compactness theorem introduced above. Before stating the precise result, let us give a preliminary definition.

Definition. *Let us define*

$$GSBD^2(\Omega) := \{u \in GSBD(\Omega) : e(u) \in L^2(\Omega, \mathbb{M}^{n\times n}_{sym}) \text{ and } \mathcal{H}^{n-1}(J_u) < \infty\}.$$

Theorem 1. *Let us assume that Ω has Lipschitz boundary. Let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$. Then there exist a sequence $u_k \in SBV^2(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ such that each J_{u_k} is contained in the union S_k of a finite number of closed connected pieces of **C^1-hypersurfaces**, each u_k belongs to $W^{1,\infty}(\Omega \setminus S_k, \mathbb{R}^n)$, and the following properties hold:*

1) $||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,
2) $||e(u_k) - e(u)||_{L^2(\Omega,\mathbb{M}^{n\times n}_{sym})} \to 0$,
3) $\mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \to 0$,
4) $\displaystyle\int_{J_{u_k}\cup J_u} |u_k^\pm - u^\pm| \wedge 1 \, d\mathcal{H}^{n-1} \to 0.$

Remark. *Combining the previous result with [5], we are able to produce a sequence of approximating functions with the property that each jump set is contained in a finite union of **$(n-1)$-dimensional closed simplexes**.*

Application 1: the Ambrosio-Tortorelli convergence in the vectorial case

Through the previous density result and an intermediate Γ-convergence step, we are able to extend the Ambrosio-Tortorelli approximation [1] to the vectorial case.

Theorem 2. *Assume that Ω has Lipschitz boundary. Let $g \in L^2(\Omega, \mathbb{R}^n)$ and let $\epsilon_k, \eta_k > 0$ be such that*

$$\epsilon_k, \eta_k \to 0 \quad \text{and} \quad \eta_k/\epsilon_k \to 0.$$

For every k, let (u_k, v_k) be a minimizer of the problem

$$\min_{(u,v)\in H^1\times V_{\eta_k}} \int_\Omega \Big(Q(v, e(u)) + \frac{(1-v)^2}{\epsilon_k} + \epsilon_k |\nabla v|^2 + |u-g|^2 \Big) dx, \tag{2}$$

where V_{η_k} is defined by

$$V_{\eta_k} := \{v \in H^1(\Omega) : \eta_k \le v \le 1 \ \mathcal{L}^n\text{-a.e. in } \Omega\},$$

and $Q\colon \mathbb{R}\times\mathbb{M}^{n\times n} \to \mathbb{R}$ is lower semicontinuous and satisfies the following properties:

1) *for every $s \in \mathbb{R}$, the function $Q(s, \cdot)$ is a quadratic form which is positive definite on the subspace $\mathbb{M}^{n\times n}_{sym}$ of $\mathbb{M}^{n\times n}$;*
2) *$c_1 s|A|^2 \le Q(s, A) \le c_2 s|A|^2$ holds as $s \in \mathbb{R}$ and $A \in \mathbb{M}^{n\times n}_{sym}$ vary, and for suitable constants $0 < c_1, c_2 < +\infty$.*

Then $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of (u_k) converges in $L^2(\Omega, \mathbb{R}^n)$ to a minimizer u of the following problem

$$\min_{u\in GSBD^2\cap L^2} \Big(\int_\Omega Q(e(u))dx + 2\mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^2 dx \Big), \tag{3}$$

where we have set $Q(e(u)) := Q(1, e(u))$. Moreover the minimum values in (2) tend to the minimum value in (3).

Application 2: approximation of fracture energies with a cohesive term

The conclusion of Theorem 2 strongly depends on the **asymptotic behavior of η_k/ϵ_k** that we considered. In [7, 9] different regimes are investigated in the **antiplane context**, that is when u is scalar and $e(u)$ is replaced by ∇u. When $\eta_k = \epsilon_k$ it is shown that the minimum problems

$$\min_{(u,v)\in H^1\times V_{\epsilon_k}} \int_\Omega \Big(v|\nabla u|^2 + \frac{(1-v)^2}{\epsilon_k} + \epsilon_k|\nabla v|^2 + |u-g|^2 \Big) dx$$

approximate, in the same sense as Theorem 2, the minimum problem

$$\min_{u\in SBV^2\cap L^2} \Big(\int_\Omega |\nabla u|^2 dx + 2\mathcal{H}^{n-1}(J_u) + 2\int_{J_u} |[u]| d\mathcal{H}^{n-1} + \int_\Omega |u-g|^2 dx \Big).$$

As for the **vectorial context**, in a recent work jointly with M. Focardi [8] the Γ-limit of the Ambrosio-Tortorelli functionals **under the regime $\eta_k = \epsilon_k$** is identified in **suitable subspaces of $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$**. Precisely, it is given in terms of the functional

$$\int_\Omega Q(e(u))dx + 2\mathcal{H}^{n-1}(J_u) + 2\int_{J_u} Q_2^{1/2}([u] \odot \nu_u) d\mathcal{H}^{n-1} + \int_\Omega |u-g|^2 dx, \qquad \text{for } u \in SBD(\Omega),$$

where it is assumed that the family $s^{-1}Q(s, \cdot)$ converges uniformly on compact sets of $\mathbb{M}^{n\times n}_{sym}$ to the function Q_2.

Here the hurdle to obtain the Γ-convergence **in the whole $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$** is the density theorem, due to the fact that Theorem 1 does not provide an approximation of the jump symmetric derivative $E^j u$. Nevertheless, the lower bound inequality is established in full generality; the sharpness of the estimate from below can be proved for **all fields in $SBV(\Omega, \mathbb{R}^n)$** by classical density theorems [5] and can be extended to **all bounded fields** by Theorem 1. Clearly, these are strong hints that the lower bound that has been derived is optimal, and that the conclusion still cannot be drawn in the general case due to difficulties probably of technical nature.

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