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TRANSPORT VIA MASS TRANSPORTATION

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Abstract. Weak topology implicit schemes based on Monge-Kantorovich or Wasserstein metrics have become prominent for their ability to solve a variety of diffusion and diffusion-like equations. They are very flexible, encompassing a wide range of nonlinear effects. They have interesting interpretations as descent algorithms in an infinite dimensional manifold setting or as dissipation principles for motion in a highly viscous environment. Transport plays a fundamental role in these schemes, as noted by Brenier and Benamou and reviewed below. The reverse implication is less explored and, at least at the outset, less obvious. Here we discuss the simplest situations in the context of systems of transport equations. We show how arbitrary Fokker-Planck Equations in one dimension conform to the mass transport paradigm. Finally, we provide some additional examples, including a simple existence result for velocity-jump processes.

1. Introduction.

Weak topology implicit schemes based on Monge-Kantorovich or Wasserstein metrics have become prominent for their ability to solve a variety of diffusion and diffusion-like equations. They are very flexible, encompassing a wide range of nonlinear effects. They have interesting interpretations as descent algorithms in an infinite dimensional manifold setting or as dissipation principles for motion in a highly viscous environment. Transport plays a fundamental role in these schemes, as noted by Brenier and Benamou and reviewed below. The reverse implication is less explored and, at least at the outset, less obvious. Here we discuss the simplest situations in the context of systems of transport equations. We show how arbitrary Fokker-Planck Equations in one dimension conform to the mass transport paradigm. Finally, we provide some additional examples, including a simple existence result for velocity-jump processes.

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For the moment we consider one space dimension. The Wasserstein metric, or 2-Wasserstein metric, on probability density functions f, f^* on $\Omega = (0,1)$, or $\mathbb{R}$ with finite variance, is given by

$$d(f,f^*)^2 = \min_P \iint_{\Omega\times\Omega} (x-y)^2 dp(x,y),$$
$$P = \textit{set of joint distributions with marginals } f \textit{ and } f^*.$$

Extended to probability measures on Ω (with finite variance if $\Omega = \mathbb{R}$), d induces the weak $\star$ topology. In this section, for simplicity of exposition we are setting Ω the unit interval. The same results hold for $\mathbb{R}$ with densities of finite variance, for example.

Let us consider the basic Wasserstein implicit scheme for a standard Fokker-Planck Equation. Let $\mathcal{M}(\Omega)$ denote the set of probability densities on Ω. For a smooth potential $\psi \geq 0$ and $\sigma > 0$, let

$$I(f) = I\left[f^*\right](f) = \frac{1}{2\tau} d(f,f^*)^2 + \int_\Omega (\psi f + \sigma f \log f)dx, \quad f, f^* \in \mathcal{M}(\Omega). \tag{1.1}$$

We now consider the implicit scheme: given an initial datum $f^{(0)} \in L^1(\Omega)$, a probability density, determine iteratively the sequence $f^{(k)} \in L^1(\Omega)$ by setting $f^* = f^{(k-1)}$ and $f^{(k)}$ the solution of

$$I(f) = \min .$$

Of course, the sequence $f^{(k)}$ depends on τ. Interpolating with

$$f^{(\tau)}(x,t) = f^{(k)}(x), \quad k\tau \leq t \leq (k+1)\tau,$$

and passing to the limit as $\tau \to 0$,

$$f^{(\tau)} \to f \text{ where}$$

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial x}(\sigma \frac{\partial}{\partial x} f + \psi' f) \text{ in } \Omega \tag{1.2}$$

$$\sigma \frac{\partial}{\partial x} f + \psi' f = 0 \text{ on } \partial\Omega. \tag{1.3}$$

For a more general example, the implicit scheme based on the functional

$$\frac{1}{2\tau} d(f,f^*)^2 + \int_\Omega (\psi f + \varphi(f))dx \tag{1.4}$$

produces a solution of the problem

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial x}(\frac{\partial}{\partial x}\alpha(f) + \psi' f) \text{ in } \Omega$$
$$\frac{\partial}{\partial x}\alpha(f) + \psi' f = 0 \text{ on } \partial\Omega,$$

where α is a Legendre transform of φ,

$$\alpha(f) = f\varphi_f - \varphi. \tag{1.5}$$

The implicit scheme is a descent algorithm in the weak $\star$ topology of probability measures on Ω. It always has a solution since the integral functional is convex and superlinear. In the case $\varphi(f) := \sigma f \log f$, it affords us a direct and natural identification of its parameters with those of the Ito diffusion, or stochastic differential equation,

$$dX = -\psi'(X)dt + \sqrt{2\sigma} dB_t, \quad X(0) = x_0, \tag{1.6}$$

where B_t is the standard Brownian motion [13].

Next we take a closer look at the Wasserstein metric itself. The Wasserstein metric minimizes (a quadratic) cost among all measure preserving transformations from f^* to f. Given f^*, f, consider all transfer functions $h : \Omega \to \Omega$ such that

$$\int_\Omega \zeta f dy = \int_\Omega \zeta(h(x)) f^*(x) dx, \quad \zeta \in C(\Omega). \tag{1.7}$$

Then

$$d(f, f^*)^2 = \int_\Omega (x - \phi(x))^2 f^*(x) dx = \min_h \int_\Omega (x - h(x))^2 f^*(x) dx. \tag{1.8}$$

In one dimension, in fact, there is only one such h, which is given by $\phi(x) = F^{*-1}(F(x))$, where F^*, F are the distribution functions of f^*, f respectively [9].

We may interpolate from f^* to f via a sequence of deformations $f(y,t)$ (Eulerian) or of transfer functions $\phi(x,t)$, $0 \le t \le \tau$ (Lagrangian) related by

$$\int_\Omega \zeta(y) f(y,t) dy = \int_\Omega \zeta(\phi(x,t)) f^*(x) dx, \quad 0 \le t \le \tau, \quad \zeta \in C(\Omega), \tag{1.9}$$

for which there is a velocity $v(y,t) = \phi_t(x,t)$. It follows from (1.9) that (f, v) satisfies

$$f_t + (vf)_x = 0 \text{ in } \Omega, \ 0 < t < \tau, \tag{1.10}$$

$$f|_{t=0} = f^*, \ f|_{t=\tau} = f. \tag{1.11}$$

This is connected to the Wasserstein metric by the result of Benamou and Brenier [3],

$$\frac{1}{\tau} d(f, f^*)^2 = \min \int_0^\tau \int_\Omega v^2 f dx dt, \tag{1.12}$$

where the minimum is taken over pairs (f, v) satisfying (1.11). In one dimension, it may be easily checked by choosing a special path (see Proposition 2).

Looking back at the implicit scheme, we see that transport is an important feature of the mass transportation approach to solving equations. But the mass transport implicit scheme cannot be directly applied to the solution of a transport equation, for example,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}(\psi' f) \text{ in } \Omega, \ t > 0, \tag{1.13}$$

since the associated functional

$$I(f) = \frac{1}{2\tau} d(f, f^*)^2 + \int_\Omega \psi f dx \tag{1.14}$$

is not superlinear in f (see next section for details). We shall establish a more intimate connection between mass transport and transport and resolve this difficulty by using a Lagrangian-style formulation.

We noted above the straight forward connection between the stochastic differential equation and the Wasserstein implicit scheme functional (1.1). A more general one dimensional Ito diffusion is the stochastic differential equation

$$dX = -\psi'(X) + a(X) dB_t, \quad X(0) = x_0.$$

The associated forward Chapman-Kolmogorov equation, or Fokker-Planck Equation, for its distribution u, is

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(Ku) + \frac{\partial}{\partial x}(\psi' u),$$

where $K(x) = a(x)^2/2$. Following the reasoning in Section 3, one can easily see that there is no readily available identification between this more general Fokker-Planck equation and the Monge-Kantorovich type variational principle (in fact, we will see, in a more general framework, that no identification is available at all). We shall show, however, that the general equation may be captured as a relaxation limit of weakly coupled transport systems. Providing a framework for the weakly coupled transport system is part of our work here.

Returning to the minimization of I from (1.14), we can easily resolve the one-dimensional problem. Given probability densities f, f^* on Ω, let h denote the transfer function (1.7). Then

$$\begin{aligned} J(h) = I(f) &= \frac{1}{2\tau} d(f, f^*)^2 + \int_\Omega \psi f dx \\ &= \int_\Omega \left\{ \frac{1}{2\tau}(x - h(x))^2 + \psi(h(x)) \right\} f^*(x) dx. \end{aligned}$$

For τ small, depending only on ψ, $J(h)$ is strictly convex and the integral can be minimized by minimizing the integrand. The minimizer $\phi = \phi(x, \tau)$ satisfies

$$-\frac{1}{\tau}(x - \phi) + \psi'(\phi) = 0 \text{ or} \tag{1.15}$$

$$x = \phi + \tau\psi'(\phi) \text{ or} \tag{1.16}$$

$$\phi^{-1}(y) = y + \tau\psi'(y) \tag{1.17}$$

and is independent of f^* and f. For this ϕ we may write, substituting $y = \phi(x)$,

$$\begin{aligned} \int_\Omega \zeta f dy &= \int_\Omega \zeta(\phi(x)) f^*(x) dx \\ &= \int_\Omega \zeta(y) f^*(\phi^{-1}(y)) \frac{d\phi^{-1}(y)}{dy} dy. \end{aligned}$$

Hence,

$$f(y) = f^*(y + \tau\psi'(y))(1 + \tau\psi''(y)), \quad y \in \Omega. \tag{1.18}$$

Note that

$$\begin{aligned} \phi_x + \tau\psi''(\phi)\phi_x &= 1, \\ \phi_\tau + \tau\psi''(\phi)\phi_\tau + \psi'(\phi) &= 0 \end{aligned}$$

whence

$$\phi_\tau = -\psi'(\phi)\phi_x,$$

so ϕ as a function of (x, τ) is just the backwards characteristic of the transport equation.

To see how the transport equation arises from this scheme, we derive its approximate Euler Equation. We have that

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega \zeta(f(x) - f^*(x)) dx + \int_\Omega \zeta'(x)\psi'(x) f(x) dx \\ &\qquad = \int_\Omega \left\{ \frac{1}{\tau}(\zeta(\phi(x)) - \zeta(x)) + \zeta'(\phi(x))\psi'(\phi(x)) \right\} f^*(x) dx. \end{aligned}$$

Substitute using the Taylor expansion

$$\zeta(x) = \zeta(y) + \zeta'(y)(x - y) + \frac{1}{2}\zeta''(z_{x,y})(x - y)^2$$

to obtain, using (1.15),

$$\begin{aligned}\frac{1}{\tau}(\zeta(\phi(x))-\zeta(x)) &= -\frac{1}{\tau}\big[\zeta'(\phi(x))(x-\phi(x))-\frac{1}{2\tau}\zeta''(z_{x,\phi(x)})(x-\phi(x))^2 \\ &= -\frac{1}{2\tau}\zeta''(z_{x,\phi(x)})(x-\phi(x))^2-\zeta'(\phi(x))\psi'(\phi(x)).\end{aligned}$$

Substituting this above gives, in view of the formula (1.8)

$$\begin{aligned}\left|\frac{1}{\tau}\int_\Omega \zeta(f-f^*)dx+\int_\Omega \zeta'\psi' f dx\right| &\le \frac{1}{2\tau}\max|\zeta''|\int_\Omega (x-\phi(x))^2 f^*(x)dx \quad (1.19)\\ &= \frac{1}{2\tau}\max|\zeta''| d(f,f^*)^2, \quad (1.20)\end{aligned}$$

the approximate Euler Equation.

It is interesting that the estimates which lead to the convergence of the scheme amount to solving the transport equation in the weak $\star$ topology.

2. **Further connections.** As noted, the mathematical importance of the time-dependent optimal transportation theory, i.e. the problem of connecting two probability measures μ_0 and μ_1 regarded as initial and final states by an optimal path (geodesic in the Wasserstein space), was introduced by Benamou and Brenier in [3], who employed an interpretation of the Wasserstein distance with a fluid mechanics flavor. In the sequel we shall employ the conventional notations.

Let ρ_0 and ρ_τ, for some relaxation time $\tau > 0$, be two compactly supported probability densities in $\mathbb{R}^N$ and consider all smooth (ρ, v) satisfying

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \\ \rho(\cdot,0) = \rho_0 \text{ in } \mathbb{R}^N, \\ \rho(\cdot,\tau) = \rho_\tau \text{ in } \mathbb{R}^N. \end{cases}$$

Then

$$\frac{1}{\tau}d(\rho_0,\rho_\tau)^2 = \inf_{(\rho,v)}\int_0^\tau\int_{\mathbb{R}^N}\rho(x,t)|v(x,t)|^2 dxdt.$$

The velocity of the optimal pair satisfies $v = \nabla\psi$, where the potential ψ is a solution for the Hamilton-Jacobi equation $\psi_t + \frac{1}{2}|\nabla\psi|^2 = 0$ (i.e. the optimal solution is given by a pressureless potential flow). In one dimension, this turns out to be Burgers' equation for v. Put it differently, the pair (ρ,ψ) given by

$$\begin{cases} \rho_t + \operatorname{div}(\rho\nabla\psi) = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \\ \psi_t + \frac{1}{2}|\nabla\psi|^2 = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \\ \rho(\cdot,0) = \rho_0 \text{ in } \mathbb{R}^N, \\ \rho(\cdot,\tau) = \rho_\tau \text{ in } \mathbb{R}^N, \end{cases}$$

also gives the Wasserstein distance between ρ_0 and ρ_T by the formula

$$\frac{1}{\tau}d(\rho_0,\rho_\tau)^2 = \int_0^\tau\int_{\mathbb{R}^N}\rho(x,t)|\nabla\psi(x,t)|^2 dxdt.$$

An excellent source for further reading on the topic is [21]. We would only like to mention two very important results. They express the important roles played by transport and Hamilton-Jacobi equations in the study of the optimality conditions for the time-dependent mass transportation. First is the Hamilton-Jacobi formulation of the Kantorovich duality:

Theorem 1. *Let $c : \mathbb{R}^N \to \mathbb{R}_+$ be a strictly convex, superlinear cost function and let μ_0, μ_1 be two probability measures on $\mathbb{R}^N$. Then the c-optimal total cost for transporting μ_0 into μ_1 satisfies*

$$\mathcal{T}_c(\mu_0, \mu_1) = \sup\left\{ \int_{\mathbb{R}^N} f(x,1)d\mu_1(x) - \int_{\mathbb{R}^N} f(x,0)d\mu_0(x)\right\},$$

where the supremum is taken over all solutions $f : \mathbb{R}^N \times [0,1] \to \mathbb{R}$ of

$$\begin{cases} f_t + c^*(\nabla_x f) = 0 \quad in\ \mathbb{R}^N \times (0,1), \\ f(\cdot, 0) = f_0 \in C_b(\mathbb{R}^N), \end{cases}$$

where c^* is the Legendre transform of c.

The following concept will be used throughout the paper.

Definition 1. We say that the measurable map $S : X \to Y$ pushes forward μ into ν and we write $S_\# \mu = \nu$ if, equivalently, one of the following holds:
(i) $\nu(A) = \mu(S^{-1}(A))$ for all Borel sets $A \subset Y$;
(ii) $\int_Y f(y)d\nu(y) = \int_X f(S(x))d\mu(x)$ for all $f \in C(Y)$.

The displacement interpolation in Eulerian formulation:

Theorem 2. *Let ρ_0, ρ_1 be two probability densities with finite second order moments in $\mathbb{R}^N$ and let $\nabla\Phi$ be a gradient of a convex function such that $\nabla\Phi_\#\rho_0 = \rho_1$ (we know, due to Brenier, that such gradient is unique and solves the Monge problem). Let $u_0 := \Phi - \mathrm{Id}^2/2$ and, for $0 < t \leq 1$, let*

$$u_t(x) := \inf_{y\in\mathbb{R}^N} \left\{ u_0(y) + \frac{|x-y|^2}{2t} \right\} \ (Hopf\!-\!Lax\ formula).$$

Moreover, let $\rho_t := T_{t\#}\rho_0$ be McCann's interpolant between ρ_0 and ρ_1, namely $T_t := (1-t)\mathrm{Id} + t\nabla\Phi$. Then u_t is locally Lipschitz for all $0 < t < 1$ and its gradient $v_t := \nabla_x u_t$ is also locally Lipschitz in t and x on $T_t(\mathbb{R}^N)$. Furthermore, $\{\rho_t\}_{0<t<1}$ satisfies the linear transport equation

$$\frac{\partial \rho_t}{\partial t} + \nabla_x \cdot (\rho_t v_t) = 0 \quad weakly\ on\ \mathbb{R}^N \times (0,1)$$

The present work explores new connections focussing on the role played by optimal mass transportation via Hamilton-Jacobi in the study of linear transport and some of its applications.

2.1. **Optimal transport.** Our plan is to examine the N-dimensional version of the minimization problem discussed in the introduction. Let $\Omega \subset \mathbb{R}^N$ be open (either bounded or $\mathbb{R}^N$) and consider a $\Psi \in C^2(\Omega)$ which is either bounded or nonnegative. Set:

$$\mathcal{M}(\Omega) := \left\{ \rho : \Omega \to \mathbb{R}^+ \;\middle|\; \rho \text{ is Lebesgue measurable and } \int_\Omega \rho(x)dx = 1 \right\}.$$

Fix $\tau > 0$, $\rho^* \in \mathcal{M}(\Omega)$ and denote by $I_\tau[\rho^*] : \mathcal{M}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ the functional given by:

$$I_\tau[\rho^*](\rho) := \frac{1}{2\tau} d(\rho, \rho^*)^2 + \int_\Omega \rho(x)\Psi(x)dx. \tag{2.1}$$

We will prove that $I_\tau[\rho^*]$ has a unique minimizer in $\mathcal{M}(\Omega)$. Note that the corresponding problem with an entropy term $E(\rho) := \int_\Omega \phi(\rho)dx$ added to $\int_\Omega \rho\Psi dx$ has been extensively studied in [13], [17], [16], [1] in some cases even for more general cost functions. The superlinear growth of ϕ or some other growth properties ([1])

cannot be employed here to obtain existence in L^1 or L^∞. As already observed, the innovative idea used in this work is *to rethink the minimization problem in ρ (probability density) and translate it into a minimization problem for the optimal map s pushing ρ^* forward to ρ.* The uniqueness part follows from the strict convexity of $I_\tau[\rho^*]$ which is a consequence of the strict convexity of $d^2(\cdot,\rho^*)$ (proved by Otto in [18]) although, in some form, it also comes out independently from our proof in the sequel.

Denote by $\mathcal{P}$ the set all Borel probability measures on Ω and by $\mathcal{P}^{ac}$ the set of probability measures on Ω that are absolutely continuous with respect to the Lebesgue measure (which may be identified with $\mathcal{M}$).

Proposition 1. *Let $\Psi \in C^2(\Omega)$ be such that $|\nabla^2\Psi| \in L^\infty(\Omega)$. If Ω is bounded, also assume $\nabla\Psi \in C_0^1(\bar\Omega;\mathbb{R}^N)$. If $\rho^* \in \mathcal{M}(\Omega)$ is positive a.e., then, for $\tau > 0$ sufficiently small, there exists a unique minimizer μ^0 over $\mathcal{P}(\Omega)$ for*

$$I_\tau[\mu^*](\mu) := \frac{1}{2\tau} d(\mu,\mu^*)^2 + \int_\Omega \Psi(x)d\mu(x),$$

where $\mu^* \in \mathcal{P}^{ac}(\Omega)$ satisfies $d\mu^* = \rho^* dx$. Furthermore, $\mu^0 \in \mathcal{P}^{ac}(\Omega)$; therefore, there exists $\rho^0 \in \mathcal{M}(\Omega)$ such that $d\mu^0 = \rho^0 dx$. If $\rho^* \in \mathcal{M}(\Omega)\cap L^p(\Omega)$ for some $1 \le p \le \infty$, then for τ small enough $\rho^0 \in \mathcal{M}(\Omega)\cap L^p(\Omega)$ and

$$(1-\alpha\tau)^{1/p'}\|\rho^*\|_{L^p(\Omega)} \le \|\rho^0\|_{L^p(\Omega)} \le (1+\alpha\tau)^{1/p'}\|\rho^*\|_{L^p(\Omega)} \tag{2.2}$$

for any given $\alpha > \|\Delta\Psi\|_{L^\infty(\Omega)}$.

Proof: Most of the ensuing argument follows closely the one dimensional case whose proof was sketched in the introduction. Assume Ω is bounded. We will only emphasize the main difficulties one comes across in multiple dimensions. First of all, for a given $\mu \in \mathcal{P}(\Omega)$, the map s such that $s_\#\mu^* = \mu$ exists (even if μ^* is not absolutely continuous, yet it does not give mass to sets of Hausdorff measure at most $N-1$; see, e.g. [21]) but may not be unique. However, the optimal one is, i.e. there exists a unique Lebesgue measurable s^μ such that $s^\mu(\Omega) \subset \Omega$ and satisfies

$$s^\mu_\#\mu^* = \mu \text{ and } d(\mu,\mu^*)^2 = \int_\Omega |x - s^\mu(x)|^2 d\mu^*(x) = \int_\Omega |x-s^\mu(x)|^2\rho^*(x)dx.$$

Therefore,

$$I_\tau[\mu^*](\mu) = \int_\Omega \left\{ \frac{|x-s^\mu(x)|^2}{2\tau} + \Psi(s^\mu(x)) \right\} \rho^*(x)dx.$$

If $\psi_\tau := \mathrm{id}_\Omega + \tau\nabla\Psi$ is a diffeomorphism of $\bar\Omega$, then we have seen that the above quantity has a unique minimizer $s^0 := \psi_\tau^{-1}$ which yields a unique minimizer $\mu^0 \in \mathcal{P}(\Omega)$ that belongs, in fact, to $\mathcal{P}^{ac}(\Omega)$. All we need to show is that $\psi_\tau : \bar\Omega \to \bar\Omega$ is onto (as the fact that it is one-to-one follows trivially for small τ). Since $\nabla\Psi \in C_0^1(\bar\Omega;\mathbb{R}^N)$, we obviously have $\psi_\tau(x) = x$ for all $x \in \partial\Omega$. Now, if we consider $y \in \Omega$ we infer $y \in \mathbb{R}^N\backslash\psi_\tau(\partial\Omega)$ which together with $\psi_\tau \equiv \mathrm{id}$ on $\partial\Omega$ leads to

$$\deg(\psi_\tau,\Omega,y) = \deg(\mathrm{id}_\Omega,\Omega,y) = 1,$$

where deg denotes the Brouwer degree. Thus, the claim that $\psi_\tau \in \mathrm{Diff}(\bar\Omega)$ (the diffeomorphisms of $\bar\Omega$) is verified. Next let $\mathcal{M}_2(\Omega) \ni \rho^0 := d\mu^0/dx$ which is, as before, given by

$$\rho^0(x) = \rho^*(x+\tau\nabla\Psi(x))\det(\mathbf{1}+\tau\nabla^2\Psi(x)) \text{ for all } x \in \Omega. \tag{2.3}$$

We have so far proved

$$I_\tau[\mu^*](\mu) \geq \int_\Omega \left\{ \frac{|x-\psi_\tau^{-1}(x)|^2}{2\tau} + \Psi(\psi_\tau^{-1}(x)) \right\} \rho^*(x)dx \geq I_\tau[\rho^*](\rho^0)$$

for all $\mu \in \mathcal{P}(\Omega)$. The second inequality is due to $d(\rho^0,\rho^*)^2 \leq \int_\Omega |x-\psi_\tau^{-1}(x)|^2\rho^*(x)dx$. Therefore,

$$I_\tau[\rho^*](\rho^0) = I_\tau[\mu^*](\mu^0) = \min_{\mu\in\mathcal{P}(\Omega)} I_\tau[\mu^*](\mu)$$

which also implies $d(\rho^0,\rho^*)^2 = \int_\Omega |x-\psi_\tau(x)|^2\rho^*(x)dx$, i.e. ψ_τ is optimal. This is in agreement with the theory of optimal transportation which asserts that ψ_τ is the gradient of a convex function; indeed $\psi_\tau = \nabla(|\mathrm{id}|^2/2 + \tau\Psi)$ and such potential is convex for the small values of τ presently considered. We conclude the proof by noting that $\det(\mathbf{1}+\tau A) = 1+\tau\mathrm{tr}A + O(\tau^2)$ for all bounded $A \in \mathbb{M}^{N\times N}$ which gives $\det(\mathbf{1}+\tau\nabla^2\Psi(x)) = 1+\tau\Delta\Psi(x)+O(\tau^2)$ for all $x\in\Omega$. We now apply (2.3) to obtain (2.2).

Note that, if $\Omega = \mathbb{R}^N$, then the fact that $\phi_y := y - \tau\nabla\Psi$ is a contraction for sufficiently small τ for all $y\in\mathbb{R}^N$ suffices to infer ψ_τ is a diffeomorphism of $\mathbb{R}^N$. Furthermore, $\nabla\Psi \equiv 0$ on $\partial\Omega$ for bounded Ω would translate into $\lim_{|x|\to\infty}\nabla\Psi(x) = 0$ in the $\Omega = \mathbb{R}^N$ case. However, this is not required in Proposition 1 and turns out to be unnecessary for proving existence of weak solutions for the transport equation. This is, to some extent, counterintuitive since a formal integration of $\rho_t = \mathrm{div}(\rho\nabla\Psi)$ may appear to yield conservation of mass for ρ only if $\nabla\Psi$ vanishes at infinity in $\mathbb{R}^N$. In fact, $\nabla\Psi$ may even be unbounded.

The map $t \to (\mathrm{id}+t\nabla\Psi)^{-1}$ is, as noted in the introduction, the backwards characteristic associated with the transport equation $\rho_t = \mathrm{div}(\rho\Psi)$. It is easy to see that its inverse in x and regarded as a function of t, i. e. $t\to \mathrm{id}+t\nabla\Psi$, realizes the optimal path (or geodesic in the Wasserstein space) connecting the minimizer ρ and the initial ρ^*. More precisely,

Proposition 2. *For sufficiently small $\tau>0$, the map $x\to x+\tau\nabla\Psi(x)$ is the optimal transfer function that pushes ρ forward ("backwards" if we regard ρ as the final object) to ρ^* while the map $x \to \big(\mathrm{id}+\tau\nabla\Psi\big)^{-1}(x)$ is the optimal transfer function that pushes ρ^* forward to ρ^0. Furthermore, the family of maps $\{\mathrm{id}+t\nabla\Psi, 0\leq t\leq\tau\}$, is optimal in the McCann interpolation sense while $\{\big(\mathrm{id}+t\nabla\Psi\big)^{-1}, 0\leq t\leq\tau\}$ is, in general, not optimal in the McCann interpolation sense.*

Proof: The first statement is obvious, since $x\to|x|^2/2+\tau\Psi(x)$ is convex for sufficiently small $\tau>0$. As in, e. g., [21], take now $T := \mathrm{id}+\tau\nabla\Psi$ which satisfies $T_\#\rho^0 = \rho^*$ and let $T_t := (1-t/\tau)\mathrm{id}+(t/\tau)T$ be McCann interpolating maps, $0\leq t\leq\tau$. Indeed, $T_0 = \mathrm{id}$, $T_\tau = T$ and, if we compute T_t, we obtain

$$T_t = (1-t/\tau)\mathrm{id}+(t/\tau)(\mathrm{id}+\tau\nabla\Psi)\mathrm{id}+t\nabla\Psi,$$

which concludes the proof of the second statement. Next, note that T^{-1} (the optimal s from the proof of Proposition 1) is the optimal map such that $T^{-1}_\#\rho^* = \rho^0$. However, we claim that the corresponding McCann interpolating maps $\big(T^{-1}\big)_t$ do not satisfy, in general, $\big(T^{-1}\big)_t \neq \big(T_t\big)^{-1}$ unless $t=\tau$. To prove this, first observe that $\big(T^{-1}\big)_t = T_{\tau-t}\circ T^{-1}$. Indeed, this follows easily from $T_{\tau-t} = (t/\tau)\mathrm{id}+(1-t/\tau)T$. Therefore, $\big(T^{-1}\big)_t = \big(T_t\big)^{-1}$ for all $0\leq t\leq\tau$ is equivalent to $T = T_t\circ T_{\tau-t}$ for all $0\leq t\leq\tau$. If we differentiate with respect to t we obtain $0 = \nabla\Psi(x)\cdot\nabla\Psi(T_{\tau-t}(x))$ for all $x\in\Omega$ and $0\leq t\leq\tau$. Let $t=\tau$ and take into account $T_0 = \mathrm{id}$ to infer $\nabla\Psi = 0$

in Ω. Consequently, Ψ must be constant in order for $t \to \big(\mathrm{id}+t\nabla\Psi\big)^{-1}$ to be optimal (the trivial case).

2.2. **Convergence of the discrete scheme.** Throughout this section $\rho_0 \in \mathcal{M}(\Omega)$ is such that $\int_\Omega \rho_0 \Psi dx < +\infty$. The implicit scheme, given $\rho_0 \in \mathcal{M}(\Omega)$ and $\tau > 0$, reads:

For every integer $k \geq 1$ we define ρ_k as the minimizer in

$$\frac{1}{2\tau} d(\rho, \rho_{k-1})^2 + \int_\Omega \Psi(x)\rho(x)dx = \min . \tag{2.4}$$

Existence and uniqueness follow by applying Proposition 1 iteratively. Obviously, the time-interpolant defined as

$$\rho^\tau(x,t) := \rho_k(x) \text{ for } k\tau \leq t < (k+1)\tau, \tag{2.5}$$

satisfies $\|\rho^\tau\|_{L^1(\Omega_T)} = T\|\rho_0\|_{L^1(\Omega)} = T$ for $T < \infty$, where $\Omega_T := \Omega \times (0,T)$ for $0 < T \leq \infty$. Furthermore, fix $n \in \mathbb{N}$ large enough and let $\tau := T/n$. We infer, due to (2.2), that

$$(1-\alpha\tau)^{k/p'} \|\rho_0\|_{L^p(\Omega)} \leq \|\rho_k\|_{L^p(\Omega)} \leq (1+\alpha\tau)^{k/p'} \|\rho_0\|_{L^p(\Omega)}. \tag{2.6}$$

Let $1 < p < \infty$ and observe that

$$\|\rho^\tau\|^p_{L^p(\Omega_T)} = \int_0^T \int_\Omega \big(\rho^\tau\big)^p dxdt = \tau \sum_{k=0}^{n-1} \|\rho_k\|^p_{L^p(\Omega)}.$$

By (2.6) we deduce

$$\begin{aligned}
\left\{\tau \sum_{k=0}^{n-1} (1-\alpha\tau)^{kp/p'}\right\}^{1/p} \|\rho_0\|_{L^p(\Omega)} \quad &\leq \quad \|\rho^\tau\|_{L^p(\Omega_T)} \\
&\leq \quad \left\{\tau \sum_{k=0}^{n-1} (1+\alpha\tau)^{kp/p'}\right\}^{1/p} \|\rho_0\|_{L^p(\Omega)}.
\end{aligned}$$

Summing leads to the inequalities

$$\begin{aligned}
\left\{\tau \frac{1-(1-\alpha\tau)^{(p-1)T/\tau}}{1-(1-\alpha\tau)^{p-1}}\right\}^{1/p} \quad &\leq \quad \|\rho^\tau\|_{L^p(\Omega_T)} / \|\rho_0\|_{L^p(\Omega)} \\
&\leq \quad \left\{\tau \frac{(1+\alpha\tau)^{(p-1)T/\tau} - 1}{(1+\alpha\tau)^{p-1} - 1}\right\}^{1/p}.
\end{aligned} \tag{2.7}$$

If $p = \infty$, (2.6) leads to

$$(1-\alpha\tau)^{T/\tau} \|\rho_0\|_{L^\infty(\Omega)} \leq \|\rho^\tau\|_{L^\infty(\Omega_T)} \leq (1+\alpha\tau)^{T/\tau} \|\rho_0\|_{L^\infty(\Omega)}. \tag{2.8}$$

We will now give the approximate Euler equation.

Proposition 3. *Let $\Psi \in C^2(\Omega)$ be such that $|\nabla^2\Psi| \in L^\infty(\Omega)$. Also, assume $\nabla\Psi \equiv 0$ on $\partial\Omega$ if Ω is bounded. Let $\rho_0 \in \mathcal{M}_2(\Omega) \cap L^p(\Omega)$ and $\{\rho_k\}_{k\in\mathbb{N}}$ be the solution of (2.4). Then $\rho_k \in \mathcal{M}_2(\Omega) \cap L^p(\Omega)$ for all $k \geq 1$ and*

$$\left| \int_\Omega \left\{ \frac{1}{\tau}(\rho_k - \rho_{k-1})\zeta + \rho_k \nabla\Psi \cdot \nabla\zeta \right\} dx \right| \leq \frac{1}{2\tau} \sup_{\mathbb{R}^N} |\nabla^2\zeta| \; d(\rho_{k-1}, \rho_k)^2, \tag{2.9}$$

for every $\zeta \in C_c^\infty(\mathbb{R}^N)$.

Proof: Recall that the optimal mass transfer map that pushes ρ_k backwards to ρ_{k-1} is $\psi_\tau = \mathrm{id}_\Omega + \tau\nabla\Psi$ for all k. It follows

$$\frac{1}{\tau}\big(\psi_\tau - \mathrm{id}_\Omega\big)\rho_k - \rho_k\nabla\Psi = 0 \text{ in } \Omega. \tag{2.10}$$

Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ and multiply (2.10) by $\nabla\zeta$ to obtain

$$\int_\Omega (\psi_\tau(x) - x)\cdot\nabla\zeta(x)\rho_k(x)dx - \tau\int_\Omega \rho_k(x)\nabla\Psi(x)\cdot\nabla\zeta(x)dx = 0. \tag{2.11}$$

The following finite Taylor expansion formula holds

$$\zeta(\psi_\tau(x)) - \zeta(x) = (\psi_\tau(x) - x)\cdot\nabla\zeta(x) + (1/2)(\psi_\tau(x) - x)^T(\nabla^2\zeta)(z_{x,\tau})(\psi_\tau(x) - x),$$

for all $x \in \Omega$ and the corresponding $z_{x,\tau} \in \mathbb{R}^N$. Along with (2.11) this yields, if we denote by I the left hand side of (2.9),

$$\begin{aligned} I &= \left|\int_\Omega [\zeta(\psi_\tau(x)) - \zeta(x) - (\psi_\tau(x) - x)\cdot\nabla\zeta(x)]\rho_k(x)dx\right| \\ &\le \frac{1}{2}\sup_{\mathbb{R}^N}|\nabla^2\zeta|\int_\Omega |\psi_\tau(x) - x|^2\rho_k(x)dx \\ &= \frac{1}{2}\|\nabla^2\zeta\|_\infty d(\rho_{k-1},\rho_k)^2 \text{ for all } \zeta \in C_c^\infty(\mathbb{R}^N), \end{aligned}$$

which is equivalent to (2.9).

The estimates (2.7) and (2.8) show that, if $1 < p < \infty$, then up to a subsequence (not relabelled) we obtain $\rho \in L^p(\Omega_T)$ such that

$$\rho^\tau \rightharpoonup \rho \text{ weakly in } L^p(\Omega_T) \text{ or weakly } \star \text{ if } p = \infty. \tag{2.12}$$

Now, following mainly [20], [17] and [16], we consider $\zeta \in C_c^\infty(\mathbb{R}^N \times [0,T))$, integrate in time from $(k-1)\tau$ to $k\tau$ the inequalities (2.9) for $k = 1..n-1$ and add them to conclude (by passing to the limit as $\tau \to 0$) that ρ is a weak solution of

$$\rho_t = \mathrm{div}(\rho\Phi) \text{ in } \Omega\times(0,\infty) \text{ with } \rho(\cdot,0) = \rho_0 \text{ in } \Omega, \tag{2.13}$$

(where $\Phi := \nabla\Psi$). As a consequence of (2.7) and of the fact that α is an arbitrary constant greater than $\alpha_0 := \|\Delta\Psi\|_\infty$, the solution satisfies

$$\left\{\frac{1 - e^{-\alpha_0 T(p-1)}}{\alpha_0(p-1)}\right\}^{1/p}\|\rho_0\|_{L^p(\Omega)} \le \|\rho\|_{L^p(\Omega_T)} \le \left\{\frac{e^{\alpha_0 T(p-1)} - 1}{\alpha_0(p-1)}\right\}^{1/p}\|\rho_0\|_{L^p(\Omega)}$$

if $1 < p < \infty$ and

$$e^{-\alpha_0 T}\|\rho_0\|_{L^\infty(\Omega)} \le \|\rho\|_{L^\infty(\Omega_T)} \le e^{\alpha_0 T}\|\rho_0\|_{L^\infty(\Omega)}$$

if $p = \infty$.

Recall ([20], [13] etc.) that essential to the existence result is the vanishing cumulative error as $\tau \downarrow 0$, i.e. we need $\sum_{k=1}^{n-1} d(\rho_k,\rho_{k-1})^2 = O(\tau)$. It is easy to see that, as ρ_k is the minimizer of $I_\tau[\rho_{k-1}]$, we obtain

$$I_\tau[\rho_{k-1}](\rho_k) \le I_\tau[\rho_{k-1}](\rho_{k-1}) = \int_\Omega \rho_{k-1}\Psi dx$$

which, by summation, yields either

$$\sum_{k=1}^\infty d(\rho_k,\rho_{k-1})^2 \le 4\tau\|\Psi\|_{L^\infty(\Omega)} \tag{2.14}$$

if Ψ is bounded or

$$\sum_{k=1}^{\infty} d(\rho_k, \rho_{k-1})^2 \leq 2\tau \int_{\Omega} \rho_0 \Psi dx \tag{2.15}$$

if Ψ is nonnegative. Furthermore, note that, for fixed $0 < T < \infty$ and $\tau := T/n$ (n is a positive integer) we can prove in some cases that $\sum_{k=1}^{n-1} d(\rho_k, \rho_{k-1})^2 = O(\tau)$ without using the variational principle for the minimum. Indeed, if $\nabla\Psi$ is bounded, then

$$\sum_{k=1}^{n-1} d(\rho_k, \rho_{k-1})^2 = \tau^2 \sum_{k=1}^{n-1} \int_{\Omega} |\nabla\Psi(x)|^2 \rho_k(x) dx \leq \tau T \|\nabla\Psi\|_{\infty}^2$$

since we can compute the Wasserstein distance directly by knowing the optimal transfer map ψ_τ at each step. It thus seems natural to attempt proving existence by using such transfer maps (they may be non-optimal) $\phi_\tau := \mathrm{id}_\Omega + \tau\Phi$ for $\Phi \in C_0^1(\bar{\Omega}; \mathbb{R}^N)$ (if Ω is bounded) which is not necessarily a gradient vector field. If $\Omega = \mathbb{R}^N$ then it suffices to have $\Phi \in C^1(\mathbb{R}^N; \mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$. For given $\rho_0 \in \mathcal{M}(\Omega)$, we define iteratively the push-forwards:

$$\rho_k := (\phi_\tau^{-1})_{\#}\rho_{k-1} \text{ for all integers } 1 \leq k < n, \tag{2.16}$$
$$\text{i. e. } \rho_k = (\rho_{k-1} \circ \phi_\tau)\det(\nabla\phi_\tau) \text{ in } \Omega.$$

Obviously, $\rho_k \in \mathcal{M}(\Omega)$ for all k. Let ρ^τ be defined as in (2.5). Clearly, the estimates (2.7) and (2.8) remain valid if we set $\alpha := 2\|\mathrm{div}\Phi\|_{L^\infty(\Omega)}$. The corresponding version of Proposition 3 with $\nabla\Psi$ replaced by Φ is also true with the error term from (2.9) (the right hand side) replaced by $(\tau/2)\|\nabla^2\zeta\|_\infty \|\Phi\|_\infty^2$. Therefore, we similarly obtain a weak solution for (2.13) for any vector field $\Phi \in C_0^1(\bar{\Omega}; \mathbb{R}^N)$ (not necessarily a gradient).
From here, it is a small step to proving (for $1 < p \leq \infty$)

Corollary 1. *The initial-value problem (2.13) admits a weak solution for every $\Phi \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ with $\Phi \equiv 0$ on $\partial\Omega$ and every $\rho_0 \in \mathcal{M}(\Omega) \cap L^p(\Omega)$.*

Proof: We consider a sequence $\{\Phi^{(n)}\}_n \subset C_0^1(\bar{\Omega}; \mathbb{R}^N)$ which converges to Φ in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ strong. This is equivalent to $\Phi^{(n)}$ and $\nabla\Phi^{(n)}$ converging uniformly to Φ and $\nabla\Phi$ respectively. Thus, we can obtain uniform bounds for $\|\rho^{(n)}\|_{L^p(\Omega_T)}$ as in (2.7) and (2.8), where $\rho^{(n)}$ is the weak solution constructed above for $\Phi^{(n)}$ and the initial value ρ_0. Consequently, $\rho^{(n)} \rightharpoonup \rho$ weakly in $L^p(\Omega_T)$ (or weak $\star$ if $p = \infty$) for some $\rho \in L^p(\Omega_T)$. Due to this and to the uniform convergence $\Phi^{(n)} \to \Phi$, we are able to pass to the limit and infer that ρ is the desired solution.

We can prove something even more general in the case $\Omega = \mathbb{R}^N$. Fix $0 < T < \infty$.

Theorem 3. *Let $1 < p \leq \infty$ and $\rho_0 \in \mathcal{M}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. The problem (2.13) admits a weak solution in $\mathbb{R}^N \times (0,T)$ provided that $\Phi \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N)$ and $\mathrm{div}\Phi \in L^\infty(\mathbb{R}^N)$.*

For the proof it suffices to approximate Φ by functions in $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ whose divergences are uniformly essentially bounded (see [7]).

2.3. **Explicit Euler schemes for characteristics.** Here we would like to stress the connection with the method of characteristics. This approach to transport is equivalent to the explicit Euler scheme for the associated characteristic equation. We argue that it is even more natural because it exploits the structure of transport as steepest descent of the potential energy in the weak topology.

We go back to (2.16) and note that

$$\rho_k := (\phi_\tau^{-1})_{\#}^{(k)}\rho_0 \text{ for all integers } 1 \le k < n,$$

$$\text{i. e. } \rho_k = [\rho_0 \circ (\phi_\tau)^{(k)}]\det[\nabla(\phi_\tau)^{(k)}] \text{ in } \Omega,$$

where $(\phi_\tau)^{(k)} := \phi_\tau \circ \phi_\tau \circ ... \circ \phi_\tau$ k times. Therefore, if $n\tau = t$, then

$$\rho^\tau = (\phi_\tau^{-1})_{\#}^{(n)}\rho_0 = [\rho_0 \circ (\phi_\tau)^{(n)}]\det[\nabla(\phi_\tau)^{(n)}] \text{ in } \Omega. \tag{2.17}$$

The solution of (2.13) can be obtained by the method of characteristics in the following way: construct a family $\{T(t;\cdot)\}_{t\ge 0}$ of diffeomorphisms solving

$$\frac{d}{dt}T(t;x) = \Phi(T(t;x)) \text{ for } t > 0,\ T(0;x) = x,\ x \in \mathbb{R}^N. \tag{2.18}$$

Then, it is well known that the solution of (2.13) is given by $\rho(\cdot,t) = T(t;\cdot)_{\#}\rho_0$. Note that the explicit Euler scheme with step τ applied to (2.18) yields the approximants T_k^τ satisfying

$$T_{k+1}^\tau(x) = T_k^\tau(x) + \tau\Phi(T_k^\tau(x)),\ k \ge 0.$$

Since $T_0^\tau(x) = T(0;x) = x$, it is immediate that $T_k^\tau = (\phi_\tau)^{(k)}$. It is known that, if we are in a bounded domain, then for fixed times $t > 0$ the Euler method converges uniformly to the solution of the ODE at rate τ from which the weak convergence of the sequence $\left\{\rho^{t/n}\right\}_{n\ge 1}$ (where $\rho^{t/n}$ is defined in (2.17)) to $\rho(\cdot,t)$ can be easily inferred.

If we differentiate (2.18) with respect to x, then we obtain the following equation for the matrix $\mathcal{U}(t;x) := \nabla_x T(t;x)$

$$\frac{d}{dt}\mathcal{U}(t;x) = \nabla\Phi(T(t;x))\mathcal{U}(t;x) \text{ for } t > 0,\ \mathcal{U}(0;x) = \mathbf{1},\ x \in \mathbb{R}^N.$$

One can deduce that, if $\nabla\Phi$ is Lipschitz continuous, we have uniform convergence of $\nabla T_n^{t/n}$ to $\nabla_x T(t;\cdot)$ (via the explicit Euler approximants $\mathcal{U}_n^{t/n}$ of $\mathcal{U}$). Thus, in this case, we have strong L^1 convergence of $\rho^{t/n}$ to $\rho(\cdot,t)$ (even uniform convergence if ρ_0 is continuous).

2.4. **Rates of convergence in the Wasserstein metric.** Let $\Phi(\rho) := \int_\Omega \Psi\rho dx$. The metric slope is defined, for $\rho^* \in \mathcal{M}$, in [2] as

$$|\partial\Phi|(\rho^*) := \limsup_{\rho\to\rho^*} \frac{\left(\Phi(\rho^*) - \Phi(\rho)\right)^+}{d(\rho^*,\rho)}, \tag{2.19}$$

where the $\rho \to \rho^*$ denotes the weak $\star$ convergence of measures. The same reference makes the following optimal uniform estimate available.

Proposition 4. *The estimate*

$$d(\rho^\tau(\cdot,T),\rho(\cdot,T)) \le \frac{\tau}{\sqrt{2}}|\partial\Phi|(\rho^0) \tag{2.20}$$

holds for any initial $\rho^0 \in \mathcal{M}$ *and every* $\tau,\ T > 0$.

We will next give a simple way of obtaining the metric slope in this case.

Lemma 1. *If* $|\nabla\Psi| \in L^2(\Omega; d\rho^*)$ *and* $|\nabla^2\Psi| \in L^\infty(\Omega)$*, then*

$$|\partial\Phi|(\rho^*) = \|\nabla\Psi\|_{L^2(\Omega;d\rho^*)}.$$

Proof: Since the limit turns out to be positive, it suffices to show that

$$\frac{\Phi(\rho^*) - \Phi(\rho)}{d(\rho^*, \rho)} \le \|\nabla\Psi\|_{L^2(\Omega; d\rho^*)} + Cd(\rho^*, \rho) \tag{2.21}$$

for all $\rho \in \mathcal{M}$ and find a sequence $\rho^\varepsilon \to \rho^*$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{\Phi(\rho^*) - \Phi(\rho^\varepsilon)}{d(\rho^*, \rho^\varepsilon)} = \|\nabla\Psi\|_{L^2(\Omega; d\rho^*)}. \tag{2.22}$$

Let s be the optimal map pushing forward ρ^* into ρ. Then

$$\begin{aligned} \Phi(\rho^*) - \Phi(\rho) &= \int_\Omega [\Psi(x) - \Psi(s(x))]\rho^*(x)dx \\ &\le \int_\Omega (x - s(x)) \cdot \nabla\Psi(x)\rho^*(x)dx + \sup|\nabla^2\Psi| d(\rho^*, \rho)^2. \end{aligned}$$

Since

$$\begin{aligned} &\int_\Omega (x - s(x)) \cdot \nabla\Psi(x)\rho^*(x)dx \\ &\qquad \le \left(\int_\Omega |x - s(x)|^2 \rho^*(x)dx\right)^{1/2} \left(\int_\Omega |\nabla\Psi(x)|^2 \rho^*(x)dx\right)^{1/2}, \end{aligned}$$

(2.21) follows with $C := \sup|\nabla^2\Psi|$.
Next let $\varepsilon > 0$ small enough such that $s^\varepsilon := \mathrm{Id} + \varepsilon\nabla\Psi$ is a diffeomorphism of Ω and let $\rho^\varepsilon := s^\varepsilon_{\#}\rho^*$. It is easy to see that (2.22) holds for this sequence.

3. **Weakly coupled transport systems. Fokker-Planck equation in one dimension.** For $i = 1, 2$ let $\Psi_i \in \mathrm{Lip}_{loc}([0, \infty); C^2(\mathbb{R}^N))$ bounded from below with $|\nabla^2\Psi_i| \in L^\infty(\mathbb{R}^N \times (0, \infty))$ and let $\nu_i : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}_+$ be essentially bounded away from zero and infinity. Choose the initial data such that

$$\rho_i^0 \in \mathcal{M}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \rho_i^0(x)\Psi_i(x, 0)dx < \infty \text{ for } i = 1, 2. \tag{3.1}$$

Consider the weakly coupled system:

$$\begin{cases} \rho_{1,t} = \mathrm{div}(\rho_1\nabla\Psi_1) - \nu_1\rho_1 + \nu_2\rho_2 \text{ in } \mathbb{R}^N \times (0, \infty), \\ \rho_{2,t} = \mathrm{div}(\rho_2\nabla\Psi_2) + \nu_1\rho_1 - \nu_2\rho_2 \text{ in } \mathbb{R}^N \times (0, \infty), \\ \rho_i(\cdot, 0) = \rho_i^{(0)} \text{ in } \mathbb{R}^N. \end{cases} \tag{WS}$$

Inspired by [5] and [19], we set up the following iterative minimization problem:
For $i = 1, 2$ and for every integer $k \ge 1$ we define ρ_i^k as the minimizer in

$$\frac{1}{2\tau} d(\rho, (\rho^{k-1}\mathbf{P}_\tau^k)_i)^2 + \int_{\mathbb{R}^N} \rho(x)\Psi_i^k(x)dx = \min, \tag{3.2}$$

where

$$\mathcal{M}_i^k(\mathbb{R}^N) := \left\{\rho : \mathbb{R}^N \to \mathbb{R}_+ \;\middle|\; \int_{\mathbb{R}^N} \rho(x)dx = \int_{\mathbb{R}^N} (\rho^k \mathbf{P}_\tau^{k+1})_i dx\right\}$$

and

$$\mathbf{P}_\tau^k := \begin{pmatrix} 1 - \tau\nu_1^k & \tau\nu_1^k \\ \tau\nu_2^k & 1 - \tau\nu_2^k \end{pmatrix} = \mathbf{1} + \tau \begin{pmatrix} -\nu_1^k & \nu_1^k \\ \nu_2^k & -\nu_2^k \end{pmatrix}. \tag{3.3}$$

We have used the notations

$$\nu_i^k(x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \nu_i(x, t)dt, \;\; \Psi_i^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \Psi_i(x, t)dt \text{ for } i = 1, 2 \text{ and } k \ge 1.$$

According to the previous results, the minimizers are given by:

$$\rho_1^k = \left\{(\mathrm{Id} + \nabla\Psi_1^k)^{-1}\right\}_{\#}[(1-\tau\nu_1^k)\rho_1^{k-1} + \tau\nu_2^k\rho_2^{k-1}]$$

and

$$\rho_2^k = \left\{(\mathrm{Id} + \nabla\Psi_2^k)^{-1}\right\}_{\#}[\tau\nu_1^k\rho_1^{k-1} + (1-\tau\nu_2^k)\rho_2^{k-1}],$$

or, equivalently,

$$\rho_1^k = \left\{[(1-\tau\nu_1^k)\rho_1^{k-1} + \tau\nu_2^k\rho_2^{k-1}]\circ(\mathrm{Id}+\nabla\Psi_1^k)\right\}\det(\mathbf{1}+\nabla^2\Psi_1^k) \tag{3.4}$$

and

$$\rho_2^k = \left\{[\tau\nu_1^k\rho_1^{k-1} + (1-\tau\nu_2^k)\rho_2^{k-1}]\circ(\mathrm{Id}+\nabla\Psi_2^k)\right\}\det(\mathbf{1}+\nabla^2\Psi_2^k). \tag{3.5}$$

Let $\lambda := \min\{ess\inf\nu_1,\ ess\inf\nu_2\}$ and $\Lambda := \max\{ess\sup\nu_1,\ ess\sup\nu_2\}$. Then we have

$$\|(1-\tau\nu_1^k)\rho_1^{k-1} + \tau\nu_2^k\rho_2^{k-1}\|_p \le (1-\lambda\tau)\|\rho_1^{k-1}\|_p + \Lambda\tau\|\rho_2^{k-1}\|_p$$

and

$$\|\tau\nu_1^k\rho_1^{k-1} + (1-\tau\nu_2^k)\rho_2^{k-1}\|_p \le \Lambda\tau\|\rho_1^{k-1}\|_p + (1-\lambda\tau)\|\rho_2^{k-1}\|_p,$$

where $\|\cdot\|_p$ denotes the standard $L^p(\mathbb{R}^N)$-norm. According to the inequalities above and (2.2), we have

$$\|\rho_1^k\|_p \le (1+\tau\alpha_1)^{1/p'}\left\{(1-\lambda\tau)\|\rho_1^{k-1}\|_p + \Lambda\tau\|\rho_2^{k-1}\|_p\right\} \tag{3.6}$$

and

$$\|\rho_2^k\|_p \le (1+\tau\alpha_2)^{1/p'}\left\{\Lambda\tau\|\rho_1^{k-1}\|_p + (1-\lambda\tau)\|\rho_2^{k-1}\|_p\right\}, \tag{3.7}$$

for given $\alpha_i > \|\Delta\Psi_i\|_\infty$. If we let $\omega_k := \left(\|\rho_1^k\|_p\ ,\ \|\rho_2^k\|_p\right)^T$, then we may write (3.6) and (3.7) combined as

$$\omega_k \le \mathcal{A}_\tau\omega_{k-1},\ \text{where } \mathcal{A}_\tau := \begin{pmatrix} (1-\lambda\tau)(1+\tau\alpha_1)^{1/p'} & \Lambda\tau(1+\tau\alpha_1)^{1/p'} \\ \Lambda\tau(1+\tau\alpha_2)^{1/p'} & (1-\lambda\tau)(1+\tau\alpha_2)^{1/p'} \end{pmatrix},$$

where "$\le$" means the suggested inequality componentwise. The simplest way to obtain a bound for ω_n for $\tau = t/n$ (and thus on the solution of the system) is to let $\alpha = \max\{\alpha_1, \alpha_2\}$, $M = \max\{\|\rho_1^0\|_p\ , \|\rho_2^0\|_p\}$. It follows

$$\|\rho_i^n\|_p \le (1+\alpha t/n)^{n/p'}\left[1 + (\Lambda-\lambda)t/n\right]^n M$$

which leads to

$$\|\rho_i\|_p \le M\exp\{[(\alpha/p') + (\Lambda-\lambda)]t\}.$$

Optimal estimates, which we do not seek here, require computing $\lim_{n\uparrow\infty}\mathcal{A}_{t/n}^n$ which is a difficult task. To prove this procedure leads to the solution for our system we employ (2.9) with $(\rho^{k-1}\mathbf{P}_\tau^k)_i$ instead of ρ_i^{k-1} for $i = 1, 2$.

Note that the time-dependence of the potential Ψ can be dealt with by using its time-average over $[k\tau, (k+1)\tau]$ in the variational principle at step k. The same "trick" works in conjunction with a possible diffusion term of type $f = f(t, \rho)$ for problems that we discuss below.

The Fokker-Planck, or more generally, the forward Kolmogorov Equation for a stochastic diffusion, is just a second order linear parabolic equation whose highest order coefficients need not be constant. At this time in the development of mass transport theory, we know that many diffusion equations can be regarded as gradient flows with respect to the Wasserstein distance, and, as mentioned earlier, there is an extensive literature about this. For the most general cases studied see [1] and [19].

Agueh utilizes general cost functions and proves convergence for doubly degenerate diffusion equations of the form:

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\{\rho \nabla c^{*}[\nabla (F'(\rho) + V)]\},$$

where c^* is the Legendre transform of a convex c having specified growth properties. In [19] the authors prove convergence for general, non-autonomous, diffusion equations of the form

$$\frac{\partial \rho}{\partial t} = \Delta f(t,\rho) + \mathrm{div}(\rho \nabla \Psi(x,t)). \tag{3.8}$$

The direct dependence of f (and Ψ) on t is treated by means of time-averaging (see [19]). However, the gradient flow method has not been extended to functions f which also depend explicitly on x. More precisely, we have the following formula for the gradient of a functional $F = F(\rho)$ in the Wasserstein space (see, e.g. [21]):

$$\mathrm{grad}_d F(\rho) = -\mathrm{div}_x \left(\rho \nabla_x \frac{\delta F}{\delta \rho} \right),$$

where $\delta F/\delta \rho$ is the gradient of F with respect to the standard L^2 distance. Therefore, a PDE of the form

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}_x \left(\rho \nabla_x \frac{\delta F}{\delta \rho} \right)$$

can be interpreted as the gradient flow of F in the Wasserstein space. Furthermore, it is apparent that such an interpretation may work even if $F = F(t,\rho)$ is of the form

$$F(t,\rho) = \int \phi(t, \rho(x,t))dx$$

and may lead to equations of type (3.8) since all the derivatives involved are spatial (see [19]).

Absent so far has been a discussion of equations whose highest order terms depend explicitly on x, even when they are lineas, e.g., equations of the general form

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}\big[\mathbf{A}(x,t)\rho\big] + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}\big\{\big[\mathcal{B}(x,t)\mathcal{B}^T(x,t)\big]_{ij}\rho\big\}. \tag{3.9}$$

The solution $\rho(\cdot,t)$ for this equation is the probability density function of the process $\mathbf{X}_t$ which satisfies the stochastic differential equation

$$d\mathbf{X} = \mathbf{A}(\mathbf{X},t)dt + \mathcal{B}(\mathbf{X},t)d\mathbf{W}_t, \tag{3.10}$$

where the N-vector-valued $\mathbf{A}$ is the drift term, the $N \times N$-matrix-valued $\mathcal{B}$ is the stochastic force and $\mathbf{W}_t$ is the standard N-dimensional Wiener process.

Our next goal is to show that, at least in one dimension, the Fokker-Planck equation arises as the limit of a sequence of weakly coupled systems of the form (WS). First, we will briefly explain why it cannot be solved directly by time-discretization in the Wasserstein space.

When $F = F(t,\rho)$ is of the form $F(t,\rho) = \int_\Omega \phi(x,t,\rho(x,t))dx$ the L^2-gradient of F becomes $\delta F/\delta \rho = \phi_\rho(x,t,\rho)$. Thus, the gradient with respect to the Wasserstein distance becomes, say, in dimension one,

$$\mathrm{grad}_d F(t,\rho)(x) := -\frac{\partial}{\partial x}\left\{\rho\left[\phi_{\rho x}(x,t,\rho) + \phi_{\rho\rho}(x,t,\rho)\frac{\partial \rho}{\partial x}\right]\right\}. \tag{3.11}$$

In dimension one, an SDE of the form

$$dX = -b(X,t)dt + a(X,t)dB_t \tag{3.12}$$

leads to the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}[b(x,t)\rho] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[a^2(x,t)\rho]. \tag{3.13}$$

It is easy to see now that, unless a is independent of x (i.e. $a = a(t)$), the right hand side of (3.13) cannot be obtained from (3.11) for any ϕ. A similar argument, although more complex, applies in arbitrary dimension. An interesting question arises at this point: is there any other type of F (not of the form $\int \phi(x,t,\rho(x,t))dx$) whose gradient flow in the Wasserstein space is precisely (3.13) for general functions a? We give a formal answer at the end of the section.

Let $\Omega \subset \mathbb{R}$ be an open interval and assume b, a are smooth functions on $\bar{\Omega}\times[0,\infty)$ such that $\alpha \le a \le \beta$ for two constants $0 < \alpha \le \beta < \infty$. Next, define ϕ, ψ on $\bar{\Omega}\times[0,\infty)$ by

$$\phi(x,t) := a^2(x,t)\exp\left(2\int_{x_0}^{x}\frac{b(y,t)}{a^2(y,t)}dy\right),\ \psi(x,t) := a^2(x,t)\exp\left(4\int_{x_0}^{x}\frac{b(y,t)}{a^2(y,t)}dy\right), \tag{3.14}$$

where $x_0 \in \Omega$ is arbitrary but fixed. Furthermore, assume $|\phi_x|$ is bounded and ψ is bounded away from 0 and ∞ in $\bar{\Omega}\times[0,T]$ for all $0 < T < \infty$.

The conditions on ϕ and ψ are automatically satisfied if, for example, Ω is a bounded interval.

Let us consider the linear system:

$$\begin{cases} \partial_t \xi_\varepsilon = -\frac{1}{\varepsilon}\partial_x(\xi_\varepsilon \phi) - \frac{1}{\varepsilon^2}\psi(\xi_\varepsilon - \eta_\varepsilon) & \text{in } \Omega\times(0,\infty), \\ \partial_t \eta_\varepsilon = \ \ \frac{1}{\varepsilon}\partial_x(\eta_\varepsilon \phi) + \frac{1}{\varepsilon^2}\psi(\xi_\varepsilon - \eta_\varepsilon) & \text{in } \Omega\times(0,\infty), \\ \quad \xi_\varepsilon = 0,\ \eta_\varepsilon = 0 & \text{on } \partial\Omega\times(0,\infty), \\ \xi_\varepsilon(\cdot,0) = \rho_0,\ \eta_\varepsilon(\cdot,0) = \rho_0 & \text{in } \Omega, \end{cases} \tag{S}$$

where $\varepsilon > 0$ is a positive integer. Obviously, this system is of the form (WS) and, since the coefficients satisfy all the requirements, it can be solved exactly as described above provided that $\rho_0 \in L^1(\Omega)$ is nonnegative.

We are inspired by [10], where we find "the simplest example that illustrates the regime in which a diffusion equation can be obtained from a velocity-jump process [...]", particles moving along the real axis at constant speed s, reversing direction at random instances according to a Poisson process with constant parameter λ. If $p^{\pm}(x,t)$ are the densities of particles at (x,t) that are moving to the right (+) and left (-), then they satisfy a system of the form (WS) with $\Psi_i = \mp s\mathrm{Id}$ and $\nu_i = \lambda$, $i = 1,2$. By letting $p = p^+ + p^-$ and $j = s(p^+ - p^-)$, a simple calculation yields

$$\begin{aligned} \partial_t p + \partial_x j &= 0, \\ \partial_t j + 2\lambda j &= -s^2\partial_x p \end{aligned}$$

which decouples to give the telegraph equation for p

$$p_{tt} + 2\lambda p_t = s^2 p_{xx}.$$

If one places $s = \sigma/\varepsilon$, $\lambda = 1/\varepsilon^2$ and lets $\varepsilon \downarrow 0$, there results the classical heat equation with diffusion constant σ^2. Next we show how the general Fokker-Planck equation is obtained by choosing appropriate variable coefficients in the system above while still maintaining the form (WS).

Theorem 4. *Let ξ_ε and η_ε be the solutions for (S) with $\rho_0 \in L^2(\Omega)$ being a probability density. Then, as $\varepsilon \downarrow 0$ both ξ_ε and η_ε converge weakly in $L^2_{loc}(\bar{\Omega} \times [0,\infty))$ to the solution of the Fokker-Planck equation (3.13) with ρ_0 as initial data.*

Proof: If we multiply the first equation by ξ_ε and the second by η_ε, then integrate twice (first in space and then in time), we obtain, using the boundary conditions,

$$\|\xi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|\eta_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \psi(x,s)\varepsilon^{-2}[\xi_\varepsilon(x,s) - \eta_\varepsilon(x,s)]^2 dxds \quad (3.15)$$

$$= 2\|\rho_0\|^2_{L^2(\Omega)} - \frac{1}{2}\int_0^t\int_\Omega \phi_x(x,s)\varepsilon^{-1}[\xi_\varepsilon(x,s) - \eta_\varepsilon(x,s)][\xi_\varepsilon(x,s) + \eta_\varepsilon(x,s)]dxds$$

for all $0 < t < \infty$. Denoting

$$u_\varepsilon(t) := \|\xi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|\eta_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)},\ v_\varepsilon(t) := \|(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon\|_{L^2(\Omega\times(0,t))},$$

we infer, due to (3.15),

$$u_\varepsilon(t) + \alpha v^2_\varepsilon(t) \le 2\|\rho_0\|^2_{L^2(\Omega)} + \frac{\|\phi_x\|_{L^\infty(\Omega_T)}}{\sqrt{2}} v_\varepsilon(t)\left(\int_0^t u_\varepsilon(s)ds\right)^{1/2} \quad (3.16)$$

for all $0 < t < T < \infty$ where $\Omega_T := \Omega \times (0,T)$. Inequality (3.16) implies, for any $\delta > 0$,

$$u_\varepsilon(t) + \alpha v^2_\varepsilon(t) \le 2\|\rho_0\|^2_{L^2(\Omega)} + \frac{\|\phi_x\|_{L^\infty(\Omega_T)}}{2\sqrt{2}}\left\{\delta v^2_\varepsilon(t) + \frac{1}{\delta}\int_0^t u_\varepsilon(s)ds\right\}. \quad (3.17)$$

By choosing δ small enough, we obtain,

$$u_\varepsilon(t) + \frac{\alpha}{2} v^2_\varepsilon(t) \le 2\|\rho_0\|^2_{L^2(\Omega)} + \frac{\|\phi_x\|_{L^\infty(\Omega_T)}}{2\sqrt{2}\delta}\int_0^t u_\varepsilon(s)ds. \quad (3.18)$$

We can neglect the terms containing v_ε and apply Gronwall's lemma to deduce that $u_\varepsilon(t)$ is bounded independently of ε. So is $\int_0^t u_\varepsilon(s)ds$ and we go back to (3.18) to infer v_ε enjoys the same property. Therefore, ξ_ε, η_ε and $(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon$ are uniformly bounded in $L^2(\Omega_T)$ for all finite $T > 0$. Up to a subsequence, they are weakly convergent in $L^2(\Omega_T)$. We do not relabel and, due to the convergence of the latter, the former two converge to the same limit, say, $\rho \in L^2(\Omega_T)$.
We will next prove that ρ is the solution for (3.13) with initial data ρ_0 (and natural BC if Ω is bounded). Let us begin by adding the first two equations to obtain

$$(\xi_\varepsilon + \eta_\varepsilon)_t = -[\phi(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon]_x.$$

Let f denote the L^2-weak limit of $(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon$ as $\varepsilon \downarrow 0$, so that

$$2\rho_t = -(f\phi)_x. \quad (3.19)$$

Note that the weak convergence of $\phi(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon$ to $f\phi$ holds due to the boundedness of ϕ.
Now we subtract the first two equations in the system to obtain

$$(\xi_\varepsilon - \eta_\varepsilon)_t = -[\phi(\xi_\varepsilon + \eta_\varepsilon)/\varepsilon]_x - \psi[(\xi_\varepsilon - \eta_\varepsilon)/\varepsilon]/\varepsilon.$$

If we multiply by ε and pass to the limit as $\varepsilon \downarrow 0$ we further obtain

$$0 = -2(\rho\phi)_x - 2\psi f. \quad (3.20)$$

We combine (3.19) and (3.20) by eliminating f to write

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\frac{\partial}{\partial x}(\phi\rho)\right).$$

It is easy to see by direct computation that this PDE is equivalent to (3.13).

Note that, if we tried to write (3.13) as the Wasserstein-gradient flow of some functional F, we would end up with the equation

$$\frac{\partial}{\partial x}\frac{\delta F}{\delta\rho}(t,\rho)(x) = -b(x,t) - a(x,t)a_x(x,t) - \frac{a^2(x,t)}{2}\frac{\partial}{\partial x}\log\rho(x,t). \tag{3.21}$$

If B is a primitive (in x) of b, then we should have

$$\frac{\delta F}{\delta\rho}(t,\rho)(x) = -B(x,t) - \frac{1}{2}a^2(x,t)[1+\log\rho(x,t)] + \int_{x_0}^{x} a(y,t)a_y(y,t)\log\rho(y,t)dy$$

for some x_0, maybe time-dependent. The first two terms from the right hand side can trivially be written as the L^2-gradients of some functionals of the form $\int\phi(x,t,\rho(x,t))dx$. However, we argue that the remaining integral term cannot be the L^2-gradient of any functional, not necessarily of the form mentioned before, unless ϕ is independent of x, rendering the term zero!

In the following formal discussion we drop the dependence on t (which is irrelevant in this context) and we let $f(x) = a(x)a'(x)$. Indeed, if there is an $\mathcal{F}$ such that

$$\frac{\delta\mathcal{F}}{\delta\rho}(\rho)(x) = \int_{x_0}^{x} f(y)\log\rho(y)dy,$$

then the Gateaux derivative of $\mathcal{F}$ in the v-direction will be

$$\langle\mathcal{F}'(\rho), v\rangle = \int_{\mathbb{R}} v(x)\int_{x_0}^{x} f(y)\log\rho(y)dydx.$$

Assume that ρ is bounded from below away from zero and all integration requirements are in place. Then the map $\rho\to\langle\mathcal{F}'(\rho), v\rangle$ is obviously C^∞, so $\mathcal{F}$ must be C^∞. Consequently, the second Gateaux derivative must be a symmetric, bilinear form. However, we have

$$\langle\mathcal{F}''(\rho), (v_1, v_2)\rangle = \int_{\mathbb{R}} v_1(x)\int_{x_0}^{x}\frac{f(y)v_2(y)}{\rho(y)}dydx$$

which, in general, violates

$$\langle\mathcal{F}''(\rho), (v_1, v_2)\rangle = \langle\mathcal{F}''(\rho), (v_2, v_1)\rangle.$$

Can Theorem 4 be extended to arbitrary dimensions? The answer is yes, it can be, but only for some special Fokker-Planck equations. Here are some details.

We rewrite (3.9) in divergence form as

$$\frac{\partial\rho}{\partial t} = \mathrm{div}\big\{-\big[\mathbf{A}(x,t)\rho\big] + \frac{1}{2}\rho\,\mathbf{div}\mathcal{Q}\big\} \tag{3.22}$$

where $\mathcal{Q}(x,t) := \mathcal{B}(x,t)\mathcal{B}^T(x,t)$ has rows denoted by Q_i and $\mathbf{div}\mathcal{Q}$ is the N-vector field whose components are $\mathrm{div}Q_i$ (all derivatives are spatial). In dimension N, ϕ from (S) is replaced by an N-vector field denoted by Φ. Thus, the N-dimensional version of (S) is

$$\begin{cases} \partial_t\xi_\varepsilon = -\varepsilon^{-1}\mathrm{div}(\xi_\varepsilon\Phi) - \varepsilon^{-2}\psi(\xi_\varepsilon-\eta_\varepsilon) \text{ in } \mathbb{R}^N\times(0,\infty), \\ \partial_t\eta_\varepsilon = \quad\varepsilon^{-1}\mathrm{div}(\eta_\varepsilon\Phi) + \varepsilon^{-2}\psi(\xi_\varepsilon-\eta_\varepsilon) \text{ in } \mathbb{R}^N\times(0,\infty), \\ \xi_\varepsilon(\cdot,0)=\rho_0,\ \eta_\varepsilon(\cdot,0)=\rho_0 \qquad \text{in } \mathbb{R}^N. \end{cases} \tag{S_N}$$

The assumption $|\phi_x|$ is bounded is now replaced by $\mathrm{div}\Phi \in L^\infty(\mathbb{R}^N \times (0,T))$ for all $T > 0$. One can retrace the proof of Theorem 4 with only minor changes to conclude that, as $\varepsilon \downarrow 0$, both ξ_ε and η_ε converge weakly in $L^2(\mathbb{R}^N \times (0,T))$ to the solution ρ for the second-order PDE

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\mathrm{div}\Big(\frac{\Phi}{\psi}\mathrm{div}(\Phi\rho)\Big). \tag{3.23}$$

It is intuitively obvious that such a PDE can be identified with (3.22) (and, thus, with (3.9)) only in some very special cases. Indeed, there are $N^2 + N$ independent entries/components in $\mathcal{B}$ and $\mathbf{A}$, whereas only $N+1$ in Φ and ψ. They only match, of course, when $N = 1$. Next we will see what is required for Φ and ψ to be determined from $\mathcal{B}$ and $\mathbf{A}$ if $N > 1$.
We equate the right hand sides of (3.22) and (3.23) to obtain, componentwise,

$$-A_i\rho + \frac{1}{2}\mathrm{div}(\rho Q_i)\frac{\Phi_i}{2\psi}\mathrm{div}(\rho\Phi)$$

for all $i = 1..N$, where A_i are the components of $\mathbf{A}$. If we impose these equations independently of ρ, we obtain, after identifying the coefficients of ρ and $\nabla\rho$,

$$-A_i + \frac{1}{2}\mathrm{div}Q_i = \frac{\Phi_i}{2\psi}\mathrm{div}\Phi \tag{3.24}$$

and

$$Q_i = \frac{\Phi_i}{\psi}\Phi \tag{3.25}$$

for all $i = 1..N$. The latter yields $Q_{ij} = \Phi_i\Phi_j/\psi$ which basically means that $Q_{ij} := q_i q_j$ for some N-vector field $\mathbf{q}$. We deduce $\Phi_i = q_i\psi^{1/2}$ for all $i = 1..N$. Going back to (3.24) we discover, after some computation,

$$\mathbf{q}\cdot\nabla\log\psi = -4A_i/q_i + 2\mathbf{q}\cdot\nabla\log q_i$$

for all $i = 1..N$. To summarize, we need $\mathcal{Q}$ of the form $(q_iq_j)_{ij}$ for some vector $\mathbf{q}$ and the following compatibility condition

$$-2A_i/q_i + \mathbf{q}\cdot\nabla\log q_i = \lambda(x,t) \text{ is independent of } i. \tag{3.26}$$

Then we choose ψ as any solution of $\mathbf{q}\cdot\nabla\log\psi = 2\lambda$.
Note that $\mathcal{B}$ has to have identical entries on each row, i.e. $B_{ij} = b_i$ for some vector field $\mathbf{b}(x,t)$, and $q_i = b_iN^{1/2}$.

4. **More applications.**

4.1. **Velocity-jump processes.** Let $V \subset \mathbb{R}^N$ be open and the *turning kernel* $\mathcal{T} : V \times V \to [0,\infty)$ (see [10]) with the following properties

$$\int_V \mathcal{T}(v,w)dw = 1 \text{ for all } v \in V \tag{4.1}$$

and

$$\int_V\int_V \mathcal{T}^2(v,w)dvdw =: M < \infty. \tag{4.2}$$

Next consider the transport equation:

$$\frac{\partial}{\partial t}\rho(x,v,t) + v\cdot\nabla_x\rho(x,v,t) = -\lambda\rho(x,v,t) + \lambda\int_V \mathcal{T}(v,w)\rho(x,w,t)dw \tag{4.3}$$

describing a *velocity-jump* process ([15]). Here $\rho(x,v,t)$ denotes the density of particles at position $x \in \mathbb{R}^N$ which move with velocity $v \in V$ at time $t \geq 0$. The

constant $\lambda > 0$ is the *turning rate* while $1/\lambda$ measures the mean run length between velocity jumps. The kernel $\mathcal{T}(v, w)$ gives the probability of a jump from w to v if a jump occurs.
Denote by $F(x, v; \rho) := -\lambda\rho + \lambda \int_V \mathcal{T}(v, w)\rho(x, w)dw$ and consider the IVP:

$$\begin{cases} \rho_t + v \cdot \nabla_x \rho = F(x, v; \rho) \text{ in } \mathbb{R}^N \times V \times (0, \infty), \\ \rho(x, v, 0) = \rho_0(x, v) \quad \text{for all } (x, v) \in \mathbb{R}^N \times V. \end{cases} \tag{4.4}$$

Consider initial data ρ_0 such that

$$\rho_0(\cdot, v) \in \mathcal{M}(\mathbb{R}^N) \text{ for almost all } v \in V, \tag{4.5}$$

and

$$\rho_0 \in L^2(\mathbb{R}^N \times V). \tag{4.6}$$

In [20], [19] the authors exploit the implicit scheme introduced by Kinderlehrer and Walkington in [14] for proving existence of solutions for nonhomogeneous diffusion problems. This approach modifies the standard schemes to accommodate the nonhomogeneous term. Here we fix $v \in V$ and consider the following version of (2.4):

For every integer $k \geq 1$ we define ρ_k as the solution of

$$\frac{1}{2\tau} d(\rho, \rho_{k-1,1})^2 + \int_{\mathbb{R}^N} (x \cdot v)\rho(x)dx = \min, \tag{4.7}$$

where $\rho_{k,1} := \rho_k + \tau F(\cdot, \cdot; \rho_k)$ for all integers $k \geq 0$.

Note that the above scheme assumes that $\rho_{k,1} \in \mathcal{M}(\mathbb{R}^N)$. Indeed, we have

$$\rho_{k,1}(x, v) = (1 - \lambda\tau)\rho_k(x, v) + \lambda\tau \int_V \mathcal{T}(v, w)\rho_k(x, w)dw. \tag{4.8}$$

By integrating (4.8) with respect to x and using induction, we see that (4.1) and (4.5) yield

$$\rho_{k,1}(\cdot, v) \in \mathcal{M}(\mathbb{R}^N) \text{ for all } k \geq 0,\ 0 < \tau \leq \frac{1}{\lambda} \text{ and for almost all } v \in V. \tag{4.9}$$

According to the previous chapter, the optimal transfer map is the translation $\psi_\tau := \mathrm{id} + \tau v$. Along with (4.8), this leads to

$$\begin{aligned} \rho_k(x, v) &:= \rho_{k-1,1}(x + \tau v, v) \\ &= (1 - h)\rho_{k-1}(x + \tau v, v) + h \int_V \mathcal{T}(v, w)\rho_{k-1}(x + \tau v, w)dw, \end{aligned} \tag{4.10}$$

where $h := \lambda\tau$. This implies

$$|\rho_k(x, v)|^2 \leq (1 + h)(1 - h)^2 |\rho_{k-1}(x + \tau v, v)|^2 + \left(1 + \frac{1}{h}\right) h^2 \left(\int_V \mathcal{T}(v, w)\rho_{k-1}(x + \tau v, w)dw \right)^2.$$

It follows, after using Holder inequality for the last term and then integrating in x over $\mathbb{R}^N$ ($\|\cdot\|_2$ denotes the L^2-norm in $\mathbb{R}^N$)

$$\|\rho_k(\cdot, v)\|_2^2 \leq (1 - h)(1 - h^2)\|\rho_{k-1}(\cdot, v)\|_2^2 + h(1 + h) \int_V \|\rho_{k-1}(\cdot, v)\|_2^2 dv \int_V \mathcal{T}^2(v, w)dw. \tag{4.11}$$

Now denote by $\|\cdot\|_{2,V}$ the L^2-norm in $\mathbb{R}^N \times V$ and integrate (4.11) over V to obtain

$$\|\rho_k\|_{2,V}^2 \leq (1 - h)(1 - h^2)\|\rho_{k-1}\|_{2,V}^2 + h(1 + h)\|\rho_{k-1}\|_{2,V}^2 \int_V \int_V \mathcal{T}^2(v, w)dvdw.$$

According to (4.2) we get

$$\|\rho_k\|_{2,V}^2 \leq \|\rho_{k-1}\|_{2,V}^2\{(1-h)(1-h^2)+Mh(1+h)\}.$$

Let $a := M-1$. Then, for small enough τ, we have (by induction)

$$\|\rho_k\|_{2,V}^2 \leq \|\rho_0\|_{2,V}^2[1+ah(1+h)+h^3]^k. \tag{4.12}$$

According to (4.2) and (4.6), $\mathcal{T}(v,\cdot) \in L^2(V)$ and $\rho_0(\cdot,v) \in L^2(\mathbb{R}^N)$ for almost all $v \in V$. Consequently, the set $V' \subset V$ defined below is of full measure:

$$V' := \left\{ v \in V \;\middle|\; \int_V \mathcal{T}^2(v,w)dw =: M(v) < \infty,\ \rho_0(\cdot,v) \in \mathcal{M}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \right\}. \tag{4.13}$$

Fix $v \in V'$. Then (4.11) and (4.12) imply

$$\begin{aligned}\|\rho_k(\cdot,v)\|_2^2 \leq{}& (1-h)(1-h^2)\|\rho_{k-1}(\cdot,v)\|_2^2 \\ &+M(v)\|\rho_0\|_{2,V}^2 h(1+h)[1+ah(1+h)+h^3]^{k-1}.\end{aligned} \tag{4.14}$$

If we let $a_k := \|\rho_k(\cdot,v)\|_2^2$, $\alpha_h := (1-h)(1-h^2)$, $\beta := M(v)\|\rho_0\|_{2,V}^2$ and $c_h := 1+ah(1+h)+h^3$, then (4.14) reads

$$a_k \leq \alpha_h a_{k-1} + \beta h(1+h)c_h^{k-1} \text{ for all integers } k \geq 1.$$

It follows

$$a_k \leq a_0\alpha_h^k + \frac{\beta}{M}(c_h^k - \alpha_h^k)\left(a_0 - \frac{\beta}{M}\right)\alpha_h^k + \frac{\beta}{M}c_h^k.$$

Therefore,

$$\begin{aligned}\|\rho^\tau(\cdot,\cdot,v)\|_{L^2}^2 &= \tau\sum_{k=0}^{n-1} a_k \\ &\leq \frac{1}{\lambda}\left\{ \left(a_0 - \frac{\beta}{M}\right)\frac{1-\alpha_h^{\lambda T/h}}{1+h-h^2} + \frac{\beta}{M}\frac{1-c_h^{\lambda T/h}}{(1-M)+h-h^2}\right\}\end{aligned} \tag{4.15}$$

which leads to the fact that $\{\rho^\tau\}_{\tau\downarrow 0}$ is bounded in $L^2(\mathbb{R}^N \times V \times (0,T))$. Indeed, (4.2) and (4.6) imply the integrability over V of the right hand side of (4.15). Next we extract (no relabelling) a subsequence $\{\rho^\tau\}_{\tau\downarrow 0}$ such that

$$\rho^\tau \rightharpoonup \rho \text{ weakly in } L^2(\mathbb{R}^N \times V \times (0,T)) \tag{4.16}$$

Again, due to (4.15) there exists a subsequence $\{\tau_v\}$ depending on v such that

$$\rho^{\tau_v}(\cdot,v,\cdot) \rightharpoonup \rho(\cdot,v,\cdot) \text{ weakly in } L^2(\mathbb{R}^N \times (0,T)). \tag{4.17}$$

By taking the limit as $\tau \downarrow 0$ in (4.15), we observe

$$\|\rho(\cdot,v,\cdot)\|_{L^2(\mathbb{R}^N\times(0,T))}^2 \leq \left(a_0 - \frac{\beta}{M}\right)\frac{1-e^{-\lambda T}}{\lambda} + \frac{\beta}{M}\mathcal{C}(M,T,\lambda), \tag{4.18}$$

where

$$\mathcal{C}(M,T,\lambda) := \frac{e^{\lambda T(M-1)}-1}{\lambda(M-1)} \text{ if } M \neq 1 \text{ and } \mathcal{C}(M,T,\lambda) := T \text{ if } M = 1.$$

Our goal is to prove the following

Proposition 5. *Let $\mathcal{T}$ and ρ_0 be nonnegative and such that (4.1), (4.2), (4.5) and (4.6) are satisfied. Then the initial-value problem (4.4) admits a weak solution in $\Omega_T := \mathbb{R}^N \times V \times (0,T)$ verifying*

$$\|\rho\|_{L^2(\Omega_T)}^2 \leq \|\rho_0\|_{2,V}^2\ \mathcal{C}(M,T,\lambda). \tag{4.19}$$

Let us now define what a weak solution is:

Definition 2. A weak solution for (4.4) is a function $\rho(x,v,t)$ satisfying
(i) $\rho \in L^2(\mathbb{R}^N \times V \times (0,T))$;
(ii) For every $\zeta \in C_c^\infty(\mathbb{R}^N \times [0,T))$ and all $v \in V'$ we have

$$\int_0^T \int_{\mathbb{R}^N} \{\rho \partial_t \zeta - (v \cdot \nabla \zeta)\rho - \lambda[\rho(\cdot,v,\cdot) \\ - \int_V \mathcal{T}(v,w)\rho(\cdot,w,\cdot)dw]\zeta\} dxdt = - \int_{\mathbb{R}^N} \rho_0(x,v)\zeta(x,0)dx.$$

We now drop the subscript in τ_v to simplify notation. According to Proposition 3, the approximate Euler equation for the variational principle (4.7) reads

$$\left| \int_\Omega \left\{ \frac{1}{\tau}(\rho_k - \rho_{k-1,1})\zeta + \rho_k(v \cdot \nabla \zeta) \right\} dx \right| \leq \frac{1}{2\tau} \sup_{\mathbb{R}^N} |\nabla^2 \zeta| \; d(\rho_{k-1,1}, \rho_k)^2, \tag{4.20}$$

for every $\zeta \in C_c^\infty(\mathbb{R}^N)$ and all integers $k \geq 1$. Taking (4.10) into account, we infer

$$d(\rho_{k-1,1}, \rho_k) = \tau |v|.$$

Subsequently, we integrate (4.20) over $[k\tau, (k+1)\tau]$ with respect to time, add for $k = 1..n-1$ and recall the definition of ρ^τ to obtain

$$\Big| - \int_\tau^{T-\tau} \int_\Omega \rho^\tau(x,v,t) \frac{1}{\tau}(\zeta(x,t+\tau) - \zeta(x,t))dxdt - \frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \rho_0(x,v)\zeta(x,t+\tau)dxdt \\ + \int_\tau^T \int_{\mathbb{R}^N} \rho^\tau(\cdot,v,\cdot)(v \cdot \nabla\zeta)dxdt + \frac{1}{\tau}\int_{T-\tau}^T \int_{\mathbb{R}^N} \rho^\tau(\cdot,v,\cdot)\zeta dxdt \\ - \int_0^{T-\tau} \int_{\mathbb{R}^N} F(\cdot,v,\rho^\tau)\zeta dxdt \Big| \leq \frac{1}{2}\|\nabla^2\zeta\|_\infty \sum_{k=1}^{n-1} \tau^2 |v|^2 \leq \frac{1}{2}\|\nabla^2\zeta\|_\infty T\tau|v|^2.$$

Let $\tau \downarrow 0$. Finally, we use (4.17) for most of the terms in the left hand side of the above display and (4.16) for the last, nonlocal term to pass to the limit as $\tau \downarrow 0$ and see that ρ is a weak solution for (4.4). Clearly, (4.19) is an immediate consequence of (4.18).

Second-order partial differential equations involving the convective derivative $\partial/\partial t + v \cdot \nabla_x$ have been studied via optimal transportation methods by Carlen and Gangbo [4], Huang and Jordan [12], Huang [11]. However, since there are no v-derivatives in (4.4), the differential operator from the left hand side represents ordinary transport in the x-direction and has nothing to do with convection.

4.2. **General first-order transport systems.** Here we analyze similar schemes for systems of first-order equations. Let Ω be either $\mathbb{R}$ or $(0,1)$. We next study the IVP

$$\rho_t = (\rho\Phi)_x \text{ in } \Omega \times (0,\infty) \text{ for all } i = 1..d \text{ and } \rho(\cdot,0) = \rho_0 \text{ in } \Omega, \tag{4.21}$$

where $\rho = (\rho^1, ..., \rho^d)$ and $\Phi \in \mathbb{M}^{d\times d}$ is a matrix function. Let $1 < p \leq \infty$. For given $\rho_0 \in \left[\mathcal{M}_2(\Omega)\right]^d \cap L^p(\Omega;\mathbb{R}^d)$ and small enough $\tau > 0$ we wish to construct iteratively ρ_k, then obtain ρ^τ by interpolating time (as in (2.5)). Next we give the definition of a weak solution for our system.

Definition 3. A weak solution for (4.21) is a vector field $\rho : \Omega \times (0,\infty) \to \mathbb{R}^d$ satisfying
(i) $\rho^i \in L^\infty_{loc}((0,\infty); L^p(\Omega))$ for all $i = 1..d$;
(ii) for every $\zeta \in C_c^\infty(\mathbb{R} \times [0,\infty); \mathbb{R}^d)$ we have

$$\sum_{i=1}^d \iint_{\Omega_\infty} \left\{ \rho^i \frac{\partial \zeta^i}{\partial t} - \sum_{j=1}^d \rho^j \Phi_{ij} \frac{\partial \zeta^i}{\partial x} \right\} dx dt = - \sum_{i=1}^d \int_\Omega \rho_0^i \zeta^i(\cdot, 0) dx. \tag{4.22}$$

The discrete scheme used in [5] to show existence only works for weakly coupled systems, i. e. Ψ is diagonal and there is a zero-order term $F(x,t,\rho)$ in the equation (if $F \equiv 0$ then the system is completely decoupled which is not interesting). Our goal here is to come up with an appropriate discrete scheme for (4.21). The trick is to first look at the approximate Euler equation (2.9) and consider what should be changed to replace it with something of the form:

$$\left| \int_\Omega \left\{ \frac{1}{\tau}(\rho_k^i - \rho_{k-1}^i)\zeta + \sum_{j=1}^d \rho_k^j \Phi_{ij} \zeta' \right\} dx \right| \le O(\tau) \sup_{\mathbb{R}} |\zeta''|, \tag{4.23}$$

for all $i = 1..d$ and all $\zeta \in C_c^\infty(\mathbb{R})$. Denote by s_i the transfer map pushing ρ_{k-1}^i into ρ_k^i. Then (4.23) becomes

$$\left| \int_\Omega \left\{ [\zeta(s_i(x)) - \zeta(x)]\rho_{k-1}^i + \tau \sum_{j=1}^d \rho_{k-1}^j \Phi_{ij}(s_j) \zeta'(s_j) \right\} dx \right| \le o(\tau) \sup_{\mathbb{R}} |\zeta''|, \tag{4.24}$$

where $\tau^{-1} o(\tau) \to 0$ as $\tau \downarrow 0$. To simplify, let us assume that $\rho^i := \rho_{k-1}^i$ are given strictly positive and smooth such that ρ^i/ρ^j is bounded away from zero and infinity for all $i,\ j$. Then, the problem is to find s_i's such that (4.24) holds for all ζ, all i and for $o(\tau)$ independent of the step $k-1 \to k$. We have $\zeta(s_i(x)) - \zeta(x) = (s_i(x) - x)\zeta'(x) + |s_i(x) - x|^2 \zeta''(y)$ for some y between x and $s_i(x)$. Also, we add and subtract $\tau \sum_{j=1}^d \rho_{k-1}^j \Phi_{ij}(s_j)\zeta'$ in the integrand from (4.24) and assume we are able to prove that

$$\int_\Omega \sum_{j=1}^d \rho_{k-1}^j \Phi_{ij}(s_j)(\zeta'(s_j) - \zeta') dx \sim O(\tau),\ d(\rho_{k-1}^i, \rho_k^i) \sim O(\tau) \tag{4.25}$$

for s_i chosen so that

$$\psi_\tau^i(s_i) + \tau \sum_{j \ne i}^d (\rho^j/\rho^i) \Phi_{ij}(s_j) = \text{id in } \Omega, \tag{4.26}$$

where $\psi_\tau^i := \text{id} + \tau \Phi_{ii}$ for all $i = 1..d$. Obviously, by assuming $\Phi_{ij} \equiv 0$ for $i \ne j$, the equations (4.26) become d completely decoupled equations with the solutions $s_i = (\psi_\tau^i)^{-1}$. Thus, we are back to single equations.

Our goal is to establish a minimal set of assumptions on the matrix Φ in order to ensure existence and uniqueness of the s_i's which must also have requisite properties, e.g., to imply (4.25).
For clarity of exposition we consider the $d = 2$ case, i.e. two by two systems. Then (4.26) becomes

$$\begin{cases} \psi_\tau^1(s_1) + \tau \frac{\rho^2(x)}{\rho^1(x)} \Phi_{12}(s_2) = x, \\ \psi_\tau^2(s_2) + \tau \frac{\rho^1(x)}{\rho^2(x)} \Phi_{21}(s_1) = x, \end{cases} \tag{4.27}$$

which is to be solved to give $s_i := s_i(x)$ for all $x \in \Omega$. Assume, as in the previous section, that $\Phi_{ii} \in C^1(\Omega) \cap W_0^{1,\infty}(\Omega)$, $i = 1, 2$ (so that, for τ sufficiently small, ψ_τ^i are both diffeomorphisms of Ω). Our goal is to prove that (4.27) has a unique solution (s_1, s_2) and then study its properties. The interesting feature of the $d = 2$ case is that (4.27) decouples and we have the equations:

$$\psi_\tau^1(s_1) + \tau \frac{\rho^2(x)}{\rho^1(x)} \Phi_{12} \circ \theta_\tau^2 (x - \tau \frac{\rho^1(x)}{\rho^2(x)} \Phi_{21}(s_1)) - x = 0 \tag{4.28}$$

and

$$\psi_\tau^2(s_2) + \tau \frac{\rho^1(x)}{\rho^2(x)} \Phi_{21} \circ \theta_\tau^1 (x - \tau \frac{\rho^2(x)}{\rho^1(x)} \Phi_{12}(s_2)) - x = 0 \tag{4.29}$$

for all $x \in \Omega$, where $\theta_\tau^i := (\psi_\tau^i)^{-1}$.
Furthermore, let us assume $\Omega = \mathbb{R}$. We can prove:

Lemma 2. *The system (4.27) has a unique solution (s_1, s_2) for sufficiently small $\tau > 0$. Also, $s_i \in \mathrm{Diff}(\mathbb{R})$ for $i = 1, 2$.*

Proof: Let us only analyze (4.28) since (4.29) is similar. For $x \in \mathbb{R}$, we define $F(x, \cdot) : \mathbb{R} \to \mathbb{R}$ by

$$F(x, s) := \theta_\tau^1 (x - \tau\beta(x)\Phi_{12} \circ \theta_\tau^2 (x - \tau\alpha(x)\Phi_{21}(s))),$$

where $\alpha := \rho^1/\rho^2$ and $\beta := \rho^2/\rho^1$. Obviously, F is differentiable in both x and s. Furthermore,

$$\frac{\partial}{\partial s} F(x, s) = \frac{\tau \Phi_{12}'(a_1)}{1 + \tau \Phi_{11}'(a_2)} \frac{\tau \Phi_{21}'(s)}{1 + \tau \Phi_{22}'(a_3)}, \tag{4.30}$$

where a_k, $k = 1, 2, 3$ are functions of $x,\ s,\ \tau$. If Φ_{ij}' are bounded in $\mathbb{R}$, then $F(x, \cdot)$ is a contraction for all $x \in \mathbb{R}$ and all sufficiently small $\tau > 0$. Therefore, the equation $s = F(x, s)$ has a unique solution $s = s_1(x)$ for each $x \in \mathbb{R}$. Therefore, there exists a unique map $s_1 : \mathbb{R} \to \mathbb{R}$ such that $s_1(x) = F(x, s_1(x))$ for all $x \in \mathbb{R}$. In order to see that $s_1 : \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, it suffices to show that $s = F(x, s)$ has a unique solution x for all $s \in \mathbb{R}$. This latter equation is equivalent to:

$$x = \psi_\tau^1(s) + \tau\beta(x)\Phi_{12} \circ \theta_\tau^2 (x - \tau\alpha(x)\Phi_{21}(s)) \tag{4.31}$$

and it is elementary to prove that the r.h.s. is a contraction in x (for yet smaller $\tau > 0$), ensuring thus the existence and uniqueness of x as desired. Therefore, $s = s_1(x)$ is the unique map satisfying $s(x) = F(x, s(x))$ for all $x \in \mathbb{R}$. The Implicit Functions Theorem yields that s is differentiable and $s'(x) = F_x(x, s(x)) + s'(x) F_s(x, s(x))$. Idem for s_2. Also, in order for s_i to be transfer maps, we need them to be strictly increasing. According to the previous proof, $s_1'(x)(1 - F_s(x, s_1(x))) = F_x(x, s_1(x))$. Since F_s is given in (4.30), let us now write F_x in detail. We have:

$$\frac{\partial}{\partial x} F(x, s) = \frac{1}{1 + \tau\Phi_{11}'(b_1)} \big\{1 - \tau\beta'(x)\Phi_{12}(b_2) - \tau\beta(x)\Phi_{12}'(b_3)[1 - \tau\alpha'(x)\Phi_{21}(s)]\big\},$$

for b_k, $k = 1, 2, 3$ functions of $x,\ s,\ \tau$. It is now clear that $s_1' = F_x/(1 - F_s) \sim 1$ for small enough $\tau > 0$. Same goes for s_2.
If we let $r_i := s_i^{-1}$, then we define $\rho_k^i := (\rho_{k-1}^i \circ r_i) r_i'$. It is now easy to prove, since $r_i \sim \mathrm{id} + O(\tau)$ and $r_i' \sim 1 + O(\tau)$, that (4.25) are true. However, the choice of the appropriate τ depends on the L^∞ norms of $\rho_{k-1}^1/\rho_{k-1}^2$, $\rho_{k-1}^2/\rho_{k-1}^1$ and, what is worse, on the norms of their derivatives.

5. **General costs.** It is interesting to realize that only for gradient vector fields do we obtain the solution approximants by steepest descent or discretized gradient flow of the functional $\rho \to \int_\Omega \rho\Psi dx$ with respect to the quadratic Wasserstein distance on $\mathcal{M}(\Omega)$. We are then led to existence of a weak solution in $\Omega \times (0,\infty)$. It may seem that no such interpretation is available for the nonconservative case. However, while the steepest descent interpretation becomes unavailable, in some special cases we may still employ time-discretized implicit schemes involving generalized time-step scaled Kantorovich distances to obtain solutions for all times. To this effect, consider the *Hamiltonian* $H : \mathbb{R}^N \to [0,\infty)$ of class C^2 satisfying

$$H(0) = 0,\ H \text{ is } \textit{strictly convex}, \text{ i.e. } H \text{ is convex and } \det(\nabla^2 H) > 0 \text{ in } \mathbb{R}^N, \quad (5.1)$$

and

$$H \text{ is } \textit{coercive}, \text{ i.e. } \lim_{|x|\uparrow\infty} \frac{H(x)}{|x|} = \infty. \quad (5.2)$$

Let $\tau > 0$ and define (see, e.g., [1])

$$W_L^\tau(\rho_0,\rho_1) := \inf_{p\in P} \iint_{\mathbb{R}^N\times\mathbb{R}^N} L\left(\frac{x-y}{\tau}\right) dp(x,y), \quad (5.3)$$

where $\rho_0,\ \rho_1 \in \mathcal{M}(\mathbb{R}^N)$, $L := H^*$ is the Legendre transform of H and P is the set of all Borel probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals $\rho_0 dx$ and $\rho_1 dx$ (see [1]). Next we analyze the following version of (2.4):

For every integer $k \geq 1$ we define ρ_k as the solution of

$$\tau W_L^\tau(\rho,\rho_{k-1}) + \int_{\mathbb{R}^N} \Psi(x)\rho(x)dx =: I_\tau[\rho_{k-1}] = \min. \quad (5.4)$$

Assume $\rho_0 > 0$ a.e. and note that, as in the proof of Proposition 1, we may write

$$I_\tau[\rho_{k-1}](\mu) = \int_{\mathbb{R}^N} \left\{\tau L\left(\frac{x - s^\mu(x)}{\tau}\right) + \Psi(s^\mu(x))\right\}\rho_{k-1}(x)dx, \quad (5.5)$$

where $\mu \in \mathcal{P}(\mathbb{R}^N)$ and s^μ is the unique transfer map pushing forward $\rho_{k-1}dx$ to μ that achieves $W_L^\tau(\mu,\rho_{k-1})$ according to its definition (5.3) (note that it can easily be extended to probability measures not necessarily absolutely continuous with respect to the Lebesgue measure). It is well-known from the classical theory of Hamilton-Jacobi equations (e.g., [8]) that, for all $x \in \mathbb{R}^N$, there exists a unique minimizer $s_\tau := (\mathrm{id} + \tau\nabla H \circ \nabla\Psi)^{-1}$ for the integrand in (5.5) which does not depend on the step k and

$$u(x,\tau) := \min_{y\in\mathbb{R}^N} \left\{\tau L\left(\frac{x-y}{\tau}\right) + \Psi(y)\right\} = \tau L\left(\frac{x - s_\tau(x)}{\tau}\right) + \Psi(s_\tau(x)) \quad (5.6)$$

is the unique semiconcave solution for the Hamilton-Jacobi IVP:

$$\begin{cases} u_t + H(\nabla u) = 0 \ \text{ in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = \Psi \text{ in } \mathbb{R}^N. \end{cases} \quad (5.7)$$

Therefore, the scheme (5.4) admits a unique solution $\{\rho_k\}_k$. Based on this, we recall the construction of the time-interpolants ρ^τ which, under some conditions, can be shown to converge to the weak solution for (2.13) in $\mathbb{R}^N \times (0,\infty)$. Let us next sketch the proof of the following:

Proposition 6. *Let $\Phi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ be of the form $\Phi = \nabla H \circ \nabla\Psi$, where $H \in C^2(\mathbb{R}^N)$ is convex with $H(0) = 0$ as its minimum and such that $H(x) \leq K|x|^2$ for all $x \in \mathbb{R}^N$ and some $K > 0$. Assume $|\nabla^2 H \circ \nabla\Psi|,\ |\nabla^2\Psi| \in L^\infty(\mathbb{R}^N)$ and*

$\rho_0 \in \mathcal{M}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ *for some* $1 < p \le \infty$. *Then there exists a weak solution for (2.13) in* $\mathbb{R}^N \times (0, \infty)$.

Proof: Note that, for every $\epsilon > 0$, $H^\epsilon(x) := H(x) + \epsilon|x|^2/2$ satisfies (5.1) and (5.2). We will show existence of solutions ρ_ϵ for (2.13) with $\Phi^\epsilon := \nabla H^\epsilon \circ \nabla\Psi$ instead of Φ. Then we will pass to the limit as $\epsilon \downarrow 0$ and obtain a weak solution for our problem in view of the uniform boundedness of ρ_ϵ in L^p. Note that

$$\begin{aligned} \mathrm{div}\Phi^\epsilon &= \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 H^\epsilon}{\partial x_i \partial x_j}(\nabla\Psi)\frac{\partial^2 \Psi}{\partial x_i \partial x_j} \\ &= (\nabla^2 H \circ \nabla\Psi + \epsilon\mathbf{1}) \cdot \nabla^2\Psi \\ &= (\nabla^2 H \circ \nabla\Psi) \cdot \nabla^2\Psi + \epsilon\Delta\Psi. \end{aligned}$$

According to the hypothesis, this equality shows that if we can prove existence of ρ_ϵ as the limit as $\tau \downarrow 0$ of the time-interpolants ρ_ϵ^τ, then (2.7) or (2.8) is sufficient to infer uniform bounds in L^p for ρ_ϵ. In order to prove existence of ρ_ϵ note that, the inequality (2.7) or (2.8) gives the boundedness of $\{\rho_\epsilon^\tau\}_\tau$ in $L^p_{loc}(\mathbb{R}^N \times (0,\infty))$. This leads to the weak (or weak $\star$) convergence of a subsequence to ρ_ϵ. To show that ρ_ϵ is the expected weak solution we can apply Proposition 3 combined with an appropriate version of (2.14) or (2.15). Note that, instead, (5.4) gives

$$\tau \sum_{k=1}^\infty W^\tau_{L^\epsilon}(\rho_{\epsilon,k}, \rho_{\epsilon,k-1}) \le 2\|\Psi\|_{L^\infty(\mathbb{R}^N)} \tag{5.8}$$

if Ψ is bounded or

$$\tau \sum_{k=1}^\infty W^\tau_{L^\epsilon}(\rho_{\epsilon,k}, \rho_{\epsilon,k-1}) \le \int_{\mathbb{R}^N} \rho_0 \Psi dx \tag{5.9}$$

if Ψ is nonnegative. We have seen that the map pushing $\rho_{\epsilon,k}$ back to $\rho_{\epsilon,k-1}$ is $\psi_\tau^\epsilon := \mathrm{id} + \tau\nabla H^\epsilon \circ \nabla\Psi$. It follows

$$W^\tau_{L^\epsilon}(\rho_{\epsilon,k}, \rho_{\epsilon,k-1}) = \int_{\mathbb{R}^N} L^\epsilon(\nabla H^\epsilon \circ \nabla\Psi(x))\rho_{\epsilon,k}(x)dx.$$

Since $0 \le H \le K|\mathrm{id}|^2$, we have $L^\epsilon(z) \ge C(\epsilon, K)|z|^2$ for certain $C(\epsilon, K) > 0$ (for small ϵ). This and the equation above lead to

$$\tau^2 \sum_{k=1}^\infty \int_{\mathbb{R}^N} |\nabla H^\epsilon \circ \nabla\Psi|^2 \rho_{\epsilon,k} dx \le \frac{\tau^2}{C(\epsilon, K)} \sum_{k=1}^\infty W^\tau_{L^\epsilon}(\rho_{\epsilon,k}, \rho_{\epsilon,k-1}).$$

In view of this and (5.8) or (5.9) we obtain

$$\sum_{k=1}^\infty \int_{\mathbb{R}^N} |x - \psi_\tau^\epsilon(x)|^2 \rho_{\epsilon,k} n(x) dx \le C\tau$$

for some constant $C = C(\epsilon, K, \Psi) > 0$. Thus, the proof is concluded.

What happens if Ω is a bounded domain? Then we have the following proposition:

Proposition 7. *Let* $\Phi \in C^1_0(\bar{\Omega}; \mathbb{R}^N)$ *be of the form* $\Phi = \nabla H \circ \nabla\Psi$, *where* $H \in C^2(\mathbb{R}^N)$ *is strictly convex with* $H(0) = 0$ *as its minimum and such that* $H(x) \le K|x|^q$ *for all* $x \in \mathbb{R}^N$ *and some* $K > 0$, $q > 1$. *Assume* $|\nabla^2 H \circ \nabla\Psi|$, $|\nabla^2\Psi| \in L^\infty(\mathbb{R}^N)$ *and* $\rho_0 \in \mathcal{M}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ *for some* $1 < p \le \infty$. *Then there exists a weak solution for (2.13) in* $\mathbb{R}^N \times (0, \infty)$.

Proof: Let $T > 0$ and $\tau > 0$ such that $T/\tau =: n$ is a positive integer. The inequality $H(x) \leq K|x|^q$ yields $L(x) \geq K'|x|^{q'}$ for some $K' > 0$ and, consequently,

$$\int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx \leq \frac{\tau^{q'}}{K'} W_L^\tau(\rho_k, \rho_{k-1}). \tag{5.10}$$

Inspired by [1], we distinguish two cases:
Case 1: $1 < q < 2$ implies $2 < q' < \infty$. Then, according to (5.10) and Jensen's inequality, we have

$$\begin{aligned}\int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx &\leq \left(\int_\Omega |x - \psi_\tau(x)|^{q'} \rho_k(x) dx \right)^{2/q'} \\ &\leq \frac{\tau^2}{(K')^{2/q'}} \left[W_L^\tau(\rho_k, \rho_{k-1}) \right]^{2/q'}.\end{aligned}$$

By summation we get

$$\sum_{k=1}^n \int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx \leq \frac{\tau^2}{(K')^{2/q'}} \left(\frac{T}{\tau} \right)^{1-2/q'} \left[\sum_{k=1}^n W_L^\tau(\rho_k, \rho_{k-1}) \right]^{2/q'}.$$

It follows

$$\sum_{k=1}^n \int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx \leq T^{1-2/q'} \frac{\tau}{(K')^{2/q'}} \left[\tau \sum_{k=1}^n W_L^\tau(\rho_k, \rho_{k-1}) \right]^{2/q'} \leq C\tau,$$

with C independent of n.
Case 2: $2 \leq q$ implies $1 < q' \leq 2$. Note that (5.10) implies

$$\begin{aligned}\int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx &\leq \left(\int_\Omega |x - \psi_\tau(x)|^{q'} \rho_k(x) dx \right)^{2/q'} \\ &\leq \frac{(\mathrm{diam}\Omega)^{2-q'}}{(K')^{2/q'}} \tau^{q'} W_L^\tau(\rho_k, \rho_{k-1}).\end{aligned}$$

Summing over k leads to

$$\sum_{k=1}^n \int_\Omega |x - \psi_\tau(x)|^2 \rho_k(x) dx \leq C\tau^{q'-1},$$

with C independent of n. Therefore, in both situations, the cumulative error term tends to zero.

Case 1 does not rely on Ω being bounded while case 2 does. However, in case 2 we deal with a summable series (the constant C is independent not only of n but also of T).

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