Higher Order Variational Problems

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1 Introduction

Higher order variational problems appear often in the engineering literature and in connection with the so-called gradient theories of phase transitions within elasto-plastic regimes. The study of equilibria of micromagnetic materials asks for mastery of second order energies (see [51], [91]; see also [31], [38], [44], [45], [61], [77], [78], [79], [108]), and the Blake-Zisserman model for image segmentation in computer vision (see [34], [35], [36]; see also [50]) seats squarely among second-order free discontinuity models which may be recasted as higher order Griffiths' models for fracture mechanics (see [7], [24], [28],[58], [68], [69], [70], [71]). The energy functionals may include lower dimensional order terms to take into account interfacial energies and discontinuities of underlying fields (see [10]). Here we will neglect the role played by these terms and we will focus on the added difficulties inherent to the presence of derivatives of order two or more.

We consider an energy functional

$$I(u) := \int_\Omega f(x, u, \nabla u, \dots, \nabla^k u)\, dx \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $u : \Omega \to \mathbb{R}^d$, $N, d \geq 1$, $u := (u_1, \dots, u_d)$, $\nabla u \in \mathbb{R}^{d\times N}$, and $(\nabla u)_{ij} := \frac{\partial u_i}{\partial x_j}$ for $i \in \{1, \dots, d\}$, $j \in \{1, \dots, N\}$.

Many interesting phenomena are related to the nonconvexity of the bulk energy density, and this brings us to two questions commonly asked in the Calculus of Variations:

Question I: Can we find necessary and sufficient conditions ensuring lower semicontinuity of I (with respect to an appropriate topology)?
Precisely, what assumptions on f guarantee that if $\{u_n\}$ is a sequence bounded in $W^{k,1}(\Omega;\mathbb{R}^d)$ and if $u_n \to u$ in $W^{k-1}(\Omega;\mathbb{R}^d)$ then

$$I(u) \leq \liminf_{n\to+\infty} I(u_n)?$$

Question II: Is it possible to characterize the effective energy of a family $\{I_\varepsilon\}$ of functionals of the type (1.1)?
Here we search for an integral representation for the limiting energy in the sense of De Giorgi's Γ-limit (see [46], [49]), i.e.,

$$I(u) := \inf_{\{\varepsilon\},\{u_\varepsilon\}} \left\{ \liminf_{\varepsilon\to 0^+} I_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \quad \text{in } W^{k-1,1}(\Omega;\mathbb{R}^d) \right\}. \tag{1.2}$$

Multiple scales arise in problems of type (1.2) where natural scaling laws and physical parameters are taken into account by the dependence $\varepsilon \to f_\varepsilon$ (see [29], [51], [52], [54]). Also, in questions of type (1.1) and when the growth of f is at most linear at infinity, we are faced with energies which may allow for concentrations, and therefore the interaction between measures of different dimensionality becomes an issue. F

A lesson to be learned is that techniques for higher order variational problems do not reduce to mere generalizations of their counterparts for first order problems (e.g., by assuming k-quasiconvexity in place of quasiconvexity, see [42], [87], [88]). Indeed, although functionals depending *uniquely* on the highest order derivatives can be treated easily, those where lower order terms are present require new ideas and new tools to handle localization and the truncation of lower order terms. Truncating gradients so that they remain gradients may be achieved through the techniques of maximal functions and of Fourier multipliers (see [1], [99], [100]) in those cases where the bulk energy density f has superlinear growth (see the proof of Lemma 2.15 in [66]). In fact, the success of this approach relies heavily on p-equi-integrability, and thus cannot be extended to the case $p = 1$ where one replaces weak convergence in $W^{k,1}(\Omega;\mathbb{R}^n)$ with the natural convergence, i.e., strong convergence in $W^{k-1,1}(\Omega;\mathbb{R}^n)$. As it turns out, when f grows at most linearly at infinity many seemingly simple questions, long ago answered within the realm of first order problems, still defy all attempts when we deal with order two or more. As an example, a standing open problem is the following (in the case where $k = 1$ this question was answered by Fonseca and Leoni in [59] Theorem 1.8):

is it true that if $f : \Omega \times \mathbb{R}^s \times \mathbb{R}^{d\times N^k} \to [0,\infty)$ is a Borel integrand, with $s := d + d \times N + \ldots + d \times N^{k-1}$, if $f(x, \mathbf{v}, \cdot)$ is k-quasiconvex, with $k \geq 2$ and

$\mathbf{v} := (u, \nabla u, \dots, \nabla^{k-1}u)$, if f satisfies "reasonable" continuity properties with respect to x and $\mathbf{v}$, if

$$C|\xi| - \frac{1}{C} \leq f(x, \mathbf{v}, \xi) \leq C(1 + |\xi|)$$

for all $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^s$, if $u \in W^{k-1,1}(\Omega; \mathbb{R}^d)$, $\nabla^k u \in BV(\Omega; \mathbb{R}^{d \times N^k})$, and if $\{u_n\}$ is a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, then

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx?$$

2 Lower Semicontinuity Results for Higher Order Variational Problems

In this section we address Question I.

Morrey's notion of *quasiconvexity* was extended by Meyers [87] to the realm of higher-order variational problems. We recall that $f : \Omega \to [0, +\infty)$ is said to be *quasiconvex* if (see [42], [88])

$$f(\xi) \leq \int_Q f(\xi + \nabla\varphi(x))\, dx \tag{2.1}$$

for all $\varphi \in C_c^\infty(Q; \mathbb{R}^d)$, and a function $F : E_k^d \to \mathbb{R}$ is said to be k-*quasiconvex* if

$$F(\xi) \leq \int_Q F(\xi + \nabla^k w(y))\, dy \tag{2.2}$$

for all $\xi \in E_k^d$ and all $w \in C_c^\infty(Q; \mathbb{R}^d)$.

To fix notation, here and in what follows, Ω is an open, bounded domain in $\mathbb{R}^N$, $Q := (-1/2, 1/2)^N$, $C_c^\infty(\mathbb{R}^N; \mathbb{R}^d)$ is the space of infinitely differentiable $\mathbb{R}^d$-valued functions in Ω with compact support, and $C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)$ stands for the space of Q–periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$. Recall that f is said to be Q-*periodic* if $f(x + k\mathbf{e}_i) = f(x)$ for all x, all $k \in \mathbb{Z}^k$, and for all $i = 1, \dots, N$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, we set

$$\nabla^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_N,$$

and for each $j \in \mathbb{N}$ the symbol $\nabla^j u$ stands for the vector-valued function whose components are all the components of $\nabla^\alpha u$ for $|\alpha| = j$. If u is C^∞ then for $j \geq 2$ we have that $\nabla^j u(x) \in E_j^d$, where E_j^d denotes the space of symmetric j-linear maps from $\mathbb{R}^N$ into $\mathbb{R}^d$. We set $E_0^d := \mathbb{R}^d$, $E_1^d := \mathbb{R}^{d \times N}$ and

$$E_{[j-1]}^d := E_0^d \times \cdots \times E_{j-1}^d, \quad E_{[0]}^d := E_0^d.$$

For any integer $k \geq 2$ we define

$$BV^k(\Omega;\mathbb{R}^d) := \left\{u \in W^{k-1,1}(\Omega;\mathbb{R}^d) : \nabla^{k-1}u \in BV(\Omega;E^d_{k-1})\right\},$$

where here $\nabla^j u$ is the Radon–Nikodym derivative of the distributional derivative $D^j u$ of $\nabla^{j-1}u$, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$.

Meyers [87] proved that k-*quasiconvexity* is a necessary and sufficient condition for (sequential) lower semicontinuity of (1.1) with respect to weak convergence (resp. weak* convergence if $p = \infty$) in $W^{k,p}(\Omega;\mathbb{R}^d)$ under appropriate growth and continuity conditions on the integrand f. Meyers' argument uses results of Agmon, Douglis and Nirenberg [2] concerning Poisson kernels for elliptic equations, and later Fusco [72] gave a simpler proof using De Giorgi's Slicing Lemma. He also extended the result to Carathéodory integrands when $p = 1$, while the case $p > 1$ has been recently established by Guidorzi and Poggiolini [75], who relied heavily on a p-Lipschitz assumption, i.e.,

$$|f(x,\mathbf{v},\xi) - f(x,\mathbf{v},\xi_1)| \leq C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})|\xi - \xi_1|.$$

As it turns out, k-quasiconvex integrands with p-growth are p-Lipschitz. This assertion was established by Marcellini [84] for $k = 1$, the case $k = 2$ was proven in [75], and recently Santos and Zappale [97] extend it to arbitrary k.

To date, the most general results concerning lowersemicontinuity and relaxation for higher-order variational problems with superlinear growth were obtained by Braides, Fonseca and Leoni in [30], where these questions may be seen as corollaries of very broad results casted for variational problems under pde constraints (here curl= 0), the $\mathcal{A}$-*quasiconvexity theory.* In a recent paper Fonseca and Müller [66] proved that $\mathcal{A}$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$(u,v) \mapsto \int_\Omega f(x,u(x),v(x))\,dx,$$

whenever $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory integrand satisfying

$$0 \leq f(x,u,v) \leq a(x,u)\,(1 + |v|^q)\,,$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^m \times \mathbb{R}^d$, where $1 \leq q < \infty$, $a \in L^\infty_{\rm loc}(\Omega\times\mathbb{R};[0,\infty))$, $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^q(\Omega;\mathbb{R}^d)$ and $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega;\mathbb{R}^l)$ (see also [43]). In the sequel $\mathcal{A} : L^q(\Omega;\mathbb{R}^d) \to W^{-1,q}(\Omega;\mathbb{R}^l)$, $\quad \mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}$ is a constant–rank, first order linear partial differential operator, with $A^{(i)} : \mathbb{R}^d \to \mathbb{R}^l$ linear transformations, $i = 1,\dots,N$. We recall that $\mathcal{A}$ satisfies the *constant-rank* property if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}w = r \quad \text{for all } w \in S^{N-1}, \tag{2.3}$$

where

$$\mathbb{A}w := \sum_{i=1}^N w_i\,A^{(i)}, \qquad w \in \mathbb{R}^N.$$

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(v) \leq \int_Q f(v + w(y))\, dy$$

for all $v \in \mathbb{R}^d$ and all $w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$.

The relevance of this general framework, as emphasized by Tartar (see [102], [103], [104], [105], [106], [107]; see also [42], [43], [92]) lies on the fact that in continuum mechanics and electromagnetism pdes other than $\operatorname{curl} v = 0$ arise naturally, and this calls for a relaxation theory which encompasses pde constraints of the type $\mathcal{A}v = 0$. Some important examples included in this general setting are given by:

(a) [Unconstrained Fields]

$$\mathcal{A}v \equiv 0.$$

Here, due to Jensen's inequality A-quasiconvexity reduces to convexity.

(b) [Divergence Free Fields]

$$\mathcal{A}v = 0 \quad \text{if and only if } \operatorname{div} v = 0,$$

where $v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ (see also [95]).

(c) [Magnetostatics Equations]

$$\mathcal{A}\begin{pmatrix} m \\ h \end{pmatrix} := \begin{pmatrix} \operatorname{div}(m+h) \\ \operatorname{curl} h \end{pmatrix} = 0,$$

where $m : \mathbb{R}^3 \to \mathbb{R}^3$ is the *magnetization* and $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the *induced magnetic field* (see [51, 106]); often these are also called Maxwell's Equations in the micromagnetics literature.

(d) [Gradients]

$$\mathcal{A}v = 0 \quad \text{if and only if } \operatorname{curl} v = 0.$$

Note that $w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ is such that curl $w = 0$ and $\int_Q w(y)\, dy = 0$ if and only if there exists $\varphi \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^n)$ such that $\nabla\varphi = v$, where $d = n \times N$. In this case we recover the notion of quasiconvexity (see (2.1)) for continuous integrands. Indeed, it suffices to prove that if f is continuous and satisfies (2.1) for test functions $\varphi \in C^\infty_{\text{c}}(Q; \mathbb{R}^d)$, then (2.1) remains valid if $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)$. Fix $\xi \in \mathbb{R}^{d\times N}$ and consider $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)$. Let $\varepsilon > 0$ and let $\theta \in C^\infty_{\text{c}}(Q; [0, 1])$ be a test function such that $\theta = 1$ in $(1 - \varepsilon)Q$, and $||\nabla\theta||_\infty \leq 2/\varepsilon$. Setting $\varphi_n(x) := \frac{1}{n}\theta(x)\varphi\left(\frac{n}{x}\right)$ we have

$$\begin{aligned} f(\xi) &\leq \liminf_{n\to\infty} \int_Q f\left(\xi + \nabla\varphi_n(x)\right)\, dx \\ &\leq \lim_{n\to\infty} \int_Q f(\xi + \nabla\varphi(nx))\, dx + \limsup_{n\to\infty} \int_{Q\setminus(1-\varepsilon)Q} f(\xi + \nabla\varphi(nx))\, dx. \end{aligned}$$

In view of the Q-periodicity of $\nabla\varphi$, by the Riemann-Lebesgue Lemma

$$\lim_{n\to\infty} \int_Q f(\xi + \nabla\varphi(nx))\, dx = \int_Q f(\xi + \nabla\varphi(y))\, dy.$$

On the other hand, for $n > 1/\varepsilon$ and for all $x \in Q$

$$\begin{aligned} |\xi + \nabla\varphi_n(x)| &= \left|\xi + \frac{1}{n}\nabla\theta(x)\varphi(nx) + \theta(x)\nabla\varphi(nx)\right| \\ &\leq C \end{aligned}$$

where the constant C is independent of n and ε. We deduce, therefore, that

$$f(\xi) \leq \int_Q f(\xi + \nabla\varphi(y))\, dy + CO(\varepsilon).$$

Letting $\varepsilon \to 0^+$ we conclude that

$$f(\xi) \leq \int_Q f(\xi + \nabla\varphi(y))\, dy.$$

(e) [Higher Order Gradients] Replacing the target space $\mathbb{R}^d$ by the finite dimensional vector space E^n_s, it is possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in L^p(\Omega; E^n_s)$ and $\mathcal{A}v = 0$ if and only if there exists $\varphi \in W^{s,q}(\Omega; \mathbb{R}^n)$ such that $v = \nabla^s\varphi$ (see Theorem 2.4). Here A-quasiconvexity reduces to k-quasiconvexity (see (2.2)) when the energy density is continuous.

Let $1 \leq p < \infty$ and $1 < q < \infty$, and consider the functional

$$F : L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^d) \times \mathcal{O}(\Omega) \to [0, \infty)$$

defined by

$$F((u, v); D) := \int_D f(x, u(x), v(x))\, dx,$$

where $\mathcal{O}(\Omega)$ is the collection of all open subsets of Ω, and the density f satisfies the following hypothesis:

(H) $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty)$ is a Carathéodory function, and

$$0 \leq f(x, u, v) \leq C\,(1 + |u|^p + |v|^q)$$

for a.e. $x \in \Omega$ and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, and for some constant $C > 0$.

For $D \in \mathcal{O}(\Omega)$ and $(u, v) \in L^p(\Omega; \mathbb{R}^m) \times \big(L^q(\Omega; \mathbb{R}^d) \cap \operatorname{Ker} \mathcal{A}\big)$ define

$$\begin{aligned} \mathcal{F}((u, v); D) := \inf\Big\{ \liminf_{n\to\infty} F((u_n, v_n); D) :\ & (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d), \\ & u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d), \\ & \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,q}(D; \mathbb{R}^l) \Big\}. \end{aligned} \tag{2.4}$$

It can be shown that the condition $\mathcal{A}v_n \to 0$ imposed in (2.4) may be equivalently replaced by requiring that v_n satisfy the homogeneous pde $\mathcal{A}v = 0$. Precisely,

$$\mathcal{F}((u,v);D) = \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) : (u_n,v_n) \in L^p(D;\mathbb{R}^m) \times L^q(D;\mathbb{R}^d),$$
$$u_n \to u \text{ in } L^p(D;\mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D;\mathbb{R}^d), \quad \mathcal{A}v_n = 0 \Big\}. \tag{2.5}$$

The following integral representation for the relaxed energy $\mathcal{F}$ was obtained in [30].

Theorem 2.1 ([30], Theorem 1.1) *Under condition* (H) *and the constant-rank hypothesis* (2.3)*, for all* $D \in \mathcal{O}(\Omega)$*,* $u \in L^p(\Omega;\mathbb{R}^m)$*, and* $v \in L^q(\Omega;\mathbb{R}^d) \cap \operatorname{Ker}\mathcal{A}$*, we have*

$$\mathcal{F}((u,v);D) = \int_D \mathrm{Q}_{\mathcal{A}} f(x,u(x),v(x))\,dx$$

where, for each fixed $(x,u) \in \Omega \times \mathbb{R}^m$*, the function* $\mathrm{Q}_{\mathcal{A}} f(x,u,\cdot)$ *is the* $\mathcal{A}$*-quasiconvexification of* $f(x,u,\cdot)$*, namely*

$$\mathrm{Q}_{\mathcal{A}} f(x,u,v) := \inf\left\{ \int_Q f(x,u,v+w(y))\,dy : w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \operatorname{Ker}\mathcal{A},\right.$$
$$\left. \int_Q w(y)\,dy = 0 \right\}$$

for all $v \in \mathbb{R}^d$*.*

The proof of this theorem relies heavily on the use of Young measures (see [15], [109]), together with the blow-up method introduced by Fonseca and Müller in [64], and the arguments developed in [66] (see also [13], [81]) .

Remark 2.2 (i) Note that in the degenerate case where $\mathcal{A} = 0$, $\mathcal{A}$-quasiconvex functions are convex and Theorem 2.1 together with condition (2.6) below yield a convex relaxation result with respect to $L^p \times L^q$(weak) convergence. See the monograph of Buttazzo [32] for related results in this context.

(ii) If the function f also satisfies a growth condition of order q from below in the variable v, that is

$$f(x,u,v) \geq \frac{1}{C}|v|^q - C \tag{2.6}$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^m \times \mathbb{R}^d$, then a simple diagonalization argument shows that $(u,v) \mapsto \mathcal{F}((u,v);D)$ is $L^p \times (L^q\text{-weak})$ lower semicontinuous, i.e.,

$$\int_D \mathrm{Q}_{\mathcal{A}} f(x,u(x),v(x))\,dx \leq \liminf_{n\to\infty} \int_D \mathrm{Q}_{\mathcal{A}} f(x,u_n(x),v_n(x))\,dx \tag{2.7}$$

whenever $u_n \in L^p(\Omega;\mathbb{R}^m)$, $v_n \in L^q(\Omega;\mathbb{R}^d) \cap \operatorname{Ker}\mathcal{A}$, $u_n \to u$ in $L^p(D;\mathbb{R}^m)$, $v_n \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$. In particular $\mathrm{Q}_{\mathcal{A}} f$ is $\mathcal{A}$-quasiconvex if f is continuous and

$$\frac{1}{C}|v|^q - C \leq f(v) \leq C(1+|v|^q)$$

for some constant $C > 0$, and all $v \in \mathbb{R}^d$.

The lower semicontinuity result (2.7) is not covered by Theorem 3.7 in [66], where it is assumed that the integrand be $\mathcal{A}$-quasiconvex and *continuous* in the v variable. Indeed, and as remarked in [66], in the realm of general $\mathcal{A}$-quasiconvexity the function $\mathrm{Q}_{\mathcal{A}} f(x,u,\cdot)$ may not be continuous, even if $f(x,u,\cdot)$ is. To illustrate this, it suffices to consider the degenerate case $\operatorname{Ker}\mathcal{A} = \{0\}$ all functions are $\mathcal{A}$-quasiconvex. Also, when $N = 1$, $d = 2$, and $v = (v_1, v_2)$, let

$$\mathcal{A}v := \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.$$

Then for $w \in \mathbb{R}$

$$\mathbb{A}w = \begin{pmatrix} 0 & w \end{pmatrix}$$

and thus when $|w| = 1$ the matrix $\mathbb{A}w$ has constant rank 1. For any given function f the $\mathcal{A}$-quasiconvex envelope of f is obtained by convexification in the first component, so that by considering e.g. (see [66], [85])

$$f_1(v) := e^{-|v_1|\, v_2^2}, \qquad f_2(v) := (1+|v_1|)^{|v_2|},$$

one gets

$$\mathrm{Q}_{\mathcal{A}} f_1(v) = \begin{cases} 0 & \text{if } v_2 \neq 0, \\ 1 & \text{if } v_2 = 0, \end{cases} \qquad \mathrm{Q}_{\mathcal{A}} f_1(v) = \begin{cases} (1+|v_1|)^{|v_2|} & \text{if } |v_2| \geq 1, \\ 1 & \text{if } |v_2| < 1. \end{cases}$$

(iii) The continuity of f with respect to v is essential to ensure the representation of $\mathcal{F}$ provided in Theorem 2.1, in contrast with the curl-free case where $\mathcal{A}v = 0$ if and only if $\operatorname{curl} v = 0$. In fact, if $f : \mathbb{R}^{n\times N} \to [0,\infty)$ is a Borel function satisfying the growth condition

$$0 \leq f(v) \leq C(1+|v|^q)$$

for $C > 0$, $1 \leq q < \infty$, $v \in \mathbb{R}^{n\times N}$, then it can be shown easily that

$$\mathcal{F}(w;D) = \int_D Qf(\nabla w(x))\,dx \tag{2.8}$$

for all $D \in \mathcal{O}(\Omega)$, $w \in W^{1,q}(\Omega;\mathbb{R}^n)$, where Qf is the quasiconvex envelope of f. Indeed, Qf is a (continuous) quasiconvex function satisfying (H) (see [56], [25] Theorem 4.3); therefore by Theorem 2.1

$$w \mapsto \int_D Qf(\nabla w(x))\,dx$$

is $W^{1,q}$-sequentially weakly lower semicontinuous, and so

$$\int_D Qf(\nabla w(x))\,dx \leq \mathcal{F}(w;D).$$

Conversely, under hypothesis (H) it is known that $\mathcal{F}(v;\cdot)$ admits an integral representation (see Theorem 9.1 in [27], Theorem 20.1 in [46])

$$\mathcal{F}(w;D) = \int_D \varphi(\nabla w(x))\,dx,$$

where φ is a quasiconvex function, and $\varphi(v) \leq f(v)$ for all $v \in \mathbb{R}^{n\times N}$. Hence $\varphi \leq Qf$ and we conclude that (2.8) holds.

For general constant-rank operators $\mathcal{A}$, and if f is not continuous with respect to v, it may happen that $\mathcal{F}_0((u,v);\cdot)$ is not even the trace of a Radon measure in $\mathcal{O}(\Omega)$, and thus (2.5) fails, where we define

$$\mathcal{F}_0((u,v);D) = \inf\Big\{\liminf_{n\to\infty} F((u_n,v_n);D) : (u_n,v_n) \in L^p(D;\mathbb{R}^m)\times L^q(D;\mathbb{R}^d),$$
$$u_n \to u \text{ in } L^p(D;\mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D;\mathbb{R}^d), \quad \mathcal{A}v_n = 0\Big\}.$$

As an example, consider $d=2$, $N=1$, $\Omega := (0,1)$, $v=(v_1,v_2)$, and let $\mathcal{A}(v)=0$ if and only if $v_2'=0$ as in (ii) above. Let

$$f(v) := \begin{cases} (v_1-1)^2+v_2^2 & \text{if } v_2 \in \mathbb{Q}, \\ (v_1+1)^2+v_2^2 & \text{if } v_2 \notin \mathbb{Q}. \end{cases}$$

Although f satisfies a quadratic growth condition of the type (H), it is easy to see that for all intervals $(a,b)\subset(0,1)$,

$$\begin{aligned} \mathcal{F}_0((u,v);(a,b)) &= \mathcal{F}_0(v;(a,b)) \\ &= \min\left\{\int_a^b((v_1-1)^2+v_2^2)dx, \int_a^b((v_1+1)^2+v_2^2)dx\right\} \end{aligned}$$

which is not the trace of a Radon measure on $\mathcal{O}(\Omega)$. On the other hand, it may be shown that

$$\mathcal{F}((u,v);(a,b)) = \mathcal{F}(v;(a,b)) = \int_a^b (\psi^{**}(v_1)+v_2^2)\,dx,$$

where $\psi^{**}(v_1)$ is the convex envelope of

$$\psi(v_1) := \min\left\{(v_1-1)^2,(v_1+1)^2\right\}.$$

(iv) Using the growth condition (H), a mollification argument, and the linearity of $\mathcal{A}$, it can be shown that (see Remark 3.3 in [66])

$$\begin{aligned} \mathrm{Q}_{\mathcal{A}} f(x,u,v) &= \inf \left\{ \int_Q f(x,u,v+w(y))\,dy : w \in L^q_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \operatorname{Ker}\mathcal{A}, \right. \\ &\qquad \left. \int_Q w(y)\,dy = 0 \right\}. \end{aligned}$$

We write $w \in L^q_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \operatorname{Ker}\mathcal{A}$ when $w \in L^q_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)$ and $\mathcal{A}w = 0$ in $W^{-1,q}(Q;\mathbb{R}^l)$.

Although in Theorem 2.1 the functions u and v are not related to each other, the arguments of the proof work equally well when u and v are not independent. Indeed as corollaries, we can prove the following two theorems.

Theorem 2.3 ([30], Theorem 1.5) *Let $1 \le p \le \infty$, let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set, and suppose that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to [0,\infty)$ is a Carathéodory function satisfying*

$$0 \le f(x,u,v) \le C\left(1 + |u|^p + |v|^p\right), \qquad 1 \le p < \infty,$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^N \times \mathbb{R}^{N^2}$, where $C > 0$, and

$$f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N^2};[0,\infty)) \qquad \text{if } p = \infty.$$

Then for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} u = 0$, we have

$$\begin{aligned} \int_\Omega \bar{f}(x,u(x),\nabla u(x))\,dx = \inf \left\{ \liminf_{n\to\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx : \right. \qquad (2.9) \\ \left. \{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^N), \operatorname{div} u_n = 0, u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N), (\overset{*}{\rightharpoonup} \text{ if } p = \infty) \right\}, \end{aligned}$$

where, for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^N \times \mathbb{R}^{N^2}$,

$$\bar{f}(x,u,v) := \inf \left\{ \int_Q f(x,u,v+\nabla w(y))\,dy : w \in C^\infty_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^N),\ \operatorname{div} w = 0 \right\}.$$

This result was proved in [30] in all this generality (for a different proof, with additional smoothness assumptions, see [26]). A related problem was addressed by Dal Maso, Defranceschi and Vitali in [47], where it was shown that the Γ-limit of a family of functionals of the type (2.9) may be non local if (H) is violated.

Theorem 2.4 ([30], Theorem 1.3) *Let $1 \le p \le \infty$, $s \in \mathbb{N}$, and suppose that $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ is a Carathéodory function satisfying*

$$0 \le f(x,\mathbf{u},v) \le C\left(1 + |\mathbf{u}|^p + |v|^p\right), \qquad 1 \le p < \infty,$$

for a.e. $x \in \Omega$ and all $(\mathbf{u},v) \in E^d_{[k-1]} \times E^d_k$, where $C > 0$, and

$$f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times E^d_{[k-1]} \times E^d_k;[0,\infty)) \qquad \textit{if } p = \infty.$$

Then for every $u \in W^{k,p}(\Omega;\mathbb{R}^d)$ we have

$$\int_\Omega \mathcal{Q}^k f(x,u,\dots,\nabla^k u)\,dx = \inf\Bigg\{\liminf_{n\to\infty}\int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx : \{u_n\} \subset W^{k,p}(\Omega;\mathbb{R}^d), u_n \rightharpoonup u \text{ in } W^{k,p}(\Omega;\mathbb{R}^d) \quad (\overset{*}{\rightharpoonup} \text{ if } p = \infty)\Bigg\},$$

where, for a.e. $x \in \Omega$ and all $(\mathbf{u},v) \in E^d_{[k-1]} \times E^d_k$,

$$\mathcal{Q}^k f(x,\mathbf{u},v) := \inf\left\{\int_Q f(x,\mathbf{u},v + \nabla^k w(y))\,dy : w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)\right\}.$$

When $k > 1$ earlier lower semicontinuity results related to Theorem 2.4 are due to Meyers [87], Fusco [72], and Guidorzi and Poggiolini [75] (see also [5]), while the first integral representation results for the relaxed energy when the integrand depends on the full set of variables, that is $f = f(x,u,\dots,\nabla^k u)$, were obtained by [30]. As mentioned before, classical truncation methods for $k = 1$ cannot be extended in a simple way to truncate higher order derivatives, and successful techniques often rely on p-equi-integrability, and thus cannot work in the linear growth case. Indeed, when $p = 1$ due to loss of reflexivity of the space $W^{k,1}(\Omega;\mathbb{R}^d)$ one can only conclude that an energy bounded sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$ with $\sup_n \|u_n\|_{W^{k,1}} < \infty$ admits a subsequence (not relabelled) such that

$$u_n \to u \quad \text{in } W^{k-1,1}(\Omega;\mathbb{R}^d),$$

where $u \in W^{k-1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^{k-1}u$ is a vector-valued function of bounded variation. This leads us now to seek to establish lower semicontinuity in the space $W^{k,1}(\Omega;\mathbb{R}^d)$ under this natural notion of convergence, and when $u \in BV^k(\Omega;\mathbb{R}^d)$ (see [55], [110]).

When $k = 1$ the scalar case $d = 1$ has been extensively treated, while the vectorial case $d > 1$ was first studied by Fonseca and Müller in [64] where it was proven (sequential) lower semicontinuity in $W^{1,1}(\Omega;\mathbb{R}^d)$ of a functional

$$u \mapsto \int_\Omega f(x,u(x),\nabla u(x))\,dx,$$

with respect to strong convergence in $L^1(\Omega;\mathbb{R}^d)$ (see also [8], [59], [60], [65], and the references contained therein). The approach in [64] is based on blow–up and truncation methods.

The following theorem was proved in the case $k = 1$ by Ambrosio and Dal Maso [8], while Fonseca and Müller [64] treated general integrands of the form $f = f(x, u, \nabla u)$, but their argument requires coercivity (see also [73]). The case $k \geq 2$ is due to Amar and De Cicco [4] (see [62] for a proof for all $k \geq 1$).

Proposition 2.5 ([62], Proposition 2.1) *Let $f : E^d_k \to [0,\infty)$ be a function k-quasiconvex, such that*

$$0 \leq f(\xi) \leq C\,(1 + |\xi|)\,,$$

for all $\xi \in E^d_k$. Moreover, when $k \geq 2$ assume that

$$f(\xi) \geq C_1\,|\xi| \quad \textit{for } |\xi| \textit{ large.}$$

Let $\{u_n\}$ be a sequence of functions in $W^{k,1}(Q;\mathbb{R}^d)$ converging to 0 in the space $W^{k-1,1}(Q;\mathbb{R}^d)$. Then

$$f(0) \,\leq \liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx.$$

More generally we consider the case where f depends essentially only on x and on the highest order derivatives, that is $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require to truncate the initial sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$.

Theorem 2.6 ([62], Theorem 1.1) *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ be a Borel integrand. Suppose that for all $(x_0, \mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ and $\varepsilon > 0$ there exist $\delta_0 > 0$ and a modulus of continuity ρ, with $\rho(s) \leq C_0(1 + s)$ for $s > 0$ and for some $C_0 > 0$, such that*

$$f(x_0, \mathbf{v}_0, \xi) - f(x, \mathbf{v}, \xi) \leq \varepsilon(1 + f(x, \mathbf{v}, \xi)) + \rho(|\mathbf{v} - \mathbf{v}_0|) \tag{2.10}$$

for all $x \in \Omega$ with $|x - x_0| \leq \delta_0$, and for all $(\mathbf{v}, \xi) \in E^d_{[k-1]} \times E^d_k$. Assume also that one of the following three conditions is satisfied:
(a) $f(x_0, \mathbf{v}_0, \cdot)$ is k-quasiconvex in E^d_k and

$$\frac{1}{C_1}|\xi| - C_1 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1(1 + |\xi|) \qquad \textit{for all } \xi \in E^d_k,$$

where $C_1 > 0$;
(b) $f(x_0, \mathbf{v}_0, \cdot)$ is 1-quasiconvex in E^d_k and

$$0 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1(1 + |\xi|) \qquad \textit{for all } \xi \in E^d_k,$$

where $C_1 > 0$;

(c) $f(x_0, \mathbf{v}_0, \cdot)$ is convex in E_k^d.
Let $u \in BV^k(\Omega; \mathbb{R}^d)$ and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \le \liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx.$$

Here $f(x_0, \mathbf{v}_0, \cdot)$ is said to be 1-quasiconvex if $f(x_0, \mathbf{v}_0, \cdot)$ is the trace on E_k^d of a 1-quasiconvex function $\bar{f}$ defined on $\mathbb{R}^{(d\times N^{k-1})\times N}$.

An important class of integrands which satisfy (2.10) of Theorem 2.6 is given by

$$f = f(x, \xi) := h(x) g(\xi),$$

where $h(x)$ is a nonnegative lower semicontinuous function and g is a nonnegative function which satisfies either (a) or (b) or (c). The case where $h(x) \equiv 1$ and g satisfies condition (a) was proved by Amar and De Cicco [4]. Theorem 2.6 extends a result of Fonseca and Leoni (Theorem 1.7 in [59]) to higher order derivatives, where the statement is exactly that of Theorem 2.6 setting $k = 1$ and excluding part (a). Related results when $k = 1$ where obtained previously by Serrin [98] in the scalar case $d = 1$ and by Ambrosio and Dal Maso [8] in the vectorial case $d > 1$ (see also Fonseca and Müller [64], [65]). Even in the simple case where $f = f(\xi)$ it is not known if Theorem 2.6(a) still holds without the coercivity condition

$$f(\xi) \ge \frac{1}{C_1}|\xi| - C_1.$$

The main tool in the proof of Theorem 2.6, used also in an essential way in subsequent results, is the *blow–up method* introduced by Fonseca and Müller [64], [65], which reduces the domain Ω to a ball and the target function u to a polynomial.

When the integrand f depends on the full set of variables in an essential way, the situation becomes significantly more complicated since one needs to truncate gradients and higher order derivatives in order to localize lower order terms. The following theorem was proved for $k = 1$ by Fonseca and Leoni in [59], Theorem 1.8, and extended to the higher order case in [62].

Theorem 2.7 ([62], Theorem 1.2) *Let $f : \Omega \times E_{[k-1]}^d \times E_k^d \to [0, \infty)$ be a Borel integrand, with $f(x, \mathbf{v}, \cdot)$ 1-quasiconvex in E_k^d. Suppose that for all $(x_0, \mathbf{v}_0) \in \Omega \times E_{[k-1]}^d$ either $f(x_0, \mathbf{v}_0, \cdot) \equiv 0$, or for every $\varepsilon > 0$ there exist C, $\delta_0 > 0$ such that*

$$f(x_0, \mathbf{v}_0, \xi) - f(x, \mathbf{v}, \xi) \le \varepsilon(1 + f(x, \mathbf{v}, \xi)), \tag{2.11}$$

$$C|\xi| - \frac{1}{C} \le f(x_0, \mathbf{v}_0, \xi) \le C(1 + |\xi|) \tag{2.12}$$

for all $(x, \mathbf{v}) \in \Omega \times E^d_{[k-1]}$ *with* $|x - x_0| + |\mathbf{v} - \mathbf{v}_0| \leq \delta_0$ *and for all* $\xi \in E^d_k$. *Let* $u \in BV^k(\Omega; \mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,1}(\Omega; \mathbb{R}^d)$ *converging to* u *in* $W^{k-1,1}(\Omega; \mathbb{R}^d)$. *Then*

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx.$$

A standing open problem is to decide whether Theorem 2.7 continues to hold under the weaker assumption that $f(x, \mathbf{v}, \cdot)$ is k-quasiconvex, which is the natural assumption in this context.

As in Theorem 2.6, conditions (2.11) and (2.12) can be considerably weakened if we assume that $f(x, \mathbf{v}, \cdot)$ is convex rather than 1-quasiconvex. Indeed we have the following result:

Theorem 2.8 ([62], Theorem 1.5) *Let* $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0, \infty]$ *be a lower semicontinuous function, with* $f(x, \mathbf{v}, \cdot)$ *convex in* E^d_k. *Suppose that for all* $(x_0, \mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ *either* $f(x_0, \mathbf{v}_0, \cdot) \equiv 0$, *or there exist* C_1, $\delta_0 > 0$, *and a continuous function* $g : B(x_0, \delta_0) \times B(\mathbf{v}_0, \delta_0) \to E^d_k$ *such that*

$$f(x, \mathbf{v}, g(x, \mathbf{v})) \in L^\infty\left(B(x_0, \delta_0) \times B(\mathbf{v}_0, \delta_0); \mathbb{R}\right), \tag{2.13}$$

$$f(x, \mathbf{v}, \xi) \geq C_1|\xi| - \frac{1}{C_1}$$

for all $(x, \mathbf{v}) \in \Omega \times E^d_{[k-1]}$ *with* $|x - x_0| + |\mathbf{v} - \mathbf{v}_0| \leq \delta_0$ *and for all* $\xi \in E^d_k$. *Let* $u \in BV^k(\Omega; \mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,1}(\Omega; \mathbb{R}^d)$ *converging to* u *in* $W^{k-1,1}(\Omega; \mathbb{R}^d)$. *Then*

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx.$$

Theorem 2.8 was obtained by Fonseca and Leoni (see [60], Theorem 1.1) in the case $k = 1$. It is interesting to observe that without a condition of the type (2.13) Theorem 2.8 is false in general. This has been recently proved by Černý and Malý in [37].

The proofs of Theorems 2.6(b) and (c), 2.7 and 2.8 can be deduced easily from the corresponding ones in [59], [60], where $k = 1$. It suffices to write

$$\int_\Omega f(x, u(x), \dots, \nabla^k u(x))\, dx =: \int_\Omega F(x, \mathbf{v}(x), \nabla \mathbf{v}(x))\, dx$$

with $\mathbf{v} := \left(u, \dots, \nabla^{k-1} u\right)$, and then to perturb the new integrand F in order to recover the full coercivity conditions necessary to apply the results in [59], [60]. This approach cannot be used for k-polyconvex integrands and a new proof is needed to treat this case (see [48]). Thus Theorem 2.6(a) and Theorem 2.9 below are the only truly genuine higher order results, in that they cannot be reduced in a trivial way to a first order problem.

For each $\xi \in E^d_k$ let $\mathcal{M}(\xi) \in \mathbb{R}^\tau$ be the vector whose components are all the minors of ξ.

Theorem 2.9 ([59], Theorem 1.6) *Let $h : \Omega \times E^d_{[k-1]} \times \mathbb{R}^\tau \to [0,\infty]$ be a lower semicontinuous function, with $h(x,\mathbf{v},\cdot)$ convex in $\mathbb{R}^\tau$. Suppose that for all $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $h(x_0,\mathbf{v}_0,\cdot) \equiv 0$, or there exist C, $\delta_0 > 0$, and a continuous function $g : B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0) \to \mathbb{R}^\tau$ such that*

$$\begin{aligned} &h(x,\mathbf{v},g(x,\mathbf{v})) \in L^\infty\left(B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0);\mathbb{R}\right), \\ &h(x,\mathbf{v},v) \geq C|v| - \frac{1}{C} \end{aligned}$$

for all $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ with $|x-x_0| + |\mathbf{v}-\mathbf{v}_0| \leq \delta_0$ and for all $v \in \mathbb{R}^\tau$. Let $u \in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,p}(\Omega;\mathbb{R}^d)$ which converges to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$, where p is the minimum between N and the dimension of the vectorial space E^d_{k-1}. Then

$$\begin{aligned} &\int_\Omega h\left(x,u,\ldots,\nabla^{k-1}u,\mathcal{M}(\nabla^k u)\right) dx \\ &\qquad \leq \liminf_{n\to\infty} \int_\Omega h(x,u_n,\ldots,\nabla^{k-1}u_n,\mathcal{M}(\nabla^k u_n))\, dx. \end{aligned}$$

Theorem 2.9 is closely related to a result of Ball, Currie and Olver [16], where it was assumed that

$$h(x,\mathbf{v},v) \geq \gamma\left(|v|\right) - \frac{1}{C},$$

where

$$\frac{\gamma\left(s\right)}{s} \to \infty \text{ as } s \to \infty.$$

Also, as stated above and with $k=1$, Theorem 2.9 was proved by Fonseca and Leoni in [60], Theorem 1.4.

In the scalar case $d=1$, that is when u is an $\mathbb{R}$-valued function, and for first order gradients, i.e. $k=1$, condition (2.12) can be eliminated, see Theorem 1.1 in [59]. In particular in [59] Fonseca and Leoni have shown the following result

Proposition 2.10 ([59], Corollary 1.2) *Let $g : \mathbb{R}^N \to [0,\infty)$ be a convex function, and let $h : \Omega \times \mathbb{R} \to [0,\infty)$ be a lower semicontinuous function. If $u \in BV(\Omega;\mathbb{R})$ and $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R})$ converges to u in $L^1(\Omega;\mathbb{R})$, then*

$$\int_\Omega h(x,u)g(\nabla u)\, dx \leq \liminf_{n\to\infty} \int_\Omega h(x,u_n)g(\nabla u_n)\, dx.$$

It is interesting to observe that an analog of this result is false when $k \geq 2$.

Theorem 2.11 ([62], Theorem 1.4) *Let $\Omega := (0,1)^N$, $N \geq 3$, and let h be a smooth cut-off function on $\mathbb{R}$ with $0 \leq h \leq 1$, $h(u) = 1$ for $u \leq \frac{1}{2}$, $h(u) = 0$*

for $u \geq 1$. There exists a sequence of functions $\{u_n\}$ in $W^{2,1}(\Omega;\mathbb{R})$ converging to zero in $W^{1,1}(\Omega;\mathbb{R})$ such that $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ is uniformly bounded and

$$\limsup_{n\to\infty}\int_\Omega h(u_n)(1-\Delta u_n)^+\,dx < \int_\Omega h(0)\,dx.$$

Here Theorem 2.8 fails to apply because we do not have bounds on the full Hessian matrices $\{D^2 u_n\}$. The proof of Theorem 2.11 is hinged on the construction below.

Lemma 2.12 ([62], Lemma 4.1) *Let D be a cube with $|D| \leq 1$, and let $N \geq 3$. Then there exists constants $C > 0$, $\lambda \in (0,1)$ depending only on N, a function $u \in W^{2,\infty}(D;\mathbb{R})$ with compact support in D, and sets A, E, $G \subset D$, with $A \cup E \cup G = D$ and $|E| \leq \lambda |D|$, such that*

$$\|\Delta u\|_{L^1(D;\mathbb{R})} \leq C|D|, \quad \|u\|_{W^{1,1}(D;\mathbb{R})} \leq C|D|^{1+\frac{1}{N}}, \tag{2.14}$$

$$\Delta u = 1 \quad \text{on } A, \tag{2.15}$$

$$u = 0 \quad \text{on } E, \quad u \geq 1 \quad \text{on } G. \tag{2.16}$$

Proof. After a translation we may assume that there exists $B(0,R) \subset D$ such that

$$C^{-1}R^N \leq |D| \leq CR^N, \quad R \in (0,1/2),$$

for some $C > 0$. We search for a radial function of type

$$u(x) := \varphi(|x|),$$

where φ is a C^2-function on $(0,\infty)$ such that

$$\varphi(t) = 0 \quad \text{for } t \geq R, \tag{2.17}$$

$$\varphi'(0+) = 0. \tag{2.18}$$

Further we want that for some $a > 0$

$$\Delta u(x) = \begin{cases} -a & \text{if } |x| < r, \\ 1 & \text{if } r < |x| < R, \end{cases} \tag{2.19}$$

where r is determined by the equation

$$r^{2-N}R^N = 2N(N-2). \tag{2.20}$$

Note that $r \in (0,R)$ because $R < 1$ and $N \geq 3$. In order to find a and φ satisfying (2.17), (2.18) and (2.19), we note that

$$\Delta u(x) = \varphi''(|x|) + |x|^{-1}\,(N-1)\,\varphi'(|x|), \quad \text{for } |x| \neq 0,$$

or, equivalently,

$$\Delta u(x) = t^{1-N}(t^{N-1}\varphi'(t))'\ , \quad \text{where } t = |x|.$$

On the interval (r,R) (2.19) now yields

$$(t^{N-1}\varphi'(t))' = t^{N-1},$$

and thus, by (2.17),

$$\varphi'(t) = \frac{t}{N}\left(1-\frac{R^N}{t^N}\right). \tag{2.21}$$

On the interval $(0,r)$, and in view of (2.19), we have

$$(t^{N-1}\varphi'(t))' = -at^{N-1}$$

which, together with (2.18), implies that

$$\varphi'(t) = -\frac{at}{N}. \tag{2.22}$$

We have

$$-\frac{ar}{N} = \varphi'(r-) = \varphi'(r+) = \frac{r}{N}\left(1-\frac{R^N}{r^N}\right)$$

and thus

$$a = \left(\frac{R^N}{r^N}-1\right). \tag{2.23}$$

Now the function u is uniquely determined by its properties. Obviously we have $(2.16)_1$ by setting

$$A := B(0,R)\setminus B(0,r), \quad E := D\setminus B(0,R), \quad G := B(0,r),$$

with $|E| \le \lambda|D|$ and $\lambda = \lambda(N)$. In light of (2.19) and (2.23) we have

$$\begin{aligned}\|\Delta u\|_{L^1(D;\mathbb{R})} &\le |B(0,R)\setminus B(0,r)| + a\,|B(0,r)| \\ &= \omega_N\left(R^N - r^N + \left(\frac{R^N}{r^N}-1\right)r^N\right) \le 2\omega_N R^N,\end{aligned}$$

where $\omega_N := |B(0,1)|$. If $x\in G$, we have by (2.22), (2.21) and (2.20)

$$\begin{aligned}u(x) \ge \varphi(r) &= -\int_r^R \varphi'(t)\,dt = \frac{1}{N}\int_r^R (R^N t^{1-N} - t)\,dt \\ &\ge \frac{1}{N(N-2)}(r^{2-N}R^N - R^2) - \frac{R^2}{2N} = 2 - \frac{R^2}{2\,(N-2)} \ge 1,\end{aligned}$$

as $R\le \frac{1}{2}$ as $B(0,R)\le D$ and the side length of D does not exceed 1. This proves $(2.16)_2$. By (2.21) and (2.22),

$$\begin{aligned}\int_D |\nabla u|\,dx &\le C\int_0^R t^{N-1}|\varphi'(t)|\,dt \\ &\le C\left(\int_0^r r^{-N}R^N t^N\,dt + \int_r^R R^N\,dt\right) \le CR^{N+1},\end{aligned}$$

which, with the aid of the Poincaré inequality for zero boundary values, proves $(2.14)_2$. ■

Proof of Theorem 2.11. We set $\Omega = (0,1)^N$, and we construct the $\frac{1}{n}$-periodic sequence $\{u_n\}$ as follows: divide Ω into small cubes D_α of measure $\frac{1}{n^N}$, $\alpha \in I_n$ where the set of indices I_n has cardinality n^N. On each D_α we construct u_n as indicated in 2.12, and denote by A_α, E_α, G_α the corresponding sets. Then $u_n \to 0$ in $W^{1,1}(\Omega;\mathbb{R})$ because

$$\|u_n\|_{W^{1,1}(\Omega;\mathbb{R})} = \sum_{\alpha\in I_n} \|u_n\|_{W^{1,1}(D_\alpha;\mathbb{R})} \le n^N C \left(\frac{1}{n^N}\right)^{1+\frac{1}{N}} \to 0 \quad \text{as } n \to \infty,$$

and $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ is uniformly bounded since

$$\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})} = \sum_{\alpha\in I_n} \|\Delta u_n\|_{L^1(D_\alpha;\mathbb{R})} \le n^N C \frac{1}{n^N} = C.$$

Consider the functional

$$F(v) := \int_\Omega h(v)(1-\Delta v)^+ \, dx.$$

For $\alpha \in I_n$ we have by (2.14)–(2.16)

$$\begin{aligned} h(u_n) = 1 \quad \text{and} \quad \Delta u_n &= 0 \quad \text{on } E_\alpha, \\ \Delta u_n &= 1 \quad \text{on } A_\alpha, \\ h(u_n) &= 0 \quad \text{on } G_\alpha, \end{aligned}$$

and thus

$$\int_{D_\alpha} h(u_n)(1-\Delta u_n)^+ \, dx = |E_\alpha| \le \lambda |D_\alpha|.$$

Summing up over $\alpha \in I_n$, we conclude that

$$\int_\Omega h(u_n)(1-\Delta u_n)^+ \, dx \le \lambda < 1 = F(0).$$

■

3 On a Phase Transitions Model for Second Order Derivatives

We now turn to Question II where we will focus on a singular perturbations model for a double-well potential W. The asymptotic behavior of functionals of the type

$$J_\varepsilon (v;\Omega) := \int_\Omega \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2 \, dx. \tag{3.1}$$

has been exploited in theories of phase transitions, and it was first studied analytically by Modica and Mortola [90], and subsequently it was applied by Modica [89] to the van der Waals–Cahn-Hilliard theory of fluid-fluid phase transitions to solve an "optimal design" problem proposed by Gurtin [76]:

$$\text{Minimize } \int_\Omega W(u)\, dx,$$

under the density constraint

$$\frac{1}{|\Omega|}\int_\Omega u\, dx = \theta a + (1-\theta)b,$$

for some $\theta \in (0,1)$, and where W is a nonnegative bulk energy density with $\{W = 0\} = \{a, b\}$, $a, b \in \mathbb{R}$, $a < b$. It is easy to see that this problem admits infinitely many solutions, and a selection criteria for physically preferred solutions takes into account interfacial energy, thus adding a gradient term which, upon rescaling, leads to (3.1).

Using De Giorgi's notion of Γ-convergence ([49]; see also [3], [27], [46]), it was shown in [89], [90], that

$$\Gamma - \lim_{\varepsilon\to 0^+} J_\varepsilon\left(u_0;\Omega\right) = \begin{cases} K_0 \mathrm{Per}_\Omega(E) & \text{if} \quad u = \chi_E a + (1-\chi_E)b,\ |E| = \theta\,|\Omega| \\ & \qquad u \in BV\left(\Omega;\{a,b\}\right), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

where $K_0 := \int_a^b \sqrt{W(s)}ds$. The theory of Γ-convergence guarantees that preferred designs are those which, for the given volume fraction θ, exhibit minimal interfacial area.

Generalizations of (3.1)-(3.2) were obtained by Bouchitté [23] and by Owen and Sternberg [94] for the undecoupled problem, in which the integrand in J_ε has the form $\varepsilon^{-1} f(x, v(x), \varepsilon\nabla v(x))$. For the study of local minimizers we refer to Kohn and Sternberg [83].

The vector-valued setting, where $u : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^N$, $d, N > 1$, was considered in [18], [67], where K_0 is replaced by

$$K_1 := \inf\left\{\int_{-L}^{L} W\left(g\left(s\right)\right) + \left|g'\left(s\right)\right|^2 ds : L > 0,\ g \text{ piecewise } C^1, \quad g(-L) = a,\ g(L) = b\right\}.$$

The case where W has more than two wells was addressed by Baldo [14] (see also Sternberg [101]), and later generalized by Ambrosio [6].

Motivated by questions within the realm of elastic solid-to-solid phase transitions [17], [39] , [82], with $u : \Omega \to \mathbb{R}^d$ standing for the deformation, we now consider the corresponding problem for gradient vector fields, where in place of J_ε we introduce

$$I_\varepsilon\left(u;\Omega\right) := \begin{cases} \displaystyle\int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon|\nabla^2 u|^2\,dx & \text{if } u \in W^{2,2}\left(\Omega;\mathbb{R}^d\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

The analysis of this model has defied a considerable mathematical effort during the past decade. Here we will report on a recent contribution by Conti, Fonseca and Leoni [40].

An intermediate case between (3.1) and (3.3), where the nonconvex potential depends on u and the singular perturbation on $\nabla^2 u$, has been recently studied by Fonseca and Mantegazza [63] (for other generalizations see [57]). Also, in the two-dimensional case and when W vanishes on the unit circle (3.3) reduces to the so-called Eikonal functional which arises in the study of liquid crystals (see [11]) as well as in blistering of delaminated thin films (see [93]). Recently, the Eikonal problem has received considerable mathematical attention, but in spite of substantial partial progress (see [9], [12], [53], [80]) its Γ-limit remains to be identified.

Going back to the results obtained in [40] concerning (3.3), we first notice that frame-indifference requires that $W(\xi) = W(R\xi)$ for all ξ and all $R \in SO(N)$, where $SO(N)$ is the set of rotations in $\mathbb{R}^N$. Therefore, and by analogy with the hypotheses initially placed on (3.1), if we assume that $W(A) = 0 = W(B)$ then $\{W = 0\} \subset SO(N)A \cup SO(N)B$. Also, in order to guarantee the existence of "classical" (as opposed to measure-valued) non affine solutions for the limiting problem, and in view of Hadamard's compatibility condition for layered deformations (see also Ball and James [17]), the two wells must be rank-one connected. Hence, so as to be able to construct gradients taking values only on $\{A, B\}$ and layered perpendicularly to ν, we assume that

$$A - B = a \otimes \nu$$

for some $a \in \mathbb{R}^N$ and $\nu \in S^{N-1} := \partial B(0,1) \subset \mathbb{R}^N$, and we simplify (greatly!) the problem by removing the frame-indifference constraint, and assuming simply that

$$\{W = 0\} = \{A, B\}.$$

Since now interfaces of minimizers must be planar with normal ν (see [17]), at first glance the analysis may seem to be greatly simplified as compared with the initial problem (3.1) which requires handling minimal surfaces. However, it turns out that the pde constraint curl $= 0$ imposed on the admissible fields presents numerous difficulties to the characterization of the Γ-limsup. In particular, if, say, ∇u has a layered structure with two interfaces then it is possible to construct a "realizing" (effective, or recovering) sequence nearby each interface, but the task of gluing together the two sequences on a suitable low-energy intermediate layer is very delicate. In order to illustrate the difficulties encountered here, we explain briefly how we would "normally" undertake the heuristic argument to glue together two optimal sequences $\{(u_n, \varepsilon_n)\}$, corresponding to an interface of a cylindrical body Ω at a given height h, and $\{(v_n, \delta_n)\}$, corresponding to an interface at a height h', $h' > h$.

First we must convince ourselves that the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ (related to the "periodicity" of the ripples of the optimal fine structure near each interface) may be taken to be the same. This is by no means trivial (although true)! But let us take this for granted, and, as it is usual, we consider as a candidate for the two-interface situation a convex combination

$$w_{k,n} := \varphi_k u_n + (1 - \varphi_k) v_n$$

where φ_k is a smooth cut-off function, with $\{0 < \varphi_k < 1\} \subset L_{k,n}$ and $L_{k,n}$ is a horizontal layer intermediate between heights h and h' (here we are assuming that $\nu = e_N$). The crux of the problem is to choose $L_{k,n}$ in a judicious way so that no extra energy is added to the system by the new sequence $\{w_{k,n}\}$.

Using De Giorgi's Slicing Method, we slice horizontally the layer between heights h and h' into M horizontal sub-layers L_k of width $(h' - h)/M$. In view of the fact that $||\nabla\varphi||_\infty = O(M)$, we then have

$$\sum_{k=1}^{M} \int_{L_k} \frac{1}{\varepsilon_n} W(w_{k,n}) + \varepsilon_n |\nabla w_{k,n}|^2 \leq O\left(\frac{1}{\varepsilon_n}\right) + \varepsilon_n M^2 ||u_n - v_n||^2. \tag{3.4}$$

Choosing $k = k(n)$ such that

$$\int_{L_{k(n)}} \frac{1}{\varepsilon_n} W(w_{k(n),n}) + \varepsilon_n |\nabla w_{k(n),n}|^2 \leq \frac{1}{M} \sum_{k=1}^{M} \int_{L_k} \frac{1}{\varepsilon_n} W(w_{k,n}) + \varepsilon_n |\nabla w_{k,n}|^2, \tag{3.5}$$

it is clear that by setting $M = O\left(\frac{1}{\varepsilon_n \sqrt{||u_n - v_n||_{L^2}}}\right)$, and using the fact that the admissible sequences $\{u_n\}$ and $\{v_n\}$ satisfy $||u_n - v_n||_{L^2} \to 0$ in the intermediate layer between heights h and h', we may conclude that

$$\lim_{n\to\infty} \int_{L_{k(n)}} \frac{1}{\varepsilon_n} W(w_{k(n),n}) + \varepsilon_n |\nabla w_{k(n),n}|^2 = 0.$$

Suppose now that we want to extend this argument to the present setting involving second order derivatives. The estimate (3.4) now becomes

$$\sum_{k=1}^{M} \int_{L_k} \frac{1}{\varepsilon_n} W(\nabla w_{k,n}) + \varepsilon_n |\nabla^2 w_{k,n}|^2 \leq O\left(\frac{1}{\varepsilon_n}\right) + \varepsilon_n M^4 ||u_n - v_n||^2,$$

and, seeking *equi-partition of energy* as we have done above, we are led to

$$\varepsilon_n M^4 ||u_n - v_n||^2 = O\left(\frac{1}{\varepsilon_n}\right).$$

This would entail $M = O\left(\frac{1}{\sqrt{\varepsilon_n}\sqrt{||u_n - v_n||_{L^2}}}\right)$, and thus the upper bound in (3.5) would be

$$\int_{L_{k(n)}} \frac{1}{\varepsilon_n} W(w_{k,n}) + \varepsilon_n |\nabla w_{k,n}|^2 \leq \frac{1}{M} O\left(\frac{1}{\varepsilon_n}\right) \leq C\sqrt{\frac{||u_n - v_n||_{L^2}}{\varepsilon_n}}.$$

Therefore, to ensure that the extra energy in the layer $L_{k(n)}$ does not affect the optimality of the sequence, we would have to guarantee that $\{u_n - v_n\}$ goes to zero in L^2 faster than $\sqrt{\varepsilon_n}$, and whether or not this holds it remains an open question !

The restrictive constitutive hypotheses placed on W in Theorems 3.2 and 3.3 below allow us to find alternative ways in which the gluing is successful. The main idea consists in accepting the fact that matching in one single swift step is simply to abrupt for higher order problems. We proceed using two-set matchings where control of Poincaré and Poincaré-Freidrichs' constants is carefully kept.

Assume

(H_1) W is continuous, $W(\xi) = 0$ if and only if $\xi \in \{A, B\}$, where $A - B = a \otimes \nu$ for some $a \in \mathbb{R}^d \setminus \{0\}$ and $\nu \in S^{N-1}$,

and we introduce our *candidate* for Γ-limit

$$I(u;\Omega) = \begin{cases} K^* \mathcal{H}^{N-1}(S(\nabla u) \cap \Omega) & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \nabla u \in BV(\Omega;\{A,B\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where $S(\nabla u)$ is the singular set of ∇u, i.e. the collection of interfaces,

$$\begin{aligned} K^* &:= \Gamma - \liminf_{\varepsilon \to 0^+} I_\varepsilon(u_0; Q_\nu) \\ &= \inf \Big\{ \liminf_{n \to \infty} I_{\varepsilon_n}(u_n; Q_\nu) : \varepsilon_n \to 0^+,\ u_n \in W^{2,2}\left(Q_\nu; \mathbb{R}^d\right) \\ &\qquad u_n \to u_0 \text{ in } L^1\left(Q; \mathbb{R}^d\right) \Big\}, \end{aligned}$$

where Q_ν is a unit cube in $\mathbb{R}^N$ centered at the origin and with two of its faces orthogonal to ν, and

$$\nabla u_0 := \begin{cases} A & \text{if } x \cdot \nu > 0, \\ B & \text{if } x \cdot \nu < 0. \end{cases}$$

We start with the following compactness result.

Theorem 3.1 ([40], Theorem 1.1) *[Compactness] Assume that the double well potential W satisfies conditions (H_1) and*

(H_2) *there exists $C > 0$ such that*

$$W(\xi) \geq C|\xi| - \frac{1}{C}$$

for all $\xi \in \mathbb{R}^{d \times N}$.

Let $\varepsilon_n \to 0^+$. If $\{u_n\} \subset W^{2,2}\left(\Omega;\mathbb{R}^d\right)$ is such that

$$\sup_n I_{\varepsilon_n}\left(u_n;\Omega\right) < \infty,$$

then there exist a subsequence $\{u_{n_k}\}$ and a function $u \in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u \in BV\left(\Omega;\{A,B\}\right)$, such that

$$u_{n_k} - \frac{1}{|\Omega|}\int_\Omega u_{n_k}\,dx \to u \text{ in } W^{1,1}\left(\Omega;\mathbb{R}^d\right).$$

Next, and without loss of generality, we may assume that

$$A = -B = a \otimes e_N.$$

Consider

(H_2) $W(\xi) \to \infty$ as $|\xi| \to \infty$;

(H_3) $W(\xi) \geq W(0,\xi_N)$ where $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.

Note that (H_3) is satisfied by the prototype bulk energy density

$$W(\xi) := \min\left\{|\xi - A|^2, |\xi - B|^2\right\}.$$

Theorem 3.2 ([40], Theorem 1.3) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1), (H_2) and (H_3). Suppose, in addition, that W is differentiable at A and B. Let $u \in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, with $\nabla u \in BV\left(\Omega;\{A,B\}\right)$. Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon(u;\Omega) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u = (1 - \chi_E(x))\,A + \chi_E(x)\,B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

The hypothesis (H_3) entails a one dimensional character to the asymptotic problem. Indeed in this case the characterization of the constant K^* can be greatly simplified. It can be shown that K^* reduces to the analog of the constant K_1 introduced in (3), precisely, $K^* = K$ where

$$K := \inf\left\{\int_{-L}^{L} W(0, g(s)) + |g'(s)|^2\,ds : L > 0,\ g \text{ piecewise } C^1,\right.$$
$$\left. g(-L) = -a,\ g(L) = a\right\}.$$

Theorem 3.2 is related to work of Kohn and Müller [82] who studied the minimization of the functional

$$\int_{(0,L)\times(0,1)} \left(\frac{\partial u}{\partial x_1}\right)^2 + \varepsilon\left|\frac{\partial^2 u}{\partial x_2^2}\right|\,dx_1 dx_2$$

subject to the constraint $\left|\frac{\partial u}{\partial x_2}\right| = 1$ and boundary conditions.

The construction of a realizing sequence for the $\Gamma-$limsup is strongly hinged to the geometry of the domain. As an example, if we assume that

$$\begin{aligned}&\text{for each } t \in \mathbb{R} \text{ the horizontal section}\\ &\Omega_t := \{(x', x_N) \in \Omega : x_N = t\} \text{ is connected in } \mathbb{R}^N,\end{aligned}$$

and that

$$t \mapsto \mathcal{H}^{N-1}(\Omega_t) \text{ is continuous in } (\alpha, \beta),$$

where

$$\alpha := \inf \{x_N : x \in \Omega\}, \quad \beta := \sup \{x_N : x \in \Omega\},$$

(it is easy to see that convex domains or cylinders of the form $\omega \times (a, b)$, where $\omega \subset \mathbb{R}^{N-1}$, satisfy these conditions), then it is relatively easy to show that realizing sequences are one-dimensional. However, once this hypothesis is lost the analysis becomes exceedingly more complicated. We remark that these difficulties cannot be resolved by performing rotations and translations of Ω nearby the identity because the perimeter of the interface may change discontinuously under these transformations.

Next we present a situation where the transition behavior is no longer one-dimensional. Consider the *isotropy condition*

(H_4) W is even in each variable ξ_i, $i = 1, \cdots, N-1$, that is

$$W(\xi_1, \cdots, -\xi_i, \cdots, \xi_N) = W(\xi_1, \cdots, \xi_i, \cdots, \xi_N)$$

for each $i = 1, \cdots, N-1$,

where

$$\xi = (\xi_1, \cdots, \xi_N) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N \text{ times}}, \quad \xi' = (\xi_1, \cdots, \xi_N - 1) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{N-1 \text{ times}},$$

so that $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.

Theorem 3.3 ([40], Theorem 1.4) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1), (H_5), and that there exist an exponent $p \geq 2$, constants $c, C, \rho > 0$ and a convex function $g : [0, \infty) \to [0, \infty)$, with $g(s) = 0$ if and only if $s = 0$, such that g is derivable in $s = 0$, $g(2t) \leq cg(t)$ for all $0 \leq t \leq \rho$,*

$$\begin{aligned}g(|\xi - A|) \leq W(\xi) \leq cg(|\xi - A|) \ \ \textit{if} \ \ |\xi - A| \leq \rho,\\ g(|\xi - B|) \leq W(\xi) \leq cg(|\xi - B|) \ \ \textit{if} \ \ |\xi - B| \leq \rho,\end{aligned}$$

and

$$\frac{1}{C}|\xi|^p - C \leq W(\xi) \leq C(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{d\times N}$. *Let* $u \in W^{1,1}\left(\Omega;\mathbb{R}^d\right)$, *with* $\nabla u \in BV\left(\Omega;\{A,B\}\right)$. *Then*

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon\left(u;\Omega\right) = K^* \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = \left(1-\chi_E\left(x\right)\right)A + \chi_E\left(x\right)\, B$ *for* $\mathcal{L}^N$ *a.e.* $x \in \Omega$. *Moreover*

$$K^* = K_{\text{per}},$$

where

$$K_{\text{per}} := \inf\left\{\int_Q L\,W\left(\nabla v\right) + \frac{1}{L}\left|\nabla^2 v\right|^2\, dx : L>0, v \in W^{2,\infty}\left(Q;\mathbb{R}^d\right),\right.$$
$$\left.\nabla v = \pm a\otimes e_N \text{ nearby } x_N = \pm\frac{1}{2},\ v \text{ periodic of period one in } x'\right\}.$$

The condition $g\left(2t\right) \leq cg\left(t\right)$ for all $0 \leq t \leq \rho$, is called the *doubling condition* – it prevents g to be too degenerate near $t=0$; it is satisfied if $g\left(t\right) \sim$const.$\,t^p$ as $t\to 0^+$, for some $p\geq 1$, while it does not hold if g grows exponentially near the origin, i.e., $g\left(t\right) \sim$const.$\,\mathrm{e}^{-1/t^2}$ as $t\to 0^+$.

It would be interesting to know if Theorem 3.3 continues to hold without assuming the isotropy assumption $\left(H_4\right)$. We have not been able to prove this.

Most of the literature on singularly perturbed double-well potentials deals with asymptotic problems which are one-dimensional, i.e. with asymptotically optimal interface profiles which depend only on one coordinate (the distance to the interface). Such profiles have been shown to be optimal in the first-order gradient theory of phase transitions modeled by (3.1) (see also [63]), and the same has been conjectured in various other cases such as the Eikonal problem mentioned above. However, Jin and Kohn [80] have shown that a simple perturbation of the Eikonal functional has non one-dimensional minimizers. A one-dimensional *ansatz* is also often used in the physics literature on ferroelastic domain walls, even if its validity is still under debate (see e.g. [19], [33], [41], [96], and references therein).

In the problem of interest here, we defined K as the energy of the optimal one-dimensional interface profile, and K_{per} as the optimal energy of interface profiles which are periodic along the interface. As discussed above, under hypothesis (H_3) we show that $K_{\text{per}} = K$, hence that one-dimensional interface profiles are energetically preferred. However, building upon the example by Jin and Kohn [80], in [40], Section 8, we show that without hypothesis (H_3), in general, we may have

$$K_{\text{per}} < K$$

thereby proving that optimal interface profiles are, at least in some cases, not one-dimensional. This happens because generating finite in–plane gradients (i.e. having a dependence on the coordinates parallel to the interface) reduces the energy in the regions far away from the two potential wells. The zero-curl constraint leads then to an oscillatory pattern. In elasticity, this multidimensional behavior has been predicted in [33]. Similar mechanisms are at play in the theory of micromagnetism, where indeed various non-one-dimensional wall structures are known, such as cross-tie domain walls and charged zigzag walls in ferromagnetic thin films (see e.g. [77] and references therein). It would be interesting to know if K_{per} is smaller or equal to K for realistic ferroelastic potentials obtained from the Landau theory of phase transitions.

As mentioned before, gluing of recovering sequences asks for a careful handling of Poincaré and Poincaré-Friedrichs' inequalities. We conclude by presenting a generalization of Poincaré's inequality to Orlicz-Sobolev spaces, which can then be used to prove Theorem 3.3. Our argument in [40] follows that of Maz'ja [86] for the case $g(s) = |s|^p$. A first version has been proved by Bhattacharya and Leonetti [21] in the case where Ω is convex and $S = \Omega$.

First we observe that the function g introduced in Theorem 3.3 to control the behavior of W near the wells may be extended to a function G still satisfying the doubling condition, and such that $G(|\cdot - A|)$ and $G(|\cdot - B|)$ may be compared with W in the whole $\mathbb{R}^{d\times N}$.

Lemma 3.4 ([40], Lemma 7.2) *Let $g : [0,\infty) \to [0,\infty)$ be a convex function, with $g(s) = 0$ if and only if $s = 0$, such that*

$$g(2t) \leq Cg(t) \tag{3.6}$$

for all $0 \leq t \leq \rho$,

$$g(|\xi - A|) \leq W(\xi) \leq Cg(|\xi - A|) \tag{3.7}$$

for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi - A| \leq \rho$, and

$$g(|\xi - B|) \leq W(\xi) \leq Cg(|\xi - B|) \tag{3.8}$$

for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi - B| \leq \rho$, for some constant $C = C(\rho) > 0$. Then there exists a convex function $G : [0,\infty) \to [0,\infty)$ such that $G(t) = g(t)$ for all $t \in [0,\rho]$,

$$G(s+t) \leq C_1(G(s) + G(t)) \tag{3.9}$$

for all $s,t \geq 0$ and for some constant $C_1 > 0$,

$$\lim_{t\to\infty} \frac{G(t)}{t^p} = 1, \tag{3.10}$$

$$\begin{aligned} \frac{1}{C_2}G(|\xi'|) \leq \frac{1}{C_2}\min\{G(|\xi - A|), G(|\xi - B|)\} \leq W(\xi) \\ \leq C_2 \min\{G(|\xi - A|), G(|\xi - B|)\} \end{aligned} \tag{3.11}$$

for all $\xi \in \mathbb{R}^{d\times N}$ *and for some constant* $C_2 > 0$,

$$W(\xi) \leq C_3 (W(\eta) + G(|\xi - \eta|)) \tag{3.12}$$

for all $\xi, \eta \in \mathbb{R}^{d\times N}$ *and for some constant* $C_3 > 0$,

$$C_4 G(|\xi - A|) \leq W(\xi) \tag{3.13}$$

for all $\xi \in \mathbb{R}^{d\times N}$ *such that* $|\xi - A|, |\xi - B| \geq \rho$ *and for some constant* $C_4 > 0$.

Remark 3.5 In light of Lemma 3.4, and in spite of the fact that the qualitative properties of g are only given nearby zero, in the sequel, and without loss of generality, we will assume that g satisfies (3.9)-(3.11) and (3.12)-(3.13) with g in place of G.

We recall that an open set $\Omega \subset \mathbb{R}^N$ is *starshaped with respect to a set* $S \subset \Omega$ if Ω is star-shaped with respect to each point of S, i.e. if $x \in \Omega$ and $s \in S$ then $\theta x + (1-\theta) s \in \Omega$ for all $\theta \in (0,1)$.

Proposition 3.6 ([40], Proposition 9.1) *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set, star-shaped with respect to a set* $S \subset \Omega$, *with* $|S| > 0$. *Let* $g : [0,\infty) \to [0,\infty)$ *be a convex function, with* $g(0) = 0$. *Let* $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ *be such that* $g(|\nabla u|) \in L^1(\Omega)$. *Then*

$$\int_\Omega g\left(\frac{|u(x) - u_S|}{d}\right) dx \leq \left(\frac{\alpha_N\, d^N}{|\Omega|}\right)^{1-\frac{1}{N}} \frac{|\Omega|}{|S|} \int_\Omega g(|\nabla u|)\, dx,$$

where $u_S := \frac{1}{|S|}\int_S u\, dx$, d *is any number greater or equal than the diameter of* Ω, *and* α_N *is the volume of the unit ball in* $\mathbb{R}^N$.

Proof. We follow Lemma 7.16 in Gilbarg and Trudinger [74]. Assume first that $u \in W^{1,1}(\Omega; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)$. Since Ω is star-shaped with respect to $S \subset \Omega$, for $x \in \Omega$ and $y \in S$ we have

$$u(x) - u(y) = -\int_0^{|x-y|} D_r u(x + r\omega)\, dr, \quad \omega = \frac{y - x}{|y - x|}.$$

Averaging with respect to y over S yields

$$u(x) - u_S = -\frac{1}{|S|}\int_S dy \int_0^{|x-y|} D_r u(x + r\omega)\, dr.$$

Since $|x - y| \leq d$ we have

$$\frac{|u(x) - u_S|}{d} \leq \frac{1}{|S|}\int_S \frac{1}{|x-y|}\int_0^{|x-y|} |D_r u(x + r\omega)|\, dr dy.$$

As g is convex, it now follows from applying twice Jensen's inequality that

$$g\left(\frac{|u(x) - u_S|}{d}\right) \leq \frac{1}{|S|}\int_S \frac{1}{|x-y|}\int_0^{|x-y|} g(|D_r u(x + r\omega)|)\, dr dy.$$

Defining

$$V(x) = \begin{cases} |\nabla u(x)| & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

and, as g is increasing, we have

$$\begin{aligned} g\left(\frac{|u(x) - u_S|}{d}\right) &\leq \frac{1}{|S|} \int_{\{y:|x-y|<d\}} \frac{1}{|x-y|} \int_0^\infty g(V(x+r\omega))\, dr dy \\ &= \frac{1}{|S|} \int_0^\infty \int_{|\omega|=1} \int_0^d g(V(x+r\omega))\, \rho^{N-2} d\rho d\omega dr \\ &= \frac{d^{N-1}}{(N-1)|S|} \int_0^\infty \int_{|\omega|=1} g(V(x+r\omega))\, d\omega dr \\ &= \frac{d^{N-1}}{(N-1)|S|} \int_\Omega |x-y|^{1-N} g(|\nabla u(y)|)\, dy, \end{aligned}$$

where we have used the fact that $g(0) = 0$. The theory of Riesz potentials (Lemma 7.12 in Gilbarg and Trudinger [74]) now yields

$$\int_\Omega g\left(\frac{|u(x) - u_S|}{d}\right) dx \leq \frac{1}{N} (\alpha_N)^{1-\frac{1}{N}} |\Omega|^{\frac{1}{N}} \frac{d^{N-1}}{(N-1)|S|} \int_\Omega g(|\nabla u(x)|)\, dx$$

and the proof is complete. ■

Proposition 3.7 ([40], Proposition 9.2) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain having the cone property, let $g : [0,\infty) \to [0,\infty)$ be a convex function satisfying the doubling condition, with $g(0) = 0$. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ be such that $g(|\nabla u|) \in L^1(\Omega)$. Then*

$$\int_\Omega g(|u(x) - u_B|)\ dx \leq C \int_\Omega g(|\nabla u|)\ dx,$$

where

$$u_B := \frac{1}{|B|} \int_B u(y)\, dy,$$

B is any fixed ball whose closure is contained in Ω, and C is a positive constant depending only on Ω and on the ball B.

Proof. Since Ω has the cone property, it is the union of a finite number of domains starshaped with respect to a ball. Let d be a number greater than the diameter of all these domains, and let A be any of these subdomains with D being the corresponding ball. Construct a finite family of balls $B_0, \cdots, B_M$

contained in Ω and such that $B_0 = D$, $B_i \cap B_{i+1} \neq \emptyset$, $B_M = B$. Since A is star-shaped with respect to any fixed ball $\tilde{B}$ contained in $B_0 \cap B_1$, by Proposition 3.6 we obtain

$$\int_A g\left(\frac{|u(x) - u_{\tilde{B}}|}{d}\right) dx \leq \left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx.$$

By Remark 3.5 and (3.9)

$$\begin{aligned}\int_A g\left(\frac{|u(x)|}{d}\right) dx &\leq C\,|A|\, g\left(\frac{|u_{\tilde{B}}|}{d}\right) + C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx \\ &\leq C\frac{|A|}{\left|\tilde{B}\right|} \int_{\tilde{B}} g\left(\frac{|u(x)|}{d}\right) dx + C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx,\end{aligned}$$

where we have used Jensen's inequality. Hence

$$\begin{aligned}\int_A g\left(\frac{|u(x)|}{d}\right) dx \leq\ & C\frac{|A|}{\left|\tilde{B}\right|} \int_{B_0 \cap B_1} g\left(\frac{|u(x)|}{d}\right) dx \\ &+ C\left(\frac{\alpha_N\, d^N}{|A|}\right)^{1-\frac{1}{N}} \frac{|A|}{\left|\tilde{B}\right|} \int_A g(|\nabla u|)\, dx.\end{aligned}$$

Similarly, since for $i = 1, \cdots, M-1$ the ball B_i is star-shaped with respect to any fixed ball $\tilde{B}_i$ contained in $B_i \cap B_{i+1} \neq \emptyset$, we obtain

$$\begin{aligned}\int_{B_i} g\left(\frac{|u(x)|}{d}\right) dx \leq\ & C\frac{|B_i|}{\left|\tilde{B}_i\right|} \int_{B_i \cap B_{i+1}} g\left(\frac{|u(x)|}{d}\right) dy \\ &+ C\left(\frac{\alpha_N\, d^N}{|B_i|}\right)^{1-\frac{1}{N}} \frac{|B_i|}{\left|\tilde{B}_i\right|} \int_{B_i} g(|\nabla u|)\, dx.\end{aligned}$$

Therefore

$$\int_A g\left(\frac{|u(x)|}{d}\right) dx \leq C\left(\int_B g\left(\frac{|u(x)|}{d}\right) dx + \int_\Omega g(|\nabla u|)\, dx\right).$$

Summing over all A gives

$$\int_\Omega g\left(\frac{|u(x)|}{d}\right) dx \leq C\left(\int_B g\left(\frac{|u(x)|}{d}\right) dx + \int_\Omega g(|\nabla u|)\, dx\right). \tag{3.14}$$

Since B is convex, by Proposition 3.6

$$\int_B g\left(\frac{|u(x) - u_B|}{d}\right) dx \leq \left(\frac{\alpha_N\, d^N}{|B|}\right)^{1-\frac{1}{N}} \int_B g(|\nabla u|)\, dx,$$

where $u_B := \frac{1}{|B|}\int_B u\,dx$. Replacing u by $u-u_B$ in (3.14) we obtain

$$\int_\Omega g\left(\frac{|u(x)-u_B|}{d}\right)dx \le C\left(\int_B g\left(\frac{|u(x)-u_B|}{d}\right)+\int_\Omega g(|\nabla u|)\,dx\right)$$
$$\le C\int_\Omega g(|\nabla u|)\,dx.$$

Applying the latter inequality to du in place of u yields

$$\int_\Omega g(|u(x)-u_B|)\,dx \le C\int_\Omega g(d\,|\nabla u|)\,dx \le C_1\int_\Omega g(|\nabla u|)\,dx,$$

where we have used the fact that $g(dz) \le$const. $g(z)$ for all $z\ge 0$ (see Remark 3.5 and (3.9)). This concludes the proof. ■

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