

THE 1D SCHRÖDINGER EQUATION WITH A SPACETIME WHITE NOISE: THE AVERAGE WAVE FUNCTION

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ABSTRACT. For the 1D Schrödinger equation with a mollified spacetime white noise, we show that the average wave function converges to the equation with an effective potential after an appropriate renormalization.

KEYWORDS: random Schrödinger equation, renormalization, path integral.

1. MAIN RESULT

Consider the Schrödinger equation driven by a weak stationary spacetime Gaussian potential $V(t, x)$:

$$(1.1) \quad i\partial_t\phi(t, x) + \frac{1}{2}\Delta\phi(t, x) - \sqrt{\varepsilon}V(t, x)\phi(t, x) = 0, \quad t > 0, x \in \mathbb{R},$$

on the diffusive scale $(t, x) \mapsto (\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$,

$$(1.2) \quad \phi_\varepsilon(t, x) := \phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$$

satisfies

$$(1.3) \quad i\partial_t\phi_\varepsilon(t, x) + \frac{1}{2}\Delta\phi_\varepsilon(t, x) - \frac{1}{\varepsilon^{3/2}}V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)\phi_\varepsilon(t, x) = 0.$$

With appropriate decorrelating assumptions on V , the rescaled large highly oscillatory potential $\varepsilon^{-3/2}V(t/\varepsilon^2, x/\varepsilon)$ converges in distribution to a spacetime white noise, denoted by $\dot{W}(t, x)$. To the best of our knowledge, the asymptotics of ϕ_ε and making sense of the limit of (1.3), which formally reads

$$i\partial_t\Phi(t, x) + \frac{1}{2}\Delta\Phi(t, x) - \Phi(t, x)\dot{W}(t, x) = 0,$$

is an open problem. The goal of this short note is to take a first step by analyzing $\mathbb{E}[\phi_\varepsilon]$ as $\varepsilon \rightarrow 0$.

1.1. Assumptions on the randomness. We assume the spacetime white noise $\dot{W}(t, x)$ is built on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and

$$V(t, x) = \int_{\mathbb{R}^2} \varrho(t-s, x-y)\dot{W}(s, y)dyds$$

for some mollifier ϱ with $\int \varrho = 1$. By the scaling property of \dot{W} , we have

$$\begin{aligned} \frac{1}{\varepsilon^{3/2}}V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) &= \frac{1}{\varepsilon^{3/2}} \int_{\mathbb{R}^2} \varrho\left(\frac{t}{\varepsilon^2} - s, \frac{x}{\varepsilon} - y\right)\dot{W}(s, y)dyds \\ &= \frac{1}{\varepsilon^{3/2}} \int_{\mathbb{R}^2} \frac{1}{\varepsilon^3} \varrho\left(\frac{t-s}{\varepsilon^2}, \frac{x-y}{\varepsilon}\right)\dot{W}\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon}\right)dyds \\ &\stackrel{\text{law}}{=} \int_{\mathbb{R}^2} \frac{1}{\varepsilon^3} \varrho\left(\frac{t-s}{\varepsilon^2}, \frac{x-y}{\varepsilon}\right)\dot{W}(s, y)dyds, \end{aligned}$$

which converges in distribution to \dot{W} independent of the choice of ϱ . For simplicity, we choose

$$\varrho(t, x) = \frac{\eta(t)}{\sqrt{\pi}} e^{-|x|^2},$$

with $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\int \eta = 1$. The covariance function of V is

$$(1.4) \quad R(t, x) = \mathbb{E}[V(t, x)V(0, 0)] = \int_{\mathbb{R}^2} \varrho(t + s, x + y)\varrho(s, y)dyds = R_\eta(t)q(x),$$

with

$$(1.5) \quad R_\eta(t) := \int_{\mathbb{R}} \eta(t + s)\eta(s)ds, \quad q(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}}.$$

We define $\tilde{R}(\omega, \xi)$ as the Fourier transform of R in (t, x) :

$$\tilde{R}(\omega, \xi) = \int_{\mathbb{R}^2} R(t, x)e^{-i\omega t - i\xi x} dt dx.$$

We use \widehat{f} to denote the Fourier transform of f in the x variable:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x} dx.$$

1.2. Main result. Assuming the initial data $\phi_\varepsilon(0, x) = \phi_0(x) \in \mathcal{C}_c^\infty(\mathbb{R})$, so we have a low frequency wave before rescaling: $\phi(0, x) = \phi_0(\varepsilon x)$. The following is our main result:

Theorem 1.1. *There exists $z_1, z_2 \in \mathbb{C}$ depending on ϱ , given by (2.12) and (2.15), such that for any $t > 0, \xi \in \mathbb{R}$,*

$$(1.6) \quad \mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)]e^{\frac{z_1 t}{\varepsilon}} \rightarrow \widehat{\phi}_0(\xi)e^{-\frac{i}{2}|\xi|^2 t + z_2 t},$$

as $\varepsilon \rightarrow 0$.

Remark 1.2. The limit in (1.6) is the solution to

$$i\partial_t \bar{\phi} + \frac{1}{2}\Delta \bar{\phi} - iz_2 \bar{\phi} = 0, \quad \bar{\phi}(0, x) = \phi_0(x),$$

written in the Fourier domain:

$$\bar{\phi}(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}_0(\xi) e^{-\frac{i}{2}|\xi|^2 t + z_2 t} e^{i\xi x} d\xi.$$

Remark 1.3. In the parabolic setting, a Wong-Zakai theorem is proved [13, 12, 3, 11] for

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{1}{\varepsilon^{3/2}} V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) u_\varepsilon, \quad u(0, x) = u_0(x).$$

The result says that there exists $c_1, c_2 > 0$ depending on ϱ such that

$$(1.7) \quad u_\varepsilon(t, x) e^{-\frac{c_1 t}{\varepsilon} - c_2 t} \Rightarrow \mathcal{U}(t, x)$$

in distribution, where \mathcal{U} solves the stochastic heat equation

$$\partial_t \mathcal{U}(t, x) = \frac{1}{2}\Delta \mathcal{U}(t, x) + \mathcal{U}(t, x)\dot{W}(t, x), \quad \mathcal{U}(0, x) = u_0(x),$$

with the product $\mathcal{U}(t, x)\dot{W}(t, x)$ interpreted in the Itô's sense. Since $\mathbb{E}[\mathcal{U}]$ solves the unperturbed heat equation

$$\mathbb{E}[\mathcal{U}(t, x)] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} u_0(y) dy,$$

a consequence of (1.7) is

$$\mathbb{E}[\widehat{u}_\varepsilon(t, \xi)]e^{-\frac{c_1 t}{\varepsilon} - c_2 t} \rightarrow \widehat{u}_0(\xi)e^{-\frac{1}{2}|\xi|^2 t},$$

which should be compared to (1.6) in the Schrödinger setting, with $-c_1, c_2$ corresponding to z_1, z_2 .

Remark 1.4. Starting from the microscopic dynamics (1.1), if we consider a time scale that is shorter than (1.2), $(t, x) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}})$, with a low frequency initial data $\phi(0, x) = \phi_0(\sqrt{\varepsilon}x)$, a homogenization result was proved in [10]: for any $t > 0, \xi \in \mathbb{R}$,

$$(1.8) \quad \varepsilon^{\frac{d}{2}} \widehat{\phi}\left(\frac{t}{\varepsilon}, \sqrt{\varepsilon}\xi\right) \rightarrow \widehat{\phi}_0(\xi) e^{-\frac{z_1}{2}|\xi|^2 t - z_1 t}$$

in probability, where z_1 is the same constant as in Theorem 1.1. On the other hand, with a high frequency initial data $\phi(0, x) = \phi_0(x)$, a kinetic equation was derived [2]:

$$(1.9) \quad \mathbb{E}[|\widehat{\phi}\left(\frac{t}{\varepsilon}, \xi\right)|^2] \rightarrow \overline{W}(t, \xi),$$

where $\overline{W}(t, \xi) = \int_{\mathbb{R}} W(t, x, \xi) dx$ with W solving the radiative transfer equation

$$\partial_t W(t, x, \xi) + \xi \nabla_x W(t, x, \xi) = \int_{\mathbb{R}} \widetilde{R}\left(\frac{|p|^2 - |\xi|^2}{2}, p - \xi\right) (W(t, x, p) - W(t, x, \xi)) \frac{dp}{2\pi}.$$

For similar results in the case of a spatial randomness, see [14, 7, 4]. In other words, in the high frequency regime where the wave and the media fully interact, the momentum follows a jump process with the kernel $\widetilde{R}\left(\frac{|p|^2 - |\xi|^2}{2}, p - \xi\right)$. The real part of the constant z_1 , as in (1.6) and (1.8), describes the total scattering cross-section, i.e., the jumping rate, evaluated at the zero frequency for the equation satisfied by W :

$$2\text{Re}[z_1] = \int_{\mathbb{R}} \widetilde{R}\left(\frac{|p|^2}{2}, p\right) \frac{dp}{2\pi}.$$

From this perspective, the renormalization in (1.6) is to compensate the attenuation of wave propagation on the longer time scale of t/ε^2 . We emphasize that the average wave function, or more precisely the term $\mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)]\mathbb{E}[\widehat{\phi}_\varepsilon^*(t, \xi)]$, only captures the ballistic component of wave, and provides little information on the scattering components.

Remark 1.5. The convergences in (1.6), (1.8) and (1.9) hold in all dimensions $d \geq 1$, but the scaling chosen in (1.3) leads to a spacetime white noise only in $d = 1$.

Remark 1.6. When the spacetime potential $V(t, x)$ is replaced by a spatial potential $V(x)$, similar problems have been analyzed in [15, 1, 6, 9, 5] in $d = 1, 2$.

2. PROOFS

The proof contains two steps. First, we derive a probabilistic representation of the average wave function $\mathbb{E}[\widehat{\phi}(t, \xi)]$ with some auxiliary Brownian motion $\{B_t\}_{t \geq 0}$ built on another probability space $(\Sigma, \mathcal{A}, \mathbb{P}_{\mathbf{B}})$. Using this probabilistic representation, we pass to the limit using tools from stochastic analysis. Similar proofs have already appeared in [9, 11].

2.1. Probabilistic representation. Assuming $\{B_t\}_{t \geq 0}$ is a standard Brownian motion starting from the origin, defined on $(\Sigma, \mathcal{A}, \mathbb{P}_{\mathbf{B}})$. We denote the expectation with respect to $\{B_t\}_{t \geq 0}$ by $\mathbb{E}_{\mathbf{B}}$.

Lemma 2.1. *For the equation*

$$(2.1) \quad i\partial_t \psi + \frac{1}{2}\Delta \psi - V(t, x)\psi = 0, \quad t > 0, x \in \mathbb{R},$$

with $\psi(0, x) = \psi_0(x)$, we have

$$(2.2) \quad \mathbb{E}[\widehat{\psi}(t, \xi)] = \widehat{\psi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi B_t} e^{-\frac{1}{2} \int_0^t \int_0^t R(s-u, \sqrt{i}(B_s - B_u)) ds du}].$$

On the formal level, (2.2) comes from an application of the Feynman-Kac formula to (2.1) then averaging with respect to V . We write (2.1) as

$$\partial_t \psi = \frac{i}{2} \Delta \psi - iV(t, x)\psi = 0,$$

and assume the following expression:

$$\psi(t, x) = \mathbb{E}_{\mathbf{B}}[\psi_0(x + \sqrt{i}B_t)e^{-i \int_0^t V(t-s, x + \sqrt{i}B_s) ds}].$$

Averaging with respect to V and using the Gaussianity yields

$$\mathbb{E}[\psi(t, x)] = \mathbb{E}_{\mathbf{B}}[\psi_0(x + \sqrt{i}B_t)e^{-\frac{1}{2} \int_0^t \int_0^t R(s-u, \sqrt{i}(B_s - B_u)) ds du}],$$

which, after taking the Fourier transform, gives (2.2).

Proof. We follow the proof of [9, Proposition 2.1], where a similar formula is derived for spatial random potentials. For the convenience of readers, we provide all the details here.

Fix (t, ξ) , we define the function

$$F_1(z) := \mathbb{E}_{\mathbf{B}}[e^{iz\xi B_t - \frac{1}{2} \int_0^t \int_0^t R(s-u, z(B_s - B_u)) ds du}], \quad z \in \bar{D}_0,$$

with $D_0 := \{z \in \mathbb{C} : \operatorname{Re}[z^2] > 0\}$. We also define the corresponding Taylor expansion

$$F_2(z) = \sum_{n=0}^{\infty} F_{2,n}(z), \quad z \in \bar{D}_0,$$

with

$$F_{2,n}(z) := \frac{(-1)^n}{2^n (2\pi)^n n!} \int_{[0,t]^{2n}} \int_{\mathbb{R}^n} \prod_{j=1}^n \widehat{R}(s_j - u_j, p_j) \mathbb{E}_{\mathbf{B}} \left[e^{iz\xi B_t} \prod_{j=1}^n e^{izp_j(B_{s_j} - B_{u_j})} \right] dp ds du.$$

Recall that $R(t, x) = \frac{R_\eta(t)}{\sqrt{2\pi}} e^{-x^2/2}$. In the definition of F_1 , we have extended the definition so that $R(t, z) = \frac{R_\eta(t)}{\sqrt{2\pi}} e^{-z^2/2}$ for all $z \in \mathbb{C}$. We also emphasize that $\widehat{R}(t, p)$ is the Fourier transform of $R(t, x)$ in the x -variable:

$$\widehat{R}(t, p) = R_\eta(t) e^{-\frac{1}{2}p^2}.$$

It is straightforward to check that both F_1 and F_2 are analytic on D_0 and continuous on \bar{D}_0 . Note that $\sqrt{i} \in \partial D_0$. The goal is to show that

$$(2.3) \quad \mathbb{E}[\widehat{\psi}(t, \xi)] = \widehat{\psi}_0(\xi) F_1(\sqrt{i}).$$

Since $(z, s, u) \mapsto R(s-u, z(B_s - B_u))$ is bounded on $\bar{D}_0 \times \mathbb{R}_+^2$, we have

$$(2.4) \quad \begin{aligned} F_1(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \mathbb{E}_{\mathbf{B}} \left[e^{iz\xi B_t} \left(\int_{[0,t]^2} R(s-u, z(B_s - B_u)) ds du \right)^n \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \mathbb{E}_{\mathbf{B}} \left[e^{iz\xi B_t} \int_{[0,t]^{2n}} \prod_{j=1}^n R(s_j - u_j, z(B_{s_j} - B_{u_j})) ds du \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (2\pi)^n n!} \mathbb{E}_{\mathbf{B}} \left[e^{iz\xi B_t} \int_{[0,t]^{2n}} \int_{\mathbb{R}^n} \prod_{j=1}^n \widehat{R}(s_j - u_j, p_j) e^{izp_j(B_{s_j} - B_{u_j})} dp ds du \right]. \end{aligned}$$

For $z = x \in \mathbb{R}$, we can apply the Fubini theorem to see that $F_1(x) = F_2(x)$. Due to the analyticity and continuity of F_1 and F_2 , we therefore have $F_1(z) = F_2(z)$ for all $z \in \bar{D}_0$. Hence, (2.3) is equivalent to

$$(2.5) \quad \mathbb{E}[\widehat{\psi}(t, \xi)] = \widehat{\psi}_0(\xi) \sum_{n=0}^{\infty} F_{2,n}(\sqrt{i}).$$

For a fixed n , we rewrite

$$F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n (2\pi)^n n!} \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \widehat{R}(s_{2j-1} - s_{2j}, p_{2j-1}) \delta(p_{2j-1} + p_{2j}) \\ \times \mathbb{E}_{\mathbf{B}} \left[e^{i\sqrt{i}\xi B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp.$$

Let σ denote the permutations of $\{1, \dots, 2n\}$. After a relabeling of the p -variables we can write

$$(2.6) \quad F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n (2\pi)^n n!} \sum_{\sigma} \int_{[0,t]_{<}^{2n}} \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \widehat{R}(s_{\sigma(2j-1)} - s_{\sigma(2j)}, p_{\sigma(2j-1)}) \delta(p_{\sigma(2j-1)} + p_{\sigma(2j)}) \\ \times \mathbb{E}_{\mathbf{B}} \left[e^{i\sqrt{i}\xi B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp,$$

where $[0,t]_{<}^{2n} := \{(s_1, \dots, s_{2n}) : 0 \leq s_{2n} \leq \dots \leq s_1 \leq t\}$. Let \mathcal{F} denote the pairings formed over $\{1, \dots, 2n\}$. It is straightforward to check that

$$(2.7) \quad F_{2,n}(\sqrt{i}) = \frac{1}{i^{2n} (2\pi)^n} \sum_{\mathcal{F}} \int_{[0,t]_{<}^{2n}} \int_{\mathbb{R}^{2n}} \prod_{(k,l) \in \mathcal{F}} \widehat{R}(s_k - s_l, p_k) \delta(p_k + p_l) \\ \times \mathbb{E}_{\mathbf{B}} \left[e^{i\sqrt{i}\xi B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp.$$

The pre-factors in (2.6) and (2.7) differ by a factor of $2^n n!$ since $i^{-2n} = (-1)^n$, and this comes from the mapping between the sets of permutations and pairings: for a given pairing with n pairs, we have $n!$ ways of permutating the pairs, and inside each pair, we have 2 options which leads to the additional factor of 2^n .

The phase factor inside the integral in (2.7) can be computed explicitly:

$$(2.8) \quad \mathbb{E}_{\mathbf{B}} \left[e^{i\sqrt{i}\xi B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] = e^{-\frac{i}{2} |\xi|^2 (t-s_1) - \frac{i}{2} |\xi - p_1|^2 (s_1 - s_2) - \dots - \frac{i}{2} |\xi - \dots - p_{2n}|^2 s_{2n}}.$$

On the other hand, the equation (2.1) is written in the Fourier domain as

$$\partial_t \widehat{\psi} = -\frac{i}{2} |\xi|^2 \widehat{\psi} + \int_{\mathbb{R}} \frac{\widehat{V}(t, dp)}{2\pi i} \widehat{\psi}(t, \xi - p), \quad \widehat{\psi}(0, \xi) = \widehat{\psi}_0(\xi),$$

where $V(t, x)$ admits the spectral representation $V(t, x) = \int_{\mathbb{R}} \frac{\widehat{V}(t, dp)}{2\pi} e^{ipx}$. Using the above formula, we can write the solution $\widehat{\psi}(t, \xi)$ as an infinite series

$$(2.9) \quad \widehat{\psi}(t, \xi) = \sum_{n=0}^{\infty} \int_{[0,t]_{<}^n} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\widehat{V}(s_j, dp_j)}{2\pi i} e^{-\frac{i}{2} |\xi|^2 (t-s_1) - \frac{i}{2} |\xi - p_1|^2 (s_1 - s_2) - \dots - \frac{i}{2} |\xi - \dots - p_n|^2 s_n} \\ \times \widehat{\psi}_0(\xi - p_1 - \dots - p_n) ds.$$

Evaluating the expectation $\mathbb{E}[\widehat{\psi}(t, \xi)]$ in (2.9), using the Wick formula for computing the Gaussian moment

$$\mathbb{E}[\widehat{V}(s_1, dp_1) \dots \widehat{V}(s_n, dp_n)],$$

and the fact that

$$\mathbb{E}[\widehat{V}(s_i, dp_i) \widehat{V}(s_j, dp_j)] = 2\pi \widehat{R}(s_i - s_j, p_i) \delta(p_i + p_j) dp_i dp_j,$$

and comparing the result to (2.7)-(2.8), we conclude that (2.5) holds, which completes the proof. \square

2.2. Convergence of Brownian functionals. By Lemma 2.1, the interested quantity is written as

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)] = \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi B_t} e^{-\frac{1}{2} \int_0^t \int_0^t R_\varepsilon(s-u, \sqrt{i}(B_s - B_u)) ds du}],$$

with R_ε defined as the covariance function of $\varepsilon^{-3/2}V(t/\varepsilon^2, x/\varepsilon)$:

$$R_\varepsilon(t, x) = \frac{1}{\varepsilon^3} R\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right).$$

After a change of variable and using the scaling property of the Brownian motion, we have

$$\begin{aligned} & \int_0^t \int_0^t R_\varepsilon(s-u, \sqrt{i}(B_s - B_u)) ds du \\ &= \varepsilon \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} R_\eta(s-u) q(\sqrt{i}(B_{\varepsilon^2 s} - B_{\varepsilon^2 u})/\varepsilon) ds du \\ &\stackrel{\text{law}}{=} \varepsilon \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} R_\eta(s-u) q(\sqrt{i}(B_s - B_u)) ds du, \end{aligned}$$

where R_η and q were defined in (1.5). Thus, by defining

$$X_t^\varepsilon := \frac{\varepsilon}{2} \int_0^{t/\varepsilon^2} \int_0^{t/\varepsilon^2} R_\eta(s-u) q(\sqrt{i}(B_s - B_u)) ds du,$$

we have

$$(2.10) \quad \mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)] = \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi \varepsilon B_{t/\varepsilon^2} - X_t^\varepsilon}].$$

To pass to the limit of $\mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)]$, it suffices to prove the weak convergence of $(\varepsilon B_{t/\varepsilon^2}, X_t^\varepsilon)$ and some uniform integrability condition. The proof of Theorem 1.1 reduces to the following three lemmas.

Lemma 2.2. $\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \frac{z_1 t}{\varepsilon} + O(\varepsilon)$ with z_1 defined in (2.12).

Lemma 2.3. For fixed $t > 0$, as $\varepsilon \rightarrow 0$,

$$(2.11) \quad (\varepsilon B_{t/\varepsilon^2}, X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]) \Rightarrow (N_1, N_2 + iN_3)$$

in distribution, where $N_1 \sim N(0, t)$ and is independent of $(N_2, N_3) \sim N(0, t\mathbf{A})$, with the 2×2 covariance matrix \mathbf{A} defined in (2.13).

Lemma 2.4. For any $\lambda \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\mathbb{E}_{\mathbf{B}}[|e^{\lambda(X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|] \leq C$$

uniformly in $\varepsilon > 0$.

Remark 2.5. With some extra work as in [11, Proposition 2.3], the convergence in (2.11) can be upgraded to the process level. To keep the argument short, we only consider the marginal distributions, which is what we need in the proof of Theorem 1.1.

Proof of Lemma 2.2. A straightforward calculation gives

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] &= \varepsilon \int_0^{t/\varepsilon^2} ds \int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi}} \mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s - B_u|^2}] du \\ &= \varepsilon \int_0^{t/\varepsilon^2} ds \int_0^s \frac{R_\eta(u)}{\sqrt{2\pi}} \mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s - B_{s-u}|^2}] du. \end{aligned}$$

Since R_η is compactly supported, it is clear that

$$\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \frac{z_1 t}{\varepsilon} + O(\varepsilon),$$

where

$$(2.12) \quad z_1 = \int_0^\infty \frac{R_\eta(u)}{\sqrt{2\pi}} \mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_u|^2}] du = \int_0^\infty \frac{R_\eta(u)}{\sqrt{2\pi(1+iu)}} du.$$

The proof is complete. \square

Proof of Lemma 2.3. The proof is based on a martingale decomposition. Denote the Brownian filtration by \mathcal{F}_r and the Malliavin derivative with respect to dB_r by D_r . An application of the Clark-Ocone formula leads to

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \int_0^{t/\varepsilon^2} \mathbb{E}_{\mathbf{B}}[D_r X_t^\varepsilon | \mathcal{F}_r] dB_r.$$

A direct calculation gives

$$D_r X_t^\varepsilon = -i\varepsilon \int_0^{t/\varepsilon^2} \int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi}} e^{-\frac{i}{2}|B_s-B_u|^2} (B_s - B_u) 1_{[u,s]}(r) duds$$

for all $r \in [0, t/\varepsilon^2]$. Taking the conditional expectation with respect to \mathcal{F}_r yields

$$\begin{aligned} Y_r^{\varepsilon,t} &:= \varepsilon^{-1} \mathbb{E}_{\mathbf{B}}[D_r X_t^\varepsilon | \mathcal{F}_r] \\ &= -i \int_0^{t/\varepsilon^2} \int_0^s \frac{R_\eta(s-u)}{\sqrt{2\pi(1+i(s-r))}} e^{-\frac{i|B_r-B_u|^2}{2(1+i(s-r))}} (B_r - B_u) 1_{[u,s]}(r) duds. \end{aligned}$$

By the assumption, there exists $M > 0$ such that $R_\eta(s-u) = 0$ if $s-u \geq M$, so we have for $M \leq r \leq t/\varepsilon^2 - M$ that

$$\begin{aligned} Y_r^{\varepsilon,t} = Y_r &:= -i \int_r^{r+M} \int_{r-M}^r \frac{R_\eta(s-u)}{\sqrt{2\pi(1+i(s-r))}} e^{-\frac{i|B_r-B_u|^2}{2(1+i(s-r))}} (B_r - B_u) 1_{[u,s]}(r) duds \\ &= -i \int_0^M \int_0^M \frac{R_\eta(s+u)}{\sqrt{2\pi(1+is)}} e^{-\frac{i|B_r-B_{r-u}|^2}{2(1+is)}} (B_r - B_{r-u}) duds. \end{aligned}$$

We extend the definition of Y_r to $r \in \mathbb{R}$ by interpreting B as a two-sided Brownian motion. Thus, $\{Y_r\}_{r \in \mathbb{R}}$ is a stationary process with a finite range of dependence.

It is easy to check that

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] - \varepsilon \int_0^{t/\varepsilon^2} Y_r dB_r = \varepsilon \int_0^{t/\varepsilon^2} (Y_r^{\varepsilon,t} - Y_r) dB_r \rightarrow 0$$

in probability. Define $Y_{1,r} = \operatorname{Re}[Y_r]$ and $Y_{2,r} = \operatorname{Im}[Y_r]$, applying Ergodic theorem, we have

$$\varepsilon^2 \int_0^{t/\varepsilon^2} Y_{j,r} Y_{l,r} dr \rightarrow t \mathbb{E}[Y_{j,r} Y_{l,r}], \quad j, l = 1, 2,$$

and

$$\varepsilon^2 \int_0^{t/\varepsilon^2} Y_r ds \rightarrow t \mathbb{E}[Y_r] = 0,$$

almost surely. We apply the martingale central limit theorem [8, pp. 339] to derive

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon \int_0^{t/\varepsilon^2} Y_r dB_r) \Rightarrow (B_t, W_t^1 + iW_t^2)$$

in $\mathcal{C}[0, \infty)$, where B_t is a standard Brownian motion, independent of the two-dimensional Brownian motion (W_t^1, W_t^2) with the covariance matrix $\mathbf{A} = (A_{jl})_{j,l=1,2}$ given by

$$(2.13) \quad A_{jl} = \mathbb{E}[Y_{j,r} Y_{l,r}].$$

The proof is complete. \square

Proof of Lemma 2.4. We write

$$X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] = \varepsilon \int_0^{t/\varepsilon^2} \mathcal{Z}_s ds,$$

where

$$\mathcal{Z}_s := \int_0^s \frac{R_\eta(u)}{\sqrt{2\pi}} \left(e^{-\frac{i}{2}|B_s - B_{s-u}|^2} - \mathbb{E}_{\mathbf{B}}[e^{-\frac{i}{2}|B_s - B_{s-u}|^2}] \right) du.$$

Again, assuming that $R_\eta(u) = 0$ for $|u| \geq M$. Let $N_\varepsilon = \lfloor \frac{t}{M\varepsilon^2} \rfloor$, we have

$$\begin{aligned} X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon] &= \varepsilon \sum_{k=2}^{N_\varepsilon} \int_{(k-1)M}^{kM} \mathcal{Z}_s ds + \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \\ &= \varepsilon \sum_{k=2}^{N_\varepsilon} Z_k + \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \end{aligned}$$

where we defined $Z_k := \int_{(k-1)M}^{kM} \mathcal{Z}_s ds$ for $2 \leq k \leq N_\varepsilon$. Since \mathcal{Z}_s is uniformly bounded, we have

$$\left| \varepsilon \left(\int_0^M + \int_{N_\varepsilon M}^{t/\varepsilon^2} \right) \mathcal{Z}_s ds \right| \lesssim \varepsilon.$$

For the first part, we write

$$\varepsilon \sum_{k=2}^{N_\varepsilon} Z_k = \left(\sum_{k \in A_{\varepsilon,1}} + \sum_{k \in A_{\varepsilon,2}} \right) \varepsilon Z_k,$$

with $A_{\varepsilon,1} = \{2 \leq k \leq N_\varepsilon : k \text{ even}\}$ and $A_{\varepsilon,2} = \{2 \leq k \leq N_\varepsilon : k \text{ odd}\}$. By the independence of the increments of the Brownian motion, we know that $\{Z_k\}_{k \in A_{\varepsilon,j}}$ are i.i.d. for $j = 1$ and 2 . Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[|e^{\lambda(X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|] &\lesssim \mathbb{E}_{\mathbf{B}}[e^{\lambda \varepsilon \sum_{k=2}^{N_\varepsilon} \text{Re}[Z_k]}] \\ &\lesssim \sqrt{\mathbb{E}_{\mathbf{B}}[e^{2\lambda \varepsilon \sum_{k \in A_{\varepsilon,1}} \text{Re}[Z_k]}] \mathbb{E}_{\mathbf{B}}[e^{2\lambda \varepsilon \sum_{k \in A_{\varepsilon,2}} \text{Re}[Z_k]}]}. \end{aligned}$$

The proof is complete by invoking the fact that Z_k is bounded with zero mean. \square

2.3. Proof of Theorem 1.1. By (2.10), we have

$$(2.14) \quad \mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)] e^{\mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]} = \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi \varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}].$$

By Lemma 2.3, we know that, for fixed $t > 0, \xi \in \mathbb{R}$, the random variable

$$i\sqrt{i}\xi \varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon]) \Rightarrow i\sqrt{i}\xi N_1 - (N_2 + iN_3)$$

in distribution, where $N_1 \sim N(0, t)$ independent of $(N_2, N_3) \sim N(0, tA)$. Since Lemma 2.4 provides the uniform integrability:

$$\mathbb{E}_{\mathbf{B}}[|e^{i\sqrt{i}\xi \varepsilon B_{t/\varepsilon^2} - (X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|^2] \leq \sqrt{\mathbb{E}_{\mathbf{B}}[|e^{i\sqrt{i}\xi \varepsilon B_{t/\varepsilon^2}|^2] \mathbb{E}_{\mathbf{B}}[|e^{-(X_t^\varepsilon - \mathbb{E}_{\mathbf{B}}[X_t^\varepsilon])}|^2]} \lesssim 1.$$

Sending $\varepsilon \rightarrow 0$ on both sides of (2.14) and applying Lemma 2.2, we have

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t, \xi)] e^{\frac{z_1 t}{\varepsilon}} \rightarrow \widehat{\phi}_0(\xi) \mathbb{E}_{\mathbf{B}}[e^{i\sqrt{i}\xi N_1 - (N_2 + iN_3)}] = \widehat{\phi}_0(\xi) e^{-\frac{i}{2}|\xi|^2 t} e^{\frac{1}{2}(A_{11} - A_{22} + 2iA_{12})t}.$$

Define

$$(2.15) \quad z_2 = \frac{1}{2}(A_{11} - A_{22} + 2iA_{12}),$$

the proof of Theorem 1.1 is complete.

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