A model for hysteresis in mechanics using local minimizers of Young measures

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Abstract

A model for effects related to hysteresis in continuum mechanics is introduced. Its key idea is to consider local minimizers of Young measures describing the deformation gradient of a body. These Young measures are defined as limit of a quasi-static evolution by means of a gradient flow with respect to a special "regularized" Wasserstein metric. The model is described for the space-homogenous case, and some one-dimensional examples show the occurence of hysteresis and illustrate possible applications to fracture.

1 Introdcution

Different phenomena in continuum mechanics, as elasticity, fracture and the formation of microstructures, are traditionally described by different models that are often very successful for specific problems, but may make it difficult to see common properties of these phenomena. In recent years, progress has been made in particular in describing hysteresis effects [22, 13, 15, 17, 16] and fracture [9, 6] by a variational approach.

In this article we introduce a new approach to hysteresis problems, based on the notion of Young measures. A *Young measure* (or *parameterized measure*) is a family of probability measures $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^N$ associated with a sequence of measurable functions $(f_j)_{j\in\mathbb{N}}$ with $f_j : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ such that for any continuous function $\phi : \mathbb{R}^N \to \mathbb{R}$ the function

$$\overline{\phi}(x) = \int_{\mathbb{R}^N} \phi(F)\, d\nu_x(F) =: \langle \nu_x, \phi \rangle$$

is measurable, and for every weakly-converging sequence (f_j) we have

$$(\phi(f_j))_j \rightharpoonup \overline{\phi}.$$

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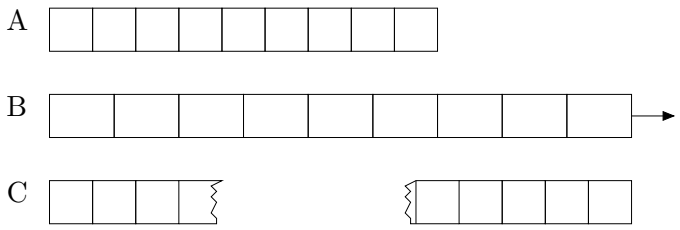

Figure 1: An elastic bar (A) can be deformed to some extend without breaking (B), although the broken configuration (C) would have a lower elastic energy. Hence it is not appropriate to consider this problem as *global* minimization of the elastic energy.

We can think of a Young measure as a one-point statistic for the sequence f_j, i.e. ν_x describes (in a certain sense which can be made mathematically precise) the probability distribution of the values of the sequence f_j at x. For an introduction into Young measures and their applications see [19, 18] and the original work by L. C. Young [25, 26].

A *gradient Young measure* is a Young measure associated with a sequence of gradients. In the one-dimensional case, the classes of gradient Young measures and Young measures coincide. In this paper we will always deal with gradient Young measures, but usually just refer to them as *Young measures*.

Young measures turned out to be a useful tool in the calculus of variations [23] and for describing microstructures in martensites [2] where assuming that the system is in a global energy minimum is a reasonable assumption, and hysteresis effects induced by the existence of local minima play a minor role. However, for other related problems as, e.g., fracture, this poses inherent difficulties: If we model, for instance, the energy density ϕ of an elastic bar as function of the deformation u_x, then we have to assume sublinear growth rate at infinity, e.g. $\phi(u_x) := \log(1 + |u_x|^2)$ or a Lennard-Jones potential, in order to obtain any fracture at all. But then the following problem arises: If we stretch the bar slightly, i.e. if we give $F = \int u_x \, dx$ a small positive value, then any minimizing sequence will in the limit have energy zero and deformation gradients concentrated in zero and infinity. This means that the elastic bar will break for arbitrarily small values of F, in contradiction with experiments (compare Fig. 1). This paradox was pointed out by Truskinovsky [24]. The problem arises because we are considering *global* energy minimization. Instead, we should look for *local* minimizers. As pointed out in [10], this might also be a useful strategy to solve the problems arising in dynamical elasticity problems where Young measure solutions (as obtained in [11, 20]) are often non-unique. But how should we introduce the notion of local minimizers in a natural way?

Recently, some progress had been made in the study of local minimizers of various variational problems in the class of *functions* [12, 7, 14, 3]. Although this approach has very interesting applications, in particular in micromagnetics, it does not use the concept of Young measures to describe possible microstructures. In particular in problems exhibiting different length scales it would be useful to introduce a notion of local minimizers for the *microstructure* itself, i.e. to define *local minimizers of Young measures*.

The fundamental idea which we propose in this article is to consider a quasi-static evolution of the Young measure following a natural gradient flow evolution. This enables us to find local minimizers that are more natural candidates for solutions of the system under study than the Young measure solutions obtained by global minimization.

In order to define a gradient flow reflecting the physical behavior of the system, it will be crucial to find an appropriate metric on the space of Young measures. The corresponding definitions will be given in Section 2. Although the concept is mathematically involved, it can be simplified in specific situations. This will be done in Section 3 where we apply the model to certain examples of hysteresis such as fracture and plasticity. In particular, we will observe that Truskinovsky's paradox disappears in our model.

The goal of this article is to introduce the main ideas, thus we address chiefly motivations and basic concepts rather than technical questions and more involved applications. In particular, we do *not* claim to be mathematically rigorous. Many natural questions connected with our model are work in progress, further studies will be necessary to evaluate the usefulness of the proposed model for hysteresis effects in continuum mechanics. However, the examples illustrated in Section 3 seem to be very promising.

2 Evolution of Young-measures

Our main idea is to describe the dynamics of a physical system via a gradient flow of Young measures with respect to a metric whose definition draws its motivation from the underlying physics. In this article we consider only problems which are homogenous in space.

The starting point for our model is to find an appropriate metric on the space of Young measures. It is natural to consider the well-studied Wasserstein metric, while considering Young measures as elements of the dual of Lipschitz continuous functions. More precisely, given two probability measures ν_1, ν_2 on the space X we set

$$d_W(\nu_1,\nu_2) := \frac{1}{2}\inf_{\mu}\left(\int_X\int_X |x-y|\,d\mu(x,y),\ \pi_1(\mu)=\nu_1,\ \pi_2(\mu)=\nu_2\right), \quad (1)$$

where $\pi_1(\mu)$ (resp. $\pi_2(\mu)$) are the projections of the measure μ on $X\times X$ on the first (resp. second) component, also known as marginals.

We will see later that only this type of Wasserstein metric (the L^1-Wasserstein metric) has the right scaling behavior needed for the gradient flow of our model.

However, it turns out that the topology induced by this Wasserstein metric is too weak. As illustrating one-dimensional example consider the two Young measures $\nu_0 := \delta_A$ and $\nu_1 := \delta_B$ for constants $A \neq B$. We can define a homotopy from ν_A to ν_B with respect to the Wasserstein metric by $\nu_\tau := (1-\tau)\delta_A + \tau\delta_B$, and if the energy at B is less than the energy at A, then the energy of ν_τ will strictly decrease in τ, thus ν_0 cannot be a local minimum in the class of Young measures with respect to d_W. Hence the Wasserstein metric does not allow for the existence of local minima in these cases. A closer look at ν_τ shows that at $\tau = 0$ a new phase B forms instantenously. In a certain sense the solution "tunnels" through a possible energy barrier between A and B, compare the similar situation of Fig. 4. In order to exclude such behavior, we "regularize" the Wasserstein metric and consider the limit where the regularizing term vanishes. In the following we denote by d_W the Wasserstein metric (1) and by d_H the Hausdorff metric.

Definition 2.1 (Regularized metrics). *Let $\varepsilon > 0$. We define the* regularized metrics *$d_{\varepsilon,R}$ and $d'_{\varepsilon,R}$ on the set of Young measures on $\mathbb{R}^{m\times n}$ with compact support by*

$$d_{\varepsilon,R}(\nu,\mu) \quad := \quad d_W(\nu,\mu) + \varepsilon\, d_H(\operatorname{supp}\nu, \operatorname{supp}\mu).$$

This definition takes into account that (if we assume continuous evolution of the Young measure) a sudden formation of a new phase (i.e. an instantenous extension of the support of the Young measure) is forbidden. We can state the following lemma:

Lemma 2.2. *d_R is a metric.*

Proof. d_W is a metric on the set of probability measures on $\mathbb{R}^{m\times n}$ with compact support and d_H is symmetric and satisfies the triangle inequality, since it is a metric for subsets of $\mathbb{R}^{m\times n}$. □

The space of Young measures equipped with the metric d_R is not necessarily complete: Take as an example the case $m = n = 1$ and $\varepsilon = 1$. The sequence $\{\nu_n\}$, where $\nu_n := \frac{1}{n+1}\delta_0 + \frac{n}{n+1}\delta_2$, is a Cauchy sequence, but does not converge: δ_2 is not the limit, since $d_R(\delta_2, \nu_n) > 1$ for all $n \in \mathbb{N}$. To fix these kinds of problems, we introduce a notion of convergence which differs slightly from the convergence induced by the metric. The physical reason for this is that, while the instantenous formation of new phases should be prohibited (for $\varepsilon > 0$), the disappearance of phases should be allowed.

Definition 2.3. *We define $\mathcal{Y}_R$ as the space of probability measures on $\mathbb{R}^{N\times N}$ equipped with the metric d_R and the following notion of convergence:*

$$(\nu_n) \subset \mathcal{Y}_R,\ \nu_n \to \nu \quad :\Longleftrightarrow$$

(i) (ν_n) *is a Cauchy sequence with respect to* d_R,

(ii) $\nu_n \to \nu$ *with respect to* d_W.

Remark 2.4. *Definition 2.3 is independend of the choice of* $\varepsilon > 0$.

We have now established a metric and a convergence on the space of Young measures. The next step would be to define a differentiation on $\mathcal{Y}_R$ which can be used for defining a gradient flow. However, the structure of the metric d_R is mathematically difficult, we will therefore only give a formal definition and apply it to special situations where the definition can be made rigorous.

Definition 2.5 (Formal). *Let* $T > 0$, *let* $F\colon \mathcal{Y}_R \to \mathbb{R}$ *be a smooth function and let* $\nu : [0,T] \to \mathcal{Y}_R$. *Then we define differentiation on* $\mathcal{Y}_R$ *by the following formulae (wherever they are well-defined):*

$$\frac{d}{dt}\nu(t) := \lim_{h\to 0} \frac{\nu(t+h)-\nu(t)}{h}, \tag{2}$$

$$-DF(\nu) := -D_\nu F(\nu) := \lim_{h\to 0} \frac{\nu-\tilde{\nu}_h}{d_R(\nu-\tilde{\nu}_h)^2}(F(\nu)-F(\tilde{\nu}_h)), \tag{3}$$

$$\text{where } \tilde{\nu}_h \to \nu.$$

Now let $N \in \mathbb{N}$, $\gamma, \varepsilon > 0$ and let $\phi\colon \mathbb{R}^{N\times N} \to \mathbb{R}$ be a smooth function. We study the initial value problem

$$\begin{cases} \gamma\nu_t(t) &= -D\langle\phi(t),\nu(t)\rangle, \\ \langle Id,\nu(t)\rangle &= F(t), \\ \nu(0) &= \nu_0. \end{cases} \tag{4}$$

Here we write $\langle\phi(t),\nu(t)\rangle := \int_{\mathbb{R}^{N\times N}} \phi(t,Y)d\nu(t)(Y)$. First let $\gamma, \varepsilon > 0$, later we will discuss the limit where $\varepsilon \to 0$ and $\gamma \to 0$ (quasi-static case). It will turn out that this model allows for the existence of local minimizers and for hysteresis effects. We will illustrate this in the following section by considering a simplified model which is accessible to direct computations. If we considered the problem analog to (4) with d_W instead of d_R as the underlying metric (i.e. with $\varepsilon = 0$), then we would get different solutions without local minimizers and hence without hysteresis effects.
Solutions of the general problem (4) might be obtained by approximating the Young measure $\nu(t)$ by a sum of finitely many Dirac masses, i.e. assuming that

$$\nu(t) \approx \sum_{i=1}^{r} \lambda_i(t)\delta_{P_i}.$$

Approximations of a similar form have been used in [5, 4]. General results on the solvability of this approximating problem and the convergence of the approximating solutions to solutions of the full system (4) are work in progress.

3 One-dimensional systems

In this section we apply our ideas to a simplified situation which can be handled mathematically relatively easy. We consider only one-dimensional cases, i.e. $m = n = 1$ and we assume that the Young measure at every time t consists at most of two Dirac masses at $A(t)$ and $B(t)$, i.e.

$$\nu(t) = \lambda(t)\delta_A(t) + (1 - \lambda(t))\delta_B(t).$$

Since $\nu(t)$ is positive we have $\lambda \in [0, 1]$. Moreover, since $\langle Id, \nu(t)\rangle = F(t)$ given, we have either $A(t) \le F(t) \le B(t)$ or $B(t) \le F(t) \le A(t)$. Without loss of generality we assume $A(t) \le F(t) \le B(t)$.

Under these assumptions on $\nu(t)$ (which can be motivated for the systems under investigation), the definition of the metric d_R gives immediately that λ, A and B are continuous functions of t as long as $\lambda \in (0, 1)$. If $\lambda(t) \to 0$ or $\lambda(t) \to 1$ as $t \to t_0$, then the definition of convergence yields that A (resp. B) may have a discontinuity at t_0 and $A(t_0) = B(t_0) = F(t_0)$. (This reflects the disappearing of a phase.)

In this simplified model we can derive evolution equations for A and B, valid whenever $\lambda(t) \in (0, 1)$. The variable λ is fixed by the condition $\langle Id, \nu(t)\rangle = F(t)$ which yields

$$\lambda(t) = \frac{F(t) - B(t)}{A(t) - B(t)}.$$

From now on, we often write A instead of $A(t)$ etc., and we use the abbreviation

$$\nu_{A,B} \quad = \quad \lambda\delta_A + (1 - \lambda)\delta_B. \tag{5}$$

We can rewrite (4) as a pair of two dynamical equations for A and B. We want to find the equation for A. To derive it, we first need to calculate the distance between $\nu_{A+h,B}$ and $\nu_{A,B}$ for a small $h > 0$. The regularizing part of $d_{\varepsilon,R}(\nu_{A+h,B}, \nu_{A,B})$ is easy and gives εh. To calculate the Wasserstein part we observe that the optimal probability measure μ in (1) can have a support only on the four points $(A + h, A)$, $(A + h, B)$, (B, A) and (B, B). Hence we have

$$\mu = \alpha_1\delta_{(A+h,A)} + \alpha_2\delta_{(A+h,B)} + \alpha_3\delta_{(B,A)} + \alpha_4\delta_{(B,B)},$$

and $\alpha_i \in [0, 1]$, $\sum_i \alpha_i = 1$. Thus we have to minimize

$$\int\int |x - y|\, d\mu(x, y) = \alpha_1 h + \alpha_2|A + h - B| + \alpha_3|B - A|.$$

Using the constraints on α_i and assuming that $h > 0$ is sufficiently small, we find that this expression is minimal if

$$\alpha_1 = \frac{B - F}{B - A},\ \alpha_2 = \frac{(B - F)h}{(B - (A + h))(B - A)},\ \alpha_3 = 0,$$

and a small calculation yields

$$d_W(\nu_{A+h,B},\nu_{A,B}) = \frac{B-F}{B-A}\,h.$$

Now we can calculate the gradient of the energy $\langle\phi,\nu\rangle$ to get

$$\begin{aligned}
\frac{d}{dt}A(t) &= -\nabla\langle\phi(t),\nu_{A(t),B(t)}\rangle \\
&= -\lim_{h\to 0}\frac{\langle\phi(t),\nu_{A(t)+h,B(t)}\rangle - \langle\phi(t),\nu_{A(t),B(t)}\rangle}{d_R(\nu_{A(t)+h,B(t)},\nu_{A(t),B(t)})} \\
&= -\lim_{h\to 0}\frac{\langle\phi(t),\nu_{A(t)+h,B(t)}\rangle - \langle\phi(t),\nu_{A(t),B(t)}\rangle}{d_W(\nu_{A(t)+h,B(t)},\nu_{A(t),B(t)}) + h} \\
&= -\lim_{h\to 0}\left[\left(\frac{B-A}{B-(A+h)}\phi(A+h) + \frac{F-(A+h)}{B-(A+h)}\phi(B)\right.\right. \\
&\quad \left.\left. -\frac{B-F}{B-A}\phi(A) - \frac{F-A}{B-A}\phi(B)\right)\left(\frac{B-F}{B-A}+\varepsilon\right)^{-1} h^{-1}\right].
\end{aligned}$$

We use the identity

$$\left(\frac{B-F}{B-A}+\varepsilon\right)^{-1} = \frac{B-A}{B-F} - \varepsilon\frac{B-A}{(B-F)\left(\frac{B-F}{B-A}+\varepsilon\right)}$$

and estimate

$$\left|\varepsilon\frac{B-A}{(B-F)\left(\frac{B-F}{B-A}+\varepsilon\right)}\right| \le \varepsilon\left|\frac{B-A}{B-F}\right|^2.$$

Hence we get

$$\begin{aligned}
\frac{d}{dt}A(t) &= -\lim_{h\to 0}\left[\left((\phi(A+h)-\phi(A)) + \frac{h\phi(A+h)}{B-(A+h)}\right.\right. \\
&\quad \left.\left. +\frac{h(F-A)\phi(B) - h(B-A)\phi(B)}{(B-(A+h))(B-F)}\right)\frac{1}{h}\right] + O(\varepsilon) \\
&= -\phi'(A(t)) - \frac{\phi(A)-\phi(B)}{B-A} + O(\varepsilon).
\end{aligned}$$

If we had chosen an L^p-Wasserstein metric with $p>1$, the limit $h\to 0$ would be zero, since no mass transport between A and B would be allowed. Hence we had to choose $p=1$ in the original definition.

Now we take the limit $\varepsilon\to 0$ and let the regularizing term in the metric d_R vanish to obtain

$$\frac{d}{dt}A(t) = -\phi'(A(t)) - \frac{\phi(A)-\phi(B)}{B-A}. \tag{6}$$

Similarly, we deduce that

$$\frac{d}{dt}B(t) \;=\; -\phi'(B(t)) - \frac{\phi(A)-\phi(B)}{B-A}. \tag{7}$$

Using standard ODE theory we get:

Lemma 3.1. *If $\phi \in \mathcal{C}^1$ has quadratic growth, ϕ' has linear growth and ϕ is onvex outside some interval $(-M,+M)$ and if F is smooth, then the system (6)–(7) admits a global solution.*

Proof: Local existence is clear. Assume $B > A$ and $A, B \notin (-M,+M)$, then we have by convexity of ϕ

$$\begin{aligned}\frac{d}{dt}B &= -\phi'(B) - \frac{\phi(A)-\phi(B)}{B-A} \\ &\leq -\phi'(tA+(1-t)B) - \frac{\phi(A)-\phi(B)}{B-A}\end{aligned}$$

for all $t \in [0,1]$. Choosing t appropriately we get $\frac{d}{dt}B \leq 0$. Thus we can prove that A and B are bounded. From this we obtain global existence. □

However, in general this solution does not satisfy the constraint $\lambda(t) \in (0,1)$. If we define a solution according to the definition of convergence in $\mathcal{Y}_R$, we can solve this problem:

Definition 3.2. *Let $(A(t),B(t))_{t\geq 0}$ satisfy (6)–(7) whenever $\lambda(t)$ as given by (5) is contained in $(0,1)$ and let $A(t) = B(t) = F(t)$ else. Suppose that $(A(t),B(t))$ is continuous for all t with $\lambda(t) \in (0,1)$ and let $\lambda(t)$ be continuous for all $t \geq 0$. Moreover, assume that $A(0) = A_0$ and $B(0) = B_0$ are given. Then we call $(A(t),B(t))_{t\geq 0}$ a solution of the problem stated above.*

With this definition the following proposition is an immediate consequence of our derivation and Lemma 3.1.

Proposition 3.3. *If the assumptions of Lemma 3.1 hold and if $A_0 \leq F(0) \leq B_0$, then there exist a solution in the sense of Definition 3.2.*

Particularly interesting are stable solutions. We make the following observation for F constant and $\nu(0) = \delta_F$:

Lemma 3.4. *If ϕ is smooth and locally strictly convex at F, then the solution $\nu(t) = \delta_F$ is stable, i.e. there exists a neighborhood U around F such that every solution $\mu(t) = \lambda(t)\delta_{A(t)} + (1-\lambda(t))\delta_{B(t)}$ with $\mu(0) \in U$ converges to $\nu(t)$. If ϕ is locally nonconvex at F, then the solution $\nu(t) = \delta_F$ cannot be stable.*

Proof. Let ϕ be locally strictly convex at F. Consider $\mu(0) = \lambda(0)\delta_{A(0)} + (1-\lambda(0))\delta_{B(0)}$, where $|A(0)-F|, |B(0)-F| < \eta$. Let $\mu(t) = \lambda(t)\delta_{A(t)} + (1-\lambda(t))\delta_{B(t)}$. Since

$$\phi(B) = \phi(A) + \phi'(A)(B-A) + \frac{1}{2}\phi''(A)(B-A)^2 + O(|B-A|^3),$$

we obtain from (6) that

$$\frac{d}{dt}A(t) = \frac{1}{2}\phi''(A)(B-A)^2 + O(\eta^3).$$

If $\eta > 0$ is sufficiently small, then we have $\phi''(A) \geq C > 0$ and $O(\eta^3) < C\eta^2$, hence

$$\frac{d}{dt}A(t) \geq \frac{C}{4}(B-A)^2,$$

and thus, by deducing the analogous formula for $B(t)$, we get $A(t), B(t) \to F$, so we have proved $\mu(t) \to \nu(t) = \delta_F$.

If, on the other hand, ϕ is locally nonconvex, then the same argument yields that $\mu(t)$ does not converge to $\nu(t) = \delta_F$. □

We will apply (6)-(7) in the following sections to different potentials ϕ to describe physical phenomena associated with nonconvexity of the underlying energy density.

4 Examples

4.1 Microstructures and hysteresis

As a first example we study the formation of microstructures in an elastic bar with an energy density described by a two-well potential $\phi_1(X) := (1-X^2)^2$. Since the dynamical equations (6)–(7) do not depend on F, we can directly calculate the gradient field, see Fig. 2. From this we see that $\nu(t) = \delta_F$ is only a stable solution if $|F|$ is larger than some constant, in agreement with Lemma 3.4, since ϕ_1 is locally strictly convex for all F with $|F| > 1/\sqrt{3} =: F_{crit}$.

If we choose $F(t)$ non-constant on time, we can observe hysteresis effects: The quasistatic solution $\nu(t)$ for $F(t) := 1 - \sin^2(\pi\, t)$ and $t \in [0,1]$ is plotted in Fig. 3.

Hence the formation of microstructures takes place instantenously – but only after the deformation (measured with respect to the deformation of the uniform state $\nu = \delta_{+1}$, i.e. $F = 1$) excedes a certain critical value, i.e. when $F < F_{crit}$. On the other hand, the microstructure vanishes only after the whole process is reversed, i.e. when $F = 1$.

How can we explain this behavior microscopically? Our model takes into account that to form a microstructure the crystal has to "evolve" through

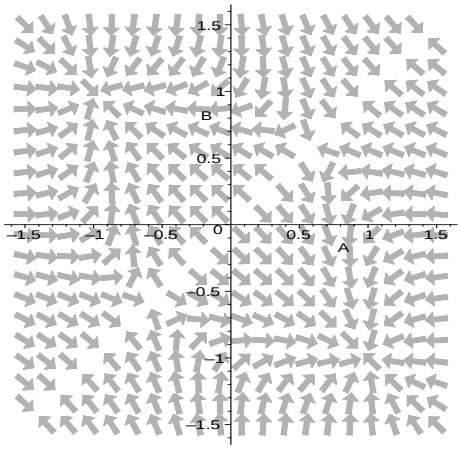

Figure 2: Phase diagramm for $\phi_1(X) := (1 - X^2)^2$.

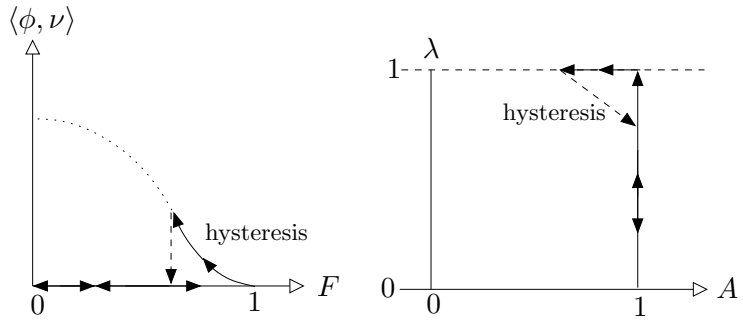

Figure 3: Quasistatic evolution with hysteresis loop, energy vs. deformation (left) and (A, λ)-plot (right) where $F(t) = \langle Id, \nu(t) \rangle$ and $\nu(t) = \lambda(t)\delta_{A(t)} + (1 - \lambda(t))\delta_{B(t)}$.

intermediate states (see Fig. 4). If they have a higher energy than the original state without microstructure, the system cannot overcome the energy barrier and stays in the local minimum, hence no microstructure forms. In our model example this is the case for $F > F_{crit}$.

Here, some comparison to observed hysteresis loops is in order:

First, the formation of microstructures often shows very small hysteresis. Because of this, classical Young measure models using global energy minimization can give a useful description of such materials. However, our model captures additionally hysteresis effects. The amount of hysteresis is thereby solely defined by the form of the energy density ϕ. A steep form with a small convex region results in a small hysteresis effect, a large convex region results in a large hysteresis effect.

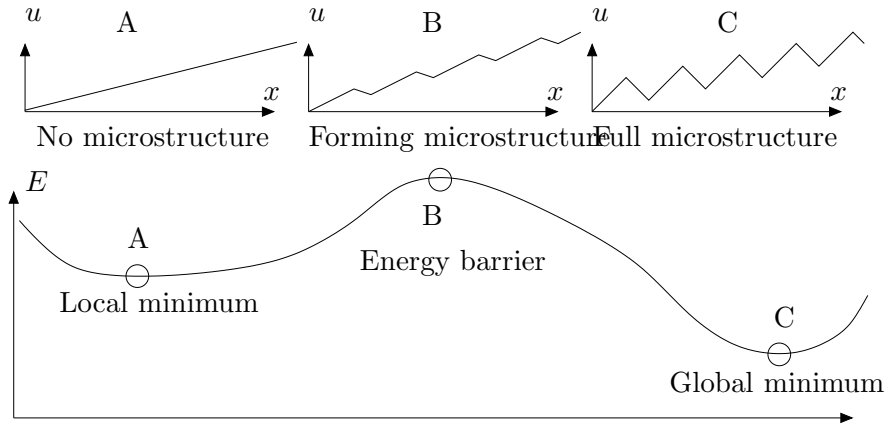

Figure 4: To form a microstructure with lower energy, a state with higher energy would have to be crossed.

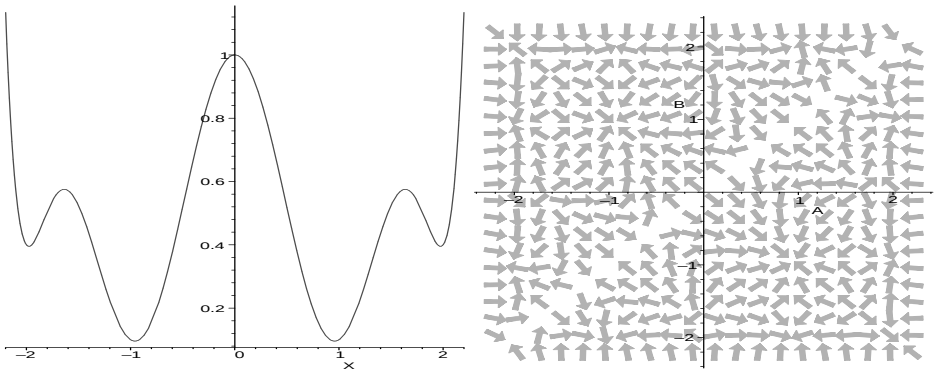

Figure 5: The function $\phi_2(X) := (1 - X^2)^2(1 - (X/2)^2)^2 + \frac{1}{10}X^2$ and the corresponding phase diagramm.

Second, the form of the hysteresis loops of Fig. 3 is oncomplete when compared to experiments. This can be explained by one oversimplification in our model: We have restricted ourselves to space-homogeneous situations. However, the onset of microstructures in a real material will be local in space. An extension of our model to non-homogeneous situations could therefore reproduce hysteresis loops corresponding to the experiments.

We conclude this subsection with a look at a more involved problem where the energy ϕ_2 is given by a four-well potential (see Fig. 4.1).

The phase diagramm (Fig. 5) shows the existence of various metastable states. Therefore the hysteresis loops for this system are of a more complicated nature.

4.2 Fracture

We consider an elastic bar and assume space-homogenous deformations. We can describe the deformation of the bar at a given time $t \geq 0$ by $F(t)$ where $F(t) = 0$ is the undeformed state. We model the elastic energy by a Perona-Malik type function $\phi(F) := \log(1 + |F|^2)$, and thus we are in the same situation as in the Truskinovsky paradox. We consider the case where the bar is slowly stretched with $F(t) = t$. The initial data is $\nu(0) = \delta_0$. It is now easy to see that the stable solution of our quasistatic limit problem (4) is $\nu(t) = \delta_{F(t)}$ as long as ϕ is locally strictly convex at $F(t)$. Hence the bar is not breaking in this range of deformations and its behavior is completely elastic. However beyond the critical point on which ϕ becomes locally nonconvex our solution becomes unstable. This corresponds to a fracture of the bar.

To model this situation rigorously, we had to use the notion of Young measure varifolds [1, 8] to capture the resulting infinite deformation gradients at the fracture, but even without using this notion we see that the Truskinovsky paradox does not occur in our model. This means that our model can describe the breaking of a bar, although it solely relies on the elastic energy without additional (phenomenological) "surface terms" for the energy contribution of a fracture.

An extension of this model to non-homogeneous situations is work in progress. First numerical experiments seem to show that in generic situations (i.e. when the problem is not set to be artificially uniform in space) the onset of fracture occurs in a single point when the stress exceeds a critical value [21].

It would be interesting to see whether also some forms of plasticity can be handled in this framework using a periodic energy density ϕ.

5 Open problems

The outline of ideas presented in the last sections leave a lot of questions open. Of particular interest is the extension of the concept to non-homogenous, i.e. space dependent, situations. The quasistatic evolution of the Young measure can be coupled with a quasistatic or dynamic evolution of the displacement. Questions like existence and (partial) uniqueness arise naturally in this context.

Another focus point is the problem of fracture. Here the concept has to be extended to Young measure varifolds to capture the formation of jumps in the displacement vector field. It is very interesting to study situations where the model can be simplified. This is ongoing work with Johannes Zimmer [21].

Finally, numerical approximations pose a lot of interesting problems. Although it seems difficult to perform the neccessary calculations for arbitrary

space dimensions (due to the lack of a constructive characterization of the space of gradient Young measures in higher dimensions), it seems to be possible to get results for specific problems.

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