An introduction to SOBOLEV spaces and interpolation spaces

I once heard my advisor, Jacques-Louis LIONS, mention that once the detailed plan of a book is made, the book is almost written, and as he had already written a few books he was certainly speaking of experience. He gave me the impression that he could write directly a very reasonable text, which he gave to a secretary for typing; maybe he gave then chapters to one of his students, as he did with me for one of his books[1], and very few technical details had to be fixed. His philosophy seemed to be that there is no need to spend too much time polishing the text or finding the best possible statement, as the goal is to take many readers to the front of research, or to be more precise to one front of research, because in the beginning he seems to have changed topics every two years. As for myself, I have not yet written a book, and a first reason is that I am quite unable to write in advance a precise plan of what I am going to talk about; also, I have never been very good at writing even in my mother tongue, which is French, and again and again I need to read what I have already written until I find the text acceptable; that notion of acceptability evolves with time, and I am horrified by my style of twenty years ago; obviously this way of writing is quite inefficient and writing a book would be prohibitively long.

One solution would be not to write books, and when I go into a library I am already amazed by the number of books which have been written on so many subjects, and which I have not read of course, because I never read much. I am even more amazed by the number of books which are not in the library, and although I have access to a good inter-library loan service, I am concerned with how difficult it is for faraway students to have access to scientific knowledge (and I do consider Mathematics as part of Science).

In the Spring 1999, I found the right solution for me, which is to give a course and to prepare lecture notes for the students, trying to write down after each course the two or three pages describing what I had just taught; for such short texts my problems about writing are not too acute. I could hardly have known at the beginning of the Semester how much an introduction to Oceanography my course would be, and when after a short introduction and the description of some classical methods for solving NAVIER-STOKES equations (in the over-simplified version which mathematicians usually consider), it was time to describe some of the models considered in Oceanography, I realized that I did not believe too much in the derivation of these models, and I prefered to finish the course by describing some of the general mathematical tools for studying the nonlinear partial differential equations of Continuum Mechanics, some of which I have developped myself. The resulting set of lecture notes is not as good as I would have liked, but an important point was to make this introductory course available on the web page of the Center for Nonlinear Analysis of the Department of Mathematical Sciences at CARNEGIE-MELLON University (http://www.math.cmu.edu/cna/publications.html#notes "Introduction to Oceanography").

In the Spring 2000, I taught a course divided in two parts, the first part on SOBOLEV spaces, and for the second part I chose to teach about Interpolation spaces. I also decided to add some information that one rarely finds in courses of Mathematics, something about the people.

I had the privilege to study in Paris, to have great teachers like Laurent SCHWARTZ and Jacques-Louis LIONS, and to have met many famous mathematicians. This has given me a different view of Mathematics than the one that comes from reading books and articles, which I find too dry, and I have tried to give a little more life to my story by telling a little about the actors; for those mathematicians whom I have met, I have used their first names in the text, and I have tried to give some simple biographical data for all people quoted in the text, in order to situate them, both in time and in space. For mathematicians of the past, a large part of this information comes from using "The MacTutor History of Mathematics archive" (http://www-history.mcs.st-and.ac.uk/history)[2], but for names which are not (yet) included in this archive, I did search the web for information, and it is possible that some of my information is incomplete or even inaccurate. My interest in History is not recent, but my interest in History of Mathematics has increased recently, in part from finding the above mentioned archive, but also as a result of seeing so many of my

[1] LIONS J.-L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris 1969 xx+554 pp.

[2] I am very thankful to John J. O'CONNOR and Edmund F. ROBERTSON, from the University of St Andrews in Scotland, for having created such an interesting archive.

ideas attributed to others, who often do not even understand them well; I have tried to be as accurate as possible concerning the work of others, and I have tried to learn more about the mathematicians who have introduced some of the ideas which I was taught when I was a student in Paris in the late 60s. I hope that I will be given the correct information by anyone who finds one of my mistakes, and that I will be forgiven for these unintentional errors.

I was born in France from a Syrian father and a French mother and I left my country for political reasons and I enjoy now the hospitality of an American university, and this may explain my interest in mentioning that others have worked in a different country than the one where they were born; I am not interested in the precise citizenship of the people mentioned, but I wanted to convey the idea that the development of Mathematics is an international endeavour. One might observe that there have been efficient schools in some areas of Mathematics at some places and at some moments in time, but although the conditions might be less favourable outside these important centers, I would like to think that a lot of good work can be done elsewhere, and it reminds me of what an Italian friend told me a few years ago: he went to teach in Somalia, for six months if I remember well, and a student came one day to explain to him that he should not be upset when some of the students fell asleep during his lectures, as the reason was not their lack of interest for the subject, but that sometimes they had eaten nothing for a week. I would like to think that my lecture notes could arrive freely in such remote places where there are such courageous students who are trying to acquire some precious knowledge about Mathematics, despite the enormous difficulties that they encounter in their everyday life.

I hope that my lack of organization skills will not bother too much the readers. I consider teaching courses like leading groups of newcomers into unknown countries, not unknown to me as I have often wandered around; some members of a group who have already read about the region or have been in other expeditions with guides much more organized than me might feel disoriented by my choice of places to visit, and indeed I may have forgotten to show a few interesting places, but my goal is to familiarize the readers with the subject and not to write a definitive account.

There are results which are mentioned without proof, and sometimes they are proven later but sometimes they are not, and as no references are given one should remember that I have been trained as a mathematician, and the statements without proofs have indeed been proven in a mathematical sense, because if they had not I would have called them conjectures instead; however, I am also human and my memory is not perfect and I may have made mistakes. I believe that the right attitude in Mathematics is to be able to explain all the statements that one makes, but in a course one has to assume that the reader already has some basic knowledge of Mathematics, and some proofs of a more elementary nature are omitted. Here and there I mention a result that I have heard of but for which I never read a proof or did not make up my own proof, and I usually say so. Actually, and I think that my advisor mentioned that to me, it is useful to read only the statement of a theorem and one should read the proof only if one cannot supply one[3].

After hearing about SOBOLEV spaces in seminars by Jacques-Louis LIONS or some of his students at Ecole Polytechnique in 1966, I learned a little more in his courses at the university in the following years, and I read his 1962 course[4] in Montreal, Canada, and a book by Shmuel AGMON[5]. I had learned about

[3] The MacTutor archive mentions an interesting anecdote in this respect concerning a visit of Antoni ZYGMUND to the University of Buenos Aires, Argentina, in 1948; Alberto CALDERÓN was a student there and he was puzzled by a question that ZYGMUND had asked, and he said that the answer was in ZYGMUND's own book "Trigonometric series", but ZYGMUND disagreed; after discussions the matter became clear, as CALDERÓN had read a statement in the book and supplied his own proof, which was more general that the one that ZYGMUND had written; CALDERÓN's proof answered the question that ZYGMUND had just asked, but CALDERÓN had not looked at ZYGMUND's proof and had not known before that his proof was different from the one in the book.

[4] LIONS J.-L., *Problèmes aux limites dans les équations aux dérivées partielles*, Deuxième édition, Séminaire de Mathématiques Supérieures, No. 1 (Été, 1962) Les Presses de l'Université de Montréal, Montreal, Que. 1965, 176 pp.

[5] AGMON S., *Lectures on elliptic boundary value problems*, William Marsh Rice University, Houston, Tex.

distributions in Laurent SCHWARTZ's course at Ecole Polytechnique in 1965/66 and then I read his book[6]. I first read about interpolation in a book that Jacques-Louis LIONS wrote with Enrico MAGENES[7] and then he gave me his article with Jaak PEETRE to read, and later he asked me to solve some problems in Interpolation for my thesis in 1971, and around that time I did read a few articles on Interpolation. I also learned in other courses by Jacques-Louis LIONS, and the usual process went on, learning, forgetting, inventing a new proof and rediscovering a proof, when asked a question by a fellow researcher or a student, so that for many results in these lectures I can hardly say if I have read them or filled the gaps in statements that I had heard. For the purpose of the lectures I consulted the book by J. BERGH and J. LÖFSTRÖM[8].

My personal reason for being interested in the subject of these lectures is that these questions appear in a natural way when one wants to solve partial differential equations from Continuum Mechanics or Physics. A good way to learn more in this direction is to consult the books of Robert DAUTRAY and Jacques-Louis LIONS[9].

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[6] SCHWARTZ L., *Théorie des distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entièrement corrigée, refondue et augmentée. Hermann, Paris 1966 xiii+420 pp.

[7] LIONS J.-L. & MAGENES E., *Problèmes aux limites non homogènes et applications*, Vol. 1. Travaux et Recherches Mathématiques, No. 17 Dunod, Paris 1968 xx+372 pp.

[8] BERGH J. & LÖFSTRÖM J., *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976. x+207 pp.

[9] DAUTRAY R. & LIONS J.-L., *Mathematical analysis and numerical methods for science and technology*, Vol. 1. *Physical origins and classical methods*, With the collaboration of Philippe Bénilan, Michel Cessenat, André Gervat, Alain Kavenoky and Hélène Lanchon, Translated from the French by Ian N. Sneddon. With a preface by Jean Teillac. Springer-Verlag, Berlin-New York, 1990. xviii+695 pp., Vol. 2. *Functional and variational methods*, With the collaboration of Michel Artola, Marc Authier, Philippe Bénilan, Michel Cessenat, Jean Michel Combes, Hélène Lanchon, Bertrand Mercier, Claude Wild and Claude Zuily. Translated from the French by Ian N. Sneddon. Springer-Verlag, Berlin-New York, 1988. xvi+561 pp., Vol. 3. *Spectral theory and applications*, With the collaboration of Michel Artola and Michel Cessenat. Translated from the French by John C. Amson. Springer-Verlag, Berlin, 1990. x+515 pp., Vol. 4. *Integral equations and numerical methods*, With the collaboration of Michel Artola, Philippe Bénilan, Michel Bernadou, Michel Cessenat, Jean-Claude Nédélec, Jacques Planchard and Bruno Scheurer. Translated from the French by John C. Amson. Springer-Verlag, Berlin, 1990. x+465 pp., Vol. 5. *Evolution problems. I*, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon. Translated from the French by Alan Craig. Springer-Verlag, Berlin, 1992. xiv+709 pp., Vol. 6. *Evolution problems. II*, With the collaboration of Claude Bardos, Michel Cessenat, Alain Kavenoky, Patrick Lascaux, Bertrand Mercier, Olivier Pironneau, Bruno Scheurer and Rémi Sentis. Translated from the French by Alan Craig. Springer-Verlag, Berlin, 1993. xii+485 pp., *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Vol. 7. *Évolution: Fourier, Laplace*, Reprint of the 1985 edition. INSTN: Collection Enseignement. Masson, Paris, 1988. xliv+344+xix pp., Vol. 8. *Évolution: semi-groupe, variationnel*, Reprint of the 1985 edition. INSTN: Collection Enseignement. Masson, Paris, 1988. pp. i–xliv, 345–854 and i–xix, Vol. 9. *Évolution: numérique, transport*, Reprint of the 1985 edition. INSTN: Collection Enseignement. Masson, Paris, 1988. pp. i–xliv and 855–1303.

21-724. SOBOLEV spaces

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In the mid 30s, Sergei SOBOLEV[1] introduced some functional spaces which have been very important in the development of partial differential equations, mostly those related to Continuum Mechanics or Physics. They are known as SOBOLEV spaces now, but I have heard that others have claimed to have had the same idea, like FICHERA[2] and FRIEDRICHS[3]. A similar idea was used a little after by Jean LERAY[4] in his study of weak solutions[5] of NAVIER-STOKES[6] equation.

The basic idea for defining a SOBOLEV space, consists in using weak derivatives, as Sergei SOBOLEV or Jean LERAY did in the mid 30s; it consists in giving a precise meaning to the statement that a function u belonging to the LEBESGUE[7] space $L^p(\Omega)$[8] could have all its partial derivatives $\frac{\partial u}{\partial x_j}$ belonging to $L^p(\Omega)$, for a nonempty open set Ω in R^N. They did not define partial derivatives for every function in $L^p(\Omega)$, but only said that some of these functions have partial derivatives belonging also to $L^p(\Omega)$, and it was Laurent SCHWARTZ[9] who defined more general mathematical objects, which he called distributions, which permit to

[1] Sergei L'vovich SOBOLEV, Russian mathematician, 1908-1989. I first met him when I was a student, first in Paris in 1969, then at the International Congress of Mathematicians in Nice in 1970, and conversed with him in French, which he spoke perfectly (most educated Europeans did learn French in the beginning of this Century, which only ends on December 31, 2000). I only met him once more, when I traveled with a French group from INRIA in 1976 to Akademgorodok, Novosibirsk, where he worked. There is now a SOBOLEV Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk.

[2] Gaetano FICHERA, Italian mathematician, 1922-1996. He worked at University of Rome I (La Sapienza).

[3] Kurt Otto FRIEDRICHS, German-born mathematician, 1901-1982; he emigrated to United States in 1937, to join COURANT.

Richard COURANT, German-born mathematician, 1888-1972; he emigrated to United States in 1934, and worked at New York University. The Department of Mathematics of New York University is now named the COURANT Institute of Mathematical Sciences.

[4] Jean LERAY, French mathematician, 1906-1998. He received the WOLF prize in 1979. He held a chair (Théorie des équations différentielles et fonctionnelles) at Collège de France, Paris, 1947-1978; he was also for some time a member of the Institute for Advanced Study in Princeton, NJ.

Ricardo WOLF, German-born diplomat and philanthropist, 1887-1981; he emigrated to Cuba before World War I; from 1961 to 1973 he was Cuban Ambassador to Israel, where he stayed afterwards; the WOLF foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900-1981, "to promote science and art for the benefit of mankind".

[5] Jean LERAY thought that his notion of weak solutions was related to turbulent flows; although nobody really understands what turbulence is, his ideas were later replaced by those of KOLMOGOROV.

Andrei Nikolaevich KOLMOGOROV, Russian mathematician, 1903-1987; he received the WOLF prize in 1980.

[6] Claude Louis Marie Henri NAVIER, French mathematician, 1785-1836; he introduced the equation in 1821.

George Gabriel STOKES, Irish-born mathematician, 1819-1903; he introduced later the linearized version now known as STOKES's equation, where inertial effects are neglected. STOKES held the Lucasian chair at Cambridge, 1849-1903.

Henry LUCAS, English clergyman, 1610-1663).

[7] Henri Léon LEBESGUE, French mathematician, 1875-1941. He held a chair (Mathématiques) at Collège de France, Paris, 1921-1941.

[8] It was Frigyes (Frederic) RIESZ, Hungarian mathematician, 1880-1956, who introduced the $L^p(\Omega)$ spaces for $1 \le p \le \infty$. He worked in Budapest.

[9] Laurent SCHWARTZ, French mathematician, born in 1915. He received the FIELDS medal in 1950. He works at Ecole Polytechnique, Palaiseau, and I had him as a teacher in 1965/66 (when Ecole Polytechnique was in Paris).

John Charles FIELDS, Canadian mathematician, 1863-1932.

define as many derivatives as one may want, for any locally integrable function. Laurent SCHWARTZ went further than the theory developed by Sergei SOBOLEV, which he did not know about, and he points out that BOCHNER[10] had also obtained some partial results, which he also only learned about later. Russians always quote GEL'FAND[11], for developing the theory of distributions, but Laurent SCHWARTZ told me that what GEL'FAND did was mostly to popularize the theory. Someone pointed out to me that Hermann WEYL[12] should be quoted for the theory too, but I have not checked that, and Laurent SCHWARTZ is not aware of his work.

Once distributions will be defined and their basic properties obtained, one will define the SOBOLEV spaces $W^{m,p}(\Omega)$ for a nonnegative integer m, for $1 \le p \le \infty$ and for an open set Ω of R^N: it is the space of all functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $|\alpha| \le m$, where $\alpha_j \ge 0$ for $j = 1, \ldots, N$ and $|\alpha| = \alpha_1 + \ldots + \alpha_N$, and $D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} u$ (one also denotes $\alpha! = \alpha_1! \ldots \alpha_N!$, and these simplifying notations have been introduced by WHITNEY[13]). One must be careful that other authors like Lars HÖRMANDER[14] use D to denote $\frac{1}{i}\frac{\partial}{\partial x}$ instead; one should not be surprised then if two different books contain similar formulas with different constants, and one should check what is the definition of the symbol D, and also what are the precise constants used in defining the FOURIER[15] transform.

The reason of Sergei SOBOLEV for introducing the space $W^{1,2}(\Omega)$, which is also denoted $H^1(\Omega)$ (but should not be confused with the HARDY[16] space[17], which will be denoted $\mathcal{H}^1$), is that it is a natural space for solving equations of the form $-\Delta\, u = f$ with boundary conditions, an equation named after LAPLACE[18] or POISSON[19]. It can be considered as related to the DIRICHLET[20] principle, which consists in noticing that if a function u of class C^2 minimizes the functional J defined by $J(v) = \int_\Omega |grad(v)|^2\,dx - 2\int_\Omega f\,v\,dx$ among all functions having a given boundary value, then u satisfies $-\Delta\, u = f$ in Ω. The principle was named after DIRICHLET by RIEMANN[21], as he had heard it from him, but it had been used before by GAUSS[22]

[10] Salomon BOCHNER, Polish-born mathematician, 1899-1982; he emigrated to United States after 1933, and he worked at Princeton University, NJ.

[11] Izrail Moiseevich GEL'FAND, Russian mathematician, born in 1913. He received the WOLF prize in 1978. He works at RUTGERS University, New Brunswick, NJ.

Henry RUTGERS, American colonel.

[12] Hermann Klaus Hugo WEYL, German-born mathematician, 1885-1955; he emigrated to United States in 1933 and he worked at the Institute of Advanced Study in Princeton, NJ.

[13] Hassler WHITNEY, American mathematician, 1907-1989. He received the WOLF prize in 1982. He worked at the Institute for Advanced Study in Princeton, NJ.

[14] Lars HÖRMANDER, Swedish mathematician, born in 1931. He received the FIELDS medal in 1962, and the WOLF prize in 1988. He works at Lund University.

[15] Jean-Baptiste Joseph FOURIER, French mathematician, 1768-1830. He was prefect in Grenoble under NAPOLÉON; the Institut FOURIER is the department of Mathematics of University of Grenoble I, itself named after Joseph FOURIER.

Napoléon BONAPARTE, French general, 1769-1821; he proclaimed himself emperor, under the name NAPOLÉON I, 1804-1814 (and 100 days in 1815).

[16] Godfrey Harold HARDY, English mathematician, 1877-1947. He held the Sadleirian chair of Pure Mathematics at Cambridge, 1931-1942.

Lady SADLEIR endowed the chair in 1701).

[17] The term seems to have been introduced by Marcel RIESZ, Hungarian-born mathematician, 1886-1969 (the younger brother of Frigyes RIESZ). He worked in Lund, Sweden.

[18] Pierre-Simon LAPLACE, French mathematician, 1749-1827. He worked in Paris, at the Bureau des Longitudes and the Paris Observatory. He became count in 1806 and marquis in 1817.

[19] Siméon Denis POISSON, French mathematician, 1781-1840. He worked in Paris

[20] Johann Peter Gustav LEJEUNE DIRICHLET, German mathematician, 1805-1859. He worked in Berlin, 1828-1855, and then in Göttingen.

[21] Georg Friedrich Bernhard RIEMANN, German mathematician, 1826-1866. He worked in Göttingen.

[22] Johann Carl Friedrich GAUSS, German mathematician, 1777-1855. He worked in Göttingen.

and by GREEN[23]. WEIERSTRASS[24] pointed out later that the functional might not attain its minimum; I think that the complete solution of the DIRICHLET principle was one in the famous list of problems which HILBERT[25] proposed in 1900 at the International Congress of Mathematicians in Paris; the introduction of SOBOLEV spaces, which are HILBERT spaces[26], together with some developments in Functional Analysis, by FRÉCHET[27], F. RIESZ and BANACH[28] which had paved the way, partly solved the problem. As the principle is also named after THOMSON[29], it is possible that Sergei SOBOLEV had considered the question of Electrostatics, a simplification of MAXWELL[30] equation.

HADAMARD[31] introduced the notion of well posed problems and showed that there are continuous functions f for which the solution u is not of class C^2. One way to solve this difficulty is to work with the family of spaces $C^{k,\alpha}$, where k is a nonnegative integer and $0 < \alpha \leq 1$, named after HÖLDER[32] or LIPSCHITZ[33]; questions like $f \in C^{k,\alpha}$ implies $u \in C^{k+2,\alpha}$ for $0 < \alpha < 1$ were investigated by SCHAUDER[34], but the similar statement is false for $\alpha = 1$, and the corresponding spaces should be replaced by spaces introduced by Antoni ZYGMUND[35]. An underlying question is related to singular integrals acting on spaces $C^{0,\alpha}$ for $0 < \alpha < 1$, which were extended to L^p for $1 < p < \infty$ by Alberto CALDERÓN[36] and Antoni ZYGMUND, so that for $1 < p < \infty$, $f \in L^p(\Omega)$ implies $u \in W^{2,p}_{loc}(\Omega)$; the question for boundary conditions was investigated by Shmuel AGMON[37], Avron DOUGLIS[38] and Louis NIRENBERG[39], but the case $p = 2$ was understood earlier, because one can use FOURIER transform, and there were simpler methods for proving

[23] George GREEN, English mathematician, 1793-1841. He was a miller and never held any academic position.

[24] Karl Theodor Wilhelm WEIERSTRASS, German mathematician, 1815-1897. He worked in Berlin.

[25] David HILBERT, German mathematician, 1862-1943. He worked in Göttingen.

[26] The term was coined by János (John) von NEUMANN, Hungarian-born mathematician, 1903-1957. He emigrated to United States and he worked at the Institute for Advanced Study in Princeton, NJ.

[27] Maurice René FRÉCHET, French mathematician, 1878-1973. He worked in Paris.

[28] Stefan BANACH, Polish mathematician, 1892-1945. He worked in Lvov (now in Ukraine).

[29] William THOMSON, Irish-born physicist, 1824-1907; in 1892 he was made baron KELVIN of Largs, and thereafter known as Lord KELVIN. He worked in Glasgow, Scotland.

[30] James CLERK MAXWELL, Scottish physicist, 1831-1879. He held the first CAVENDISH Professorship of Physics at Cambridge, 1871-1879.

Henry CAVENDISH, English physicist, 1731-1810.

[31] Jacques Salomon HADAMARD, French mathematician, 1865-1963. He held a chair (Mécanique analytique et mécanique céleste) at Collège de France, Paris 1909-1937.

[32] Ernst HÖLDER, German mathematician; I once saw him at a meeting in Oberwolfach, and I was told that it was his father who was known for the inequality (and some results in Algebra), Otto Ludwig HÖLDER, German mathematician, 1859-1937. They worked in Leipzig.

[33] Rudolf Otto Sigismund LIPSCHITZ, German mathematician, 1832-1903. He worked in Bonn.

[34] Juliusz Pawel SCHAUDER, Polish mathematician, 1899-1943. He worked in Lvov (now in Ukraine).

[35] Antoni Szczepan ZYGMUND, Polish-born mathematician, 1900-1992. He emigrated to United States in 1940, and he worked at University of Chicago.

[36] Alberto P. CALDERÓN, Argentinian-born mathematician, 1920-1998. He received the WOLF prize in 1989. He worked at University of Chicago, but kept strong ties with Argentina, as can be witnessed from the large number of mathematicians from Argentina having studied Harmonic Analysis, and often working now in United States.

[37] Shmuel AGMON, Israeli mathematician, born in 1922. He works at Hebrew University, Jerusalem.

[38] Avron DOUGLIS, American mathematician, 1918-1995. He worked at University of Maryland, College Park.

[39] Louis NIRENBERG, Canadian-born mathematician, born in 1925. He received the CRAFOORD prize in 1982. He works at the COURANT Institute of Mathematical Sciences, New York University.

Holger CRAFOORD, Swedish businesman, 1908-1982, and his wife Anna-Greta CRAFOORD, 1914-1994, established the prize in 1980 by a donation to the Royal Swedish Academy "to promote basic scientific research in Sweden and in other parts of the world in Mathematics and Astronomy, Geosciences, Biosciences with particular emphasis on ecology, and Rheumatoid arthritis".

regularity results in this case, by Louis NIRENBERG, by Jaak PEETRE[40]. In the late 50s and early 60s, SOBOLEV spaces were used in a more systematic way for solving linear partial differential equations from Continuum Mechanics or Physics, with suitable boundary conditions, the LAX[41]-MILGRAM[42] lemma being the cornerstone for the elliptic cases, but others had obtained the same result, like Mark VISHIK[43]; extension to evolution problems was worked out by Jacques-Louis LIONS[44], who with Jaak PEETRE improved the real methods for interpolation of normed spaces, studying the application to SOBOLEV spaces with noninteger order with Enrico MAGENES[45]; the late 60s saw the extension to nonlinear partial differential equations.

The framework of the theory of distributions of Laurent SCHWARTZ made the use of SOBOLEV spaces and the study of their properties more easy, and it could itself be considered a natural extension of the previously developed theory of RADON[46] measures, and some of the necessary results of Functional Analysis had been developped for that purpose; it is certainly much more difficult to think of that extension if one had only considered the abstract theory and the BOREL[47] measures, like probabilists do. Of course nothing could have been done without the developments of LEBESGUE, and although integration is more easy to understand than differentiation if one considers that Archimedes[48] had computed the area below a parabola, without even having at his disposal a Cartesian equation of the parabola as Analytical Geometry was only invented by DESCARTES[49], and one had to wait almost two thousand years to see the invention of Differential Calculus by NEWTON[50] and LEIBNIZ[51]. Although we usually teach first the RIEMANN integral, with DARBOUX[52] sums, there are not enough RIEMANN-integrable functions in order to make some natural spaces complete, and this can be done by using the LEBESGUE integral. Although the space $L^1(R)$ of LEBESGUE-integrable functions[53] of a real variable is complete, it is not a dual but one can consider $L^1(R)$ as a subset of the dual of $C_c(R)$, the space of continuous functions with compact support[54], and therefore bounded sequences in $L^1(R)$ may approach a RADON measure (in the weak $\star$ topology); for example the sequence u_n defined by

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40 Jaak PEETRE, Estonian-born mathematician, born in 1935. He works at Lund University, Sweden.

41 Peter D. LAX, Hungarian-born mathematician, born in 1925. He received the WOLF prize in 1987. He emigrated to United States before World War II; he works at the COURANT Institute of Mathematical Sciences, New York University.

42 Arthur Norton MILGRAM, American mathematician, born in 1912.

43 Mark Iosifovich VISHIK, Russian mathematician, born in 1921. He works at the Russian Academy of Sciences, Moscow.

44 Jacques-Louis LIONS, French mathematician, born in 1928. He received the Japan prize in 1991. He works at Collège de France, Paris, where he held a chair (Analyse mathématique des systèmes et de leur contrôle) 1973-1998. I first had him as a teacher at Ecole Polytechnique in 1966/67, and I did research under his direction, until my thesis in 1971.

45 Enrico MAGENES, Italian mathematician, born in 1923. He works in Pavia.

46 Johann RADON, Czech-born mathematician, 1887-1956. He worked in Vienna, Austria.

47 Félix Edouard Justin Emile BOREL, French mathematician, 1871-1956. He worked in Paris.

48 Archimedes, 287 BCE - 212 BCE, worked in Syracuse, then a Greek colony.

BCE = Before Common Era, a replacement for BC (which could also be taken as meaning Before Christian Era for those who insist in linking questions of datation with questions of religion).

49 René DESCARTES, French mathematician, 1596-1650.

50 Sir Isaac NEWTON, English mathematician, 1643-1727. He held the Lucasian chair at Cambridge, 1669-1702.

51 Gottfried Wilhelm von LEIBNIZ, German mathematician, 1646-1716.

52 Jean Gaston DARBOUX, French mathematician, 1842-1917. He worked in Paris.

53 They are actually equivalence classes, as one identifies two functions which only differ on a subset of measure 0, i.e. a subset which for every $\varepsilon > 0$ can be covered by intervals whose sums of lengths is less than ε.

54 For a continuous function f defined on a topological space and taking values in a vector space, the support is the closure of the set of x such that $f(x) \neq 0$.

$u_n(x) = n$ on $(0, 1/n)$ and $u_n(x) = 0$ elsewhere converges to the DIRAC[55] mass[56] at 0, and as one checks that for every $\varphi \in C_c(R)$ one has $\int_R u_n(x)\varphi(x)\,dx \to \varphi(0)$, the DIRAC mass at 0 corresponds to the linear functional $\varphi \mapsto \varphi(0)$. More generally a RADON measure μ on an open subset Ω of R^N is a linear form on $C_c(\Omega)$, the space of continuous functions with compact support in Ω, $\varphi \mapsto \langle \mu, \varphi \rangle$, such that for every compact $K \subset \Omega$, there exists a constant $C(K)$ such that $|\langle \mu, \varphi \rangle| \le C(K) \max_{x \in K} |\varphi(x)|$ for all $\varphi \in C_c(\Omega)$ having their support in K.

[55] Paul Adrien Maurice DIRAC, English physicist, 1902-1984. He received the NOBEL prize in Physics in 1933. He held the Lucasian chair at Cambridge, 1932-1969.

Alfred NOBEL, Swedish industrialist and philanthropist, 1833-1896. He created a fund to be used as awards for people whose work most benefited humanity.

[56] The intuition of a point mass (or charge) is obvious for anyone interested in Physics, but DIRAC went much further than dealing with these objects, as he was not afraid of taking derivatives of his strangely defined "function", a quite bold move which was given a precise mathematical meaning by Laurent SCHWARTZ in his theory of distributions.

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The LEBESGUE measure on R^N is invariant by translation and by rotation, i.e. by rigid displacements (and also by mirror symmetry): let $a \in R^N$ and $M \in SO(N)$, the special (i.e. having determinant $+1$) orthogonal group acting on R^N, then if A is LEBESGUE measurable and B is the image of A by the rigid displacement $x \mapsto a + M\,x$, then B is LEBESGUE measurable and has the same measure than A. One can "construct" non measurable sets by using the axiom of choice, the classical example being to start with the unit circle S^1, and to define equivalence classes, so that two points are equivalent if one can be obtained from the other by applying a rotation of an integer angle $n \in Z$; then one uses the axiom of choice in order to assert that there exists a subset A which contains exactly one element in each equivalence class, and denoting $A_n = n + A$ the subset obtaining from A by a rotation of n, one finds that S^1 is partitioned into the $A_n, n \in Z$, so that if A was LEBESGUE measurable, all the A_n would have the same measure and this measure could not be > 0 because the measure of S^1 is 2π, but if the measure was 0, S^1 would be a countable union of subsets of measure 0 and would have measure 0, and as A can have neither a positive measure nor a zero measure it only remains the possibility that it has no measure at all.

A more subtle construction was carried out in R^3 by HAUSDORFF[1], and simplified by BANACH and TARSKI[2], giving the HAUSDORFF-BANACH-TARSKI paradox: if A and B are two closed bounded sets of R^N with nonempty interior (and $N \geq 3$), then there exists a positive integer m, a partition of A into (disjoint) subsets $A_1, \ldots, A_m$, a partition of B into (disjoint) subsets $B_1, \ldots, B_m$, such that for $i = 1, \ldots, m$ the subset B_i is the image of A_i by a rigid displacement; of course some of the subsets are not measurable if A and B have different measure [I have read the statement (but not seen the proof) that for $N = 2$ there does exist a finitely additive measure defined for all subsets and invariant by translation and rotation, so such a paradox does not hold in R^2].

Up to a multiplication by a constant the LEBESGUE measure is the only nonzero RADON measure which is invariant by translation, and therefore it is uniquely defined if we add the requirement that the volume of the unit cube be 1. For any locally compact[3] commutative[4] group there exists a nonzero RADON measure which is invariant by translation, unique up to multiplication by a constant, a HAAR[5] measure of the group. For the group Z, a HAAR measure is the counting measure; for the group R^N, a HAAR measure is the LEBESGUE measure; for the multiplicative group $(0, \infty)$, a HAAR measure is dt/t.

Although the convolution product can be defined for any locally compact group for which one has chosen a HAAR measure, we shall use it mostly for R^N. For $f, g \in C_c(R^N)$, the convolution product $h = f \star g$ is defined by $h(x) = \int_{R^N} f(y)g(x-y)\,dy = \int_{R^N} f(x-y)g(y)\,dy$, showing that the convolution product is commutative. One has $f \star g \in C_c(R^N)$ and $support(f \star g) \subset support(f) + support(g)$. The convolution product is associative, i.e. for $a, b, c \in C_c(R^N)$ one has $(a \star b) \star c = a \star (b \star c)$.

Convolution of continuous functions with compact support satisfies YOUNG[6] inequality which asserts that if $1 \leq p, q, r$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then for $f, g \in C_c(R^N)$ one has $||f \star g||_r \leq ||f||_p ||g||_q$, where for $1 \leq s \leq \infty$,

[1] Felix HAUSDORFF, German mathematician, 1869-1942. He worked in Bonn.

[2] Alfred TARSKI, Polish-born mathematician, 1902-1983. He emigrated to United States and he worked at University on California, Berkeley.

[3] I have heard Laurent SCHWARTZ say that the result is not true for all noncompact groups.

[4] In the noncommutative case one distinguishes between invariance by translation on the left and invariance by translation on the right.

[5] Alfréd HAAR, Hungarian mathematician, 1885-1933. He worked at Szeged.

[6] William Henry YOUNG, English mathematician, 1863-1942. He worked in Lausanne, Switzerland. He is said to have discovered LEBESGUE integration two years before LEBESGUE. With his wife Grace CHISHOLM-YOUNG, English mathematician, 1868-1944, they worked on so many problems together, so that it is difficult to know if any result attributed to him is a joint work with his wife or not. Their son, Laurence Chisholm YOUNG, born in 1905 is known for his own mathematical results, and among them the introduction of YOUNG measures in the Calculus of Variations; I pioneered their use in partial differential equations in the late 70s, not knowing at the time that he had introduced them (although I had first met him in 1971 at

$||h||_s$ means $||h||_{L^s(R^N)}$. If p or q or r is 1, it is just an application of HÖLDER inequality and it is optimal, while for other cases one may prove it by applying a few times HÖLDER inequality, or JENSEN[7] inequality, but the constant is not optimal and the best constant $C(p,q)$ for which one has $||f\star g||_r \le C(p,q)||f||_p||g||_q$ has been found independently by William BECKNER[8] who used probabilistic methods, and by Elliot LIEB[9] and BRASCAMP[10] who used nonprobabilistic methods (equality holds for some particular Gaussians). Of course, under the preceding relation between p,q,r, the convolution products extends from $L^p(R^N)\times L^q(R^N)$ into $L^r(R^N)$ with the same inequalities, and this can be proven either directly of by using the density of $C_c(R^N)$ in $L^p(R^N)$ for $1\le p<\infty$, and the weak $\star$ density if $p=\infty$.
Note: I admit that this density has been proven when constructing the LEBESGUE measure, and although we shall study later an explicit way of approaching functions in $L^p(R^N)$ by functions in $C_c^\infty(R^N)$, the proof will use the fact that $C_c(R^N)$ is known to be dense, and will not be an independent proof of that result.

Proposition 1: (i) If $1<p<\infty$, $f\in L^p(R^N)$ and $g\in L^{p'}(R^N)$, then $f\star g\in C_0(R^N)$, the space of continuous (bounded) functions converging to 0 at infinity.

(ii) If $f\in L^1(R^N)$ and $g\in L^\infty(R^N)$, then $f\star g\in BUC(R^N)$, the space of bounded uniformly continuous functions.

Proof: YOUNG's inequality in that case follows from $|(f\star g)(x)|=\left|\int_{R^N} f(y)g(x-y)\,dy\right|\le ||f||_p||g||_q$ for all x by HÖLDER inequality. There exists a sequence $f_n\in C_c(R^N)$ converging to f in $L^p(R^N)$ strong, and a sequence $g_n\in C_c(R^N)$ converging to g in $L^{p'}(R^N)$ strong, and as $f\star g-f_n\star g_n=f\star(g-g_n)+(f-f_n)\star g_n$, one deduces that $||f\star g-f_n\star g_n||_\infty\le ||f||_p||g-g_n||_{p'}+||f-f_n||_p||g_n||_{p'}\to 0$, and therefore $f\star g$ is the uniform limit of the sequence $f_n\star g_n\in C_c(R^N)$, and therefore belongs to the closure of $C_c(R^N)$ in $L^\infty(R^N)$, which is $C_0(R^N)$.

Using a sequence $f_n\in C_c(R^N)$ converging to f in $L^1(R^N)$, one has $||f\star g-f_n\star g||_\infty\le ||f-f_n||_1||g||_\infty$, and therefore $f\star g$ is the uniform limit of the sequence $f_n\star g$, and it is enough to show that each $f_n\star g$ is bounded uniformly continuous, as a uniform limit of such functions also belongs to the same space $BUC(R^N)$. As the function f_n belongs to $C_c(R^N)$ it is uniformly continuous, so that $|f_n(a)-f_n(b)|\le\varepsilon_n(|a-b|)$ with $\lim_{t\to 0}\varepsilon_n(t)=0$. One has $(f_n\star g)(x)-(f_n\star g)(x')=\int_{R^N}\big(f_n(x-y)-f_n(x'-y)\big)g(y)\,dy$, but the integral may be restricted to the set of y such that $|y-x|\le R_n$ and $|y-x'|\le R_n$ if the support of f_n is included in the closed ball centered at 0 with radius R_n; therefore $|(f_n\star g)(x)-(f_n\star g)(x')|\le\int_{|y-x|\le R_n,|y-x'|\le R_n}|f_n(x-y)-f_n(x'-y)|\,|g(y)|\,dy\le\varepsilon(|x-x'|)||g||_\infty meas\big(B(0,R_n)\big)$, showing that $f_n\star g$ is uniformly continuous.■

Of course, the property of commutativity of the convolution product extends to the case where it is defined on $L^p(R^N)\times L^q(R^N)$ (i.e. if $p,q\ge 1$ and $\frac{1}{p}+\frac{1}{q}\ge 1$), and similarly the property of associativity of the convolution product extends to the case where the functions belong to $L^a(R^N),L^b(R^N),L^c(R^N)$ (i.e. $a,b,c\ge 1$, $\frac{1}{a}+\frac{1}{b}\ge 1$, $\frac{1}{b}+\frac{1}{c}\ge 1$, and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 2$), and can be proved directly using FUBINI[11]'s theorem.

However, one must be careful that there are other cases where the convolution products $f_1\star f_2$, $f_2\star f_3$, $(f_1\star f_2)\star f_3$ and $f_1\star(f_2\star f_3)$ may all be defined, for example if in each convolution product considered at least one of the functions has compact support, but with $(f_1\star f_2)\star f_3\ne f_1\star(f_2\star f_3)$: let $f_1=1$, $f_2\in C_c(R)$ with $\int_R f_2(x)\,dx=0$, and let f_3 be the HEAVISIDE[12] function, defined by $f_3(x)=0$ for $x<0$ and $f_3(x)=1$ for $x>0$; one sees immediately that $f_1\star f_2=0$ and $f_4=f_2\star f_3\in C_c(R)$, and one has to check that f_2 can be chosen in such a way that $\int_R f_4(x)\,dx\ne 0$; if one chooses f_2 with support in $[-1,+1]$, with $\int_{-1}^{+1} f_2(y)\,dy=0$

University of Wisconsin, Madison, where he worked), as I had heard about them as parametrized measures in seminars on control theory.

7 Johan Ludwig William Valdemar JENSEN, Danish mathematician, 1859-1925. He never held any academic position, and worked for a telephone company.

8 William BECKNER, American mathematician. He works at University of Texas, Austin.

9 Elliott H. LIEB, American mathematician. He works at Princeton University, NJ.

10 Herm Jan BRASCAMP, mathematical physicist.

11 Guido FUBINI, Italian-born mathematician, 1879-1943. He emigrated to United States in 1939, and he worked in New York.

12 Oliver HEAVISIDE, English engineer, 1850-1925. He developed an operational calculus, which was given a precise mathematical explanation by Laurent SCHWARTZ.

and $\int_{-1}^{+1}(1-y)f_2(y)\,dy \neq 0$, then one has $f_4(x) = \int_{-1}^{x} f_2(y)\,dy$, and $\int_R f_4(x)\,dx = \int_{-1}^{+1}(1-y)f_2(y)\,dy \neq 0$.

If one uses only functions in $L^1_{loc}(R)$ (i.e. integrable on every bounded interval), which vanish on $(-\infty,0)$, then the convolution product is well defined as $f\star g$ vanishes on $(-\infty,0)$ and satisfies $(f\star g)(x) = \int_0^x f(y)g(x-y)\,dy$ for $x>0$, showing that $f\star g \in L^1_{loc}(R)$. A theorem of TITCHMARSH[13] asserts that if the support of f starts at a and the support of g starts at b, then the support of $f\star g$ starts at $a+b$, and Jacques-Louis LIONS has generalized this result to R^N, proving that the closed convex hull of the support of $f\star g$ is equal to the (vector) sum of the closed convex hull of the support of f and the closed convex hull of the support of g.

An important property of convolution is that it commutes with translation; this is of course related to the fact that the LEBESGUE measure is invariant by translation.

Notation: For a vector $a\in R^N$, and $f\in L^1_{loc}(R^N)$, $\tau_a f$ denotes the function defined by $(\tau_a f)(x) = f(x-a)$ for almost every $x\in R^N$ (the graph of $\tau_a f$ is obtained from that of f by a translation of vector $(a,0)$). Of course, one has $\tau_b(\tau_a f) = \tau_{a+b} f$ for all $a,b\in R^N$.

Proposition 2: (i) If the convolution product of f and g is defined, then $\tau_a(f\star g) = (\tau_a f)\star g = f\star(\tau_a g)$ for every $a\in R^N$.

(ii) If $k\geq 0$, $f\in C^k_c(R^N)$, the space of functions of class C^k with compact support and $g\in L^1_{loc}(R^N$, then $f\star g\in C^k(R^N)$ and $D^\alpha(f\star g) = (D^\alpha f)\star g$ for all derivatives of length $\leq k$.

Proof: One has $\big((\tau_a f)\star g\big)(x) = \int_{R^N}(\tau_a f)(y)g(x-y)\,dy = \int_{R^N} f(y-a)g(x-y)\,dy$, which by a change of variable in the integral is $\int_{R^N} f(y)g(x-a-y)\,dy = (f\star g)(x-a) = \big(\tau_a(f\star g)\big)(x)$, showing that $(\tau_a f)\star g = \tau_a(f\star g)$, and that it is also $f\star(\tau_a g)$ follows by commutativity of the convolution product.

If $e_1,\ldots,e_N$ is the canonical basis of R^N, then a function h has a partial derivative $\frac{\partial h}{\partial x_j}$ at x if and only if $\frac{1}{\varepsilon}(h-\tau_{\varepsilon\,e_j}h)$ has a limit at x when ε tends to 0 (with $\varepsilon\neq 0$, of course). If $f\in C^1_c(R^N)$, then $\frac{1}{\varepsilon}(f-\tau_{\varepsilon\,e_j}f)$ converges uniformly to $\frac{\partial f}{\partial x_j}$ and therefore if one takes the convolution product with a function $g\in L^1_{loc}(R^N)$, one finds that $\frac{1}{\varepsilon}(f-\tau_{\varepsilon\,e_j}f)\star g$ converges uniformly on compact sets to $\frac{\partial f}{\partial x_j}\star g$; if one denotes $h = f\star g$, one has $\frac{1}{\varepsilon}(f-\tau_{\varepsilon\,e_j}f)\star g = \frac{1}{\varepsilon}(h-\tau_{\varepsilon\,e_j}h)$, and therefore the limit must be $\frac{\partial h}{\partial x_j}$ and it is equal to $\frac{\partial f}{\partial x_j}\star g$. A reiteration of this argument (at most k times if $f\in C^k_{loc}(R^N)$) gives then $D^\alpha(f\star g) = (D^\alpha f)\star g$ for all derivatives of order $|\alpha|\leq k$.■

If $f\in C^\infty_c(R^N)$ (which in the notations of Laurent SCHWARTZ is denoted by $\mathcal{D}(R^N)$), then $f\star g$ belongs to $C^k(R^N)$ for all k, i.e. $f\star g\in C^\infty(R^N)$ (which in the notations of Laurent SCHWARTZ is denoted by $\mathcal{E}(R^N)$). As will be shown later, there are enough functions in $C^\infty_c(R^N)$ for approaching any function in $L^p(R^N)$ for $1\leq p<\infty$, but just one particular function in $C^\infty_c(R^N)$ has to be constructed explicitly, and the properties of convolution will help for the rest of the argument.

Lemma: The function ρ defined on R^N by $\rho(x) = exp\big(\frac{-1}{1-|x|^2}\big)$ if $|x|<1$ and $\rho(x)=0$ if $|x|\geq 1$ belongs to $C^\infty_c(R^N)$.

Proof: It is nonnegative, and continuous because if $|x|\to 1$ with $|x|<1$, then $\frac{-1}{1-|x|^2}\to-\infty$ and $\rho(x)\to 0$; obviously ρ has for support the closed unit ball. One has $\frac{\partial\rho}{\partial x_j} = \frac{-2x_j}{(1-|x|^2)^2}\rho$, and by induction $D^\alpha\rho = \frac{P_\alpha(x)}{(1-|x|^2)^{2|\alpha|}}\rho$ for a polynomial P_α, and therefore when $|x|\to 1$ with $|x|<1$, the exponential wins over the term $\frac{1}{(1-|x|^2)^{2|\alpha|}}$ and therefore $D^\alpha\rho\to 0$, showing that all derivatives of ρ are continuous and therefore $\rho\in C^\infty_c(R^N)$.■

[13] Edward Charles TITCHMARSH, English mathematician, 1899-1963. He held the Savilian chair of Geometry at Oxford, that HARDY had occupied before leaving for Cambridge.

Sir Henry SAVILE, English mathematician, 1549-1622.

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Once one has constructed one nonzero function in $C_c^\infty(R^N)$, convolution will help creating automatically a lot of such functions, enough for approaching all functions in $L^p(R^N)$ for $1 \le p < \infty$. For doing this, the concept of a smoothing sequence is useful.

Definition: A smoothing sequence is a sequence $\rho_n \in C_c^\infty(R^N)$ such that $support(\rho_n) \to \{0\}$, $\int_{R^N} |\rho_n(x)|\,dx$ is bounded and $\int_{R^N} \rho_n(x)\,dx \to 1$. A special smoothing sequence is defined by $\rho_n(x) = n^N \rho_1(n\,x)$ where $\rho_1 \in C_c^\infty(R^N)$ is nonnegative, has integral 1 and has its support in the closed unit ball centered at 0.■

Starting from an arbitrary nonzero function $\varphi \in C_c^\infty(R^N)$, one may assume that it is nonnegative by replacing it eventually by φ^2, that it has its support in the closed unit ball centered at 0 by replacing it eventually by $\varphi(k\,x)$ for k large enough, and that it has integral 1 by multiplying it by a suitable constant. This gives a function ρ_1, which is then rescaled by $\rho_n(x) = n^N \rho_1(n\,x)$, so that the integral of ρ_n is 1, and its support is in the closed ball centered at 0 with radius $\frac{1}{n}$.

Proposition: (i) If $1 \le p < \infty$, $f \in L^p(R^N)$ and ρ_n is a smoothing sequence, then $f \star \rho_n \to f$ in $L^p(R^N)$ strong.

(ii) If $f \in L^\infty(R^N)$ and ρ_n is a smoothing sequence, then $f \star \rho_n \to f$ in $L^\infty(R^N)$ weak $\star$, and in $L^q_{loc}(R^N)$ strong for $1 \le q < \infty$, i.e. for every compact K one has $\int_K |f \star \rho_n - f|^q\,dx \to 0$.

Proof: There exists a sequence $f_m \in C_c(R^N)$ which converges to f in $L^p(R^N)$ strong. One writes $f \star \rho_n - f = (f - f_m) \star \rho_n + (f_m \star \rho_n - f_m) + (f_m - f)$, so that if one chooses m such that $||f - f_m||_p \le \varepsilon$, and if C is a bound for all the $L^1(R^N)$ norms of ρ_n, one has $||(f - f_m) \star \rho_n||_p \le ||f - f_m||_p ||\rho_n||_1 \le C\varepsilon$, and therefore $||f \star \rho_n - f||_p \le C\varepsilon + ||f_m \star \rho_n - f_m||_p + \varepsilon$, and it remains to show that for m fixed $f_m \star \rho_n$ converges to f_m in $L^p(R^N)$ strong as n tends to infinity. If $\int_{R^N} \rho_n(x)\,dx = \kappa_n$, one writes $f_m \star \rho_n - f_m = (f_m \star \rho_n - \kappa_n f_m) + (\kappa_n - 1) f_m$, and because $\kappa_n \to 1$ the second part tends to 0 in $L^p(R^N)$ strong, and because of the uniform continuity of f_m, the first part tends to 0 uniformly, and therefore in $L^p(R^N)$ strong because its support stays bounded as it is included in $support(f_m) + support(\rho_n)$; indeed, if $support(\rho_n)$ is included in the ball centered at 0 with radius ε_n, and η_m is the modulus of uniform continuity of f_m, one has $\left|\int_{R^N} \big(f_m(x-y) - f_m(x)\big)\rho_n(y)\,dy\right| \le \int_{R^N} |\rho_n(y)|\,dy \max_{|y| \le \varepsilon_n} |f_m(x-y) - f_m(x)| \le C\eta_m(\varepsilon_n)$.

For $f \in L^\infty(R^N)$, one wants to show that $f \star \rho_n$ converges to f in $L^\infty(R^N)$ weak $\star$, and this means that for every $g \in L^1(R^N)$ one has $\int_{R^N} (f \star \rho_n)(x) g(x)\,dx \to \int_{R^N} f(x) g(x)\,dx$. One notices that, by FUBINI's theorem, one has $\int_{R^N} (f \star \rho_n)(x) g(x)\,dx = \int_{R^N \times R^N} f(y) \rho_n(x-y) g(x)\,dx\,dy = \int_{R^N} f(y)(g \star \check{\rho_n})(y)\,dy$, where $\check{\rho_n}(y) = \rho_n(-y)$, so that $\check{\rho_n}$ is a smoothing sequence and by the first part $g \star \check{\rho_n}$ converges to g in $L^1(R^N)$ strong and therefore $\int_{R^N} f(y)(g \star \check{\rho_n}(y)\,dy \to \int_{R^N} f(y) g(y)\,dy$. In order to show that $\int_K |f \star \rho_n - f|^q\,dx \to 0$ for a compact K and $1 \le q < \infty$, one notices that the integral only uses values of f in a ball centered at 0 with radius R_0 large enough (so that the ball contains K and $K + support(\check{\rho_n})$), and therefore if $\widetilde{f}$ coincides with f inside the ball centered at 0 with radius R_0 and is 0 outside it, then the integral is $\int_K |\widetilde{f} \star \rho_n - \widetilde{f}|^q\,dx$, which does converge to 0 because $\widetilde{f} \star \rho_n$ converges to $\widetilde{f}$ in $L^q(R^N)$, as a consequence of the first part and of the fact that $\widetilde{f}$ belongs to $L^q(R^N)$.■

Of course, $f \star \rho_n$ belongs to $C^\infty(R^N)$ and $D^\alpha(f \star \rho_n) = f \star (D^\alpha \rho_n)$ for all multi-indices α, but the support of $f \star \rho_n$ is not compact in general.

Corollary: For $1 \le p < \infty$, the space $C_c^\infty(R^N)$ is dense in $L^p(R^N)$. $C_c^\infty(R^N)$ is weak $\star$ dense in $L^\infty(R^N)$.

Proof: If $1 \le p < \infty$, $f_m \star \rho_n \in C_c^\infty(R^N)$ because f_m has compact support. Because $f - f_m \star \rho_n = (f - f_m) + (f_m - f_m \star \rho_n)$, the argument used in the first part of the Proposition shows that there are sequences m_k and n_k such that $f_{m_k} \star \rho_{n_k}$ converges to f in $L^p(R^N)$ strong as $k \to \infty$.

In the case $f \in L^\infty(R^N)$, one defines g_m by $g_m(x) = f(x)$ if $|x| \le m$ and $g_m(x) = 0$ if $|x| > m$, and as $m \to \infty$ g_m converges to f in $L^\infty(R^N)$ weak $\star$ and $L^q_{loc}(R^N)$ strong for $1 \le q < \infty$; one concludes by noticing that for m fixed $g_m \star \rho_n$ converges to g_m in $L^\infty(R^N)$ weak $\star$ and $L^q_{loc}(R^N)$ strong for $1 \le q < \infty$ (and this argument uses the fact that on bounded sets of $L^\infty(R^N)$ the weak $\star$ topology is metrizable).■

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04. Monday January 24, 2000.

For reasons which will become more obvious later, it is useful to define in a more general setting the truncation step used previously.

Definition: A truncating sequence is a sequence $\theta_n \in C_c^\infty(R^N)$ such that $\theta_n(x) \to 1$ for almost every x, θ_n is bounded in $L^\infty(R^N)$, and for each multi-index α with $|\alpha| \geq 1$ the sequence $D^\alpha \theta_n$ tends to 0 in $L^\infty(R^N)$ strong. A special truncating sequence is defined by $\theta_n(x) = \theta_1\big(\frac{x}{n}\big)$ where $\theta_1 \in C_c^\infty(R^N)$, $0 \leq \theta_1(x) \leq 1$ for all x, and $\theta_1(x) = 1$ for $|x| \leq 1$.■

That such a θ_1 exists follows easily from the regularization by convolution, and more precisely one has the following result.

Lemma: Let $0 < a < b < c$, then there exists $\theta \in C_c^\infty(R^N)$ with $0 \leq \theta(x) \leq 1$ for all x, with $\theta(x) = 1$ if $|x| \leq a$, $\theta(x) = 0$ if $|x| \geq c$, and $\int_{R^N} \theta(x)\,dx = \int_{|x|\leq b} dx$.
Proof: Let f be the characteristic function of the ball centered at 0 with radius b; let ε satisfy $0 < \varepsilon < \min\{b-a, c-b\}$, and let $\rho_\varepsilon \in C_c^\infty(R^N)$ be nonnegative, with $\int_{R^N} \rho_\varepsilon(x)\,dx = 1$ and support in the ball centered at 0 with radius ε. Then $\theta = f \star \rho_\varepsilon$ satisfies all the desired properties, the last one coming from the fact that for two functions $f, g \in L^1(R^N)$, one has $\int_{R^N} (f \star g)(x)\,dx = \big(\int_{R^N} f(x)\,dx\big)\big(\int_{R^N} g(x)\,dx\big)$.■

Of course, if $h \in L^p(R^N)$ and θ_n is a truncating sequence, $h\,\theta_n$ converges almost everywhere to h and is bounded by $C|h|$, and therefore by LEBESGUE's dominated convergence theorem, $h\,\theta_n \to h$ in $L^p(R^N)$ strong if $1 \leq p < \infty$, and in $L^\infty(R^N)$ weak $\star$ and $L^q_{loc}(R^N)$ strong for $1 \leq q < \infty$ in the case $p = \infty$.

After these preliminaries, one defines RADON measures and distributions on an open set Ω of R^N in the following way.

Definition: i) A RADON measure μ in Ω is a linear form defined on $C_c(\Omega)$ (the space of continuous functions with compact support in Ω), $\varphi \mapsto \langle \mu, \varphi \rangle$, such that for every compact $K \subset \Omega$ there exists a constant $C(K)$ with $|\langle \mu, \varphi \rangle| \leq C(K)||\varphi||_\infty$ for all $\varphi \in C_c(\Omega)$ with *support*$(\varphi) \subset K$. One writes $\mu \in \mathcal{M}(\Omega)$, and the elements of $C_c(\Omega)$ are called test functions.

ii) A distribution S in Ω is a linear form defined on $C_c^\infty(\Omega)$ (the space of C^∞ functions with compact support in Ω), $\varphi \mapsto \langle S, \varphi \rangle$, such that for every compact $K \subset \Omega$ there exists a constant $C(K)$ and a nonnegative integer $m(K)$ with $|\langle S, \varphi \rangle| \leq C(K) \max_{|\alpha| \leq m(k)} ||D^\alpha \varphi||_\infty$ for all $\varphi \in C_c^\infty(\Omega)$ with *support*$(\varphi) \subset K$. One writes $T \in \mathcal{D}'(\Omega)$, and the elements of $C_c^\infty(\Omega)$ are called test functions.

iii) If one can take $m(K) = m_0$ for all compact $K \subset \Omega$, then the distribution is said to have order $\leq m_0$.■

RADON measures are distributions; they are exactly the distributions of order ≤ 0.

$L^1_{loc}(\Omega)$ denotes the space of locally integrable functions in Ω, i.e. the (equivalence classes of) LEBESGUE measurable functions which are integrable on every compact $K \subset \Omega$; it is not a BANACH space but it is a FRÉCHET space[1] (i.e. it is locally convex, metrizable and complete), and a sequence f_n converges to 0 in $L^1_{loc}(\Omega)$ if and only if for every compact $K \subset \Omega$ one has $\int_K |f_n(x)|\,dx \to 0$. One identifies any function $f \in L^1_{loc}(\Omega)$ with a RADON measure (and therefore with a distribution), which one usually also writes f, defined by the formula $\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,dx$ for all $\varphi \in C_c(\Omega)$. It is not really such a good notation, because it relies upon having selected the LEBESGUE measure dx and it would be better to call this measure (or distribution) $f\,dx$; this abuse of notation is of no consequence for open sets of R^N, and corresponds to the usual identification of $L^2(\Omega)$ with its dual, but when one deals with a differentiable manifold one should remember that there is no prefered volume form like dx.

If $a \in \Omega$, the DIRAC mass at a, which is denoted δ_a, is defined by $\langle \delta_a, \varphi \rangle = \varphi(a)$ for all $\varphi \in C_c(\Omega)$, and it is a RADON measure and therefore a distribution. If a sequence $a_n \in \Omega$ converges to the boundary $\partial\Omega$

[1] The term BANACH space was introduced by FRÉCHET; I do not know who introduced the term FRÉCHET space.

of Ω and c_n is an arbitrary sequence, then $\mu = \sum_n c_n \delta_{a_n}$ is a RADON measure in Ω because in the formula $\langle \mu, \varphi \rangle = \sum_n c_n \varphi(a_n)$, only a finite number of a_n belong to the compact support K of φ.

Physicists use the notation $\delta(x-a)$ instead of δ_a, and they define $\delta(x)$ as the "function" which is 0 for $x \neq 0$ and has integral 1; of course there is no such function and it is actually a measure, but after studying about measures and distributions one learns which formulas are right and which ones are wrong and one can then decide quickly if a formula used by physicists can be proved easily, or if it is a questionable one, either by showing that it is false or by noticing that mathematicians do not know yet how to make sense out of the formal computations used by physicists in that particular case.

One can create a lot of distributions by taking derivatives, and it is one of the important properties of distributions that they have as many derivatives as one wants, and as locally integrable functions are (measures and therefore) distributions, one has then a way to define their derivatives.

Definition: If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index with $\alpha_j \geq 0$ for $j = 1, \ldots, N$, then for any distribution $T \in \mathcal{D}'(\Omega)$ one defines the distribution $D^\alpha T$ by the formula $\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$ for all $\varphi \in C_c^\infty(\Omega)$.■

One must first check that $D^\alpha T$ defined in this way is a distribution, i.e. for any compact $K \subset \Omega$ one must bound $|\langle D^\alpha T, \varphi \rangle|$ for $\varphi \in C_c^\infty(\Omega)$ with *support*$(\varphi) \subset K$, and the bound should only involve the sup norm of a fixed finite number of derivatives of φ. Indeed $|\langle D^\alpha T, \varphi \rangle| = |(-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle| \leq C(K) \max_{|\beta| \leq m(K)} ||D^\beta (D^\alpha \varphi)||_\infty$, and as $D^{\alpha+\beta}$ is a derivation of order $\leq m(K) + |\alpha|$, this is bounded by $C(K) \max_{|\gamma| \leq m(K)+|\alpha|} ||D^\gamma \varphi||_\infty$, so $D^\alpha T$ is a distribution. One deduces that if T is a distribution of order $\leq m_0$ then $D^\alpha T$ is a distribution of order $\leq m_0 + |\alpha|$.

One must then check that the formula is compatible with the notion of derivative for smooth functions, i.e. if $f \in C^1(\Omega)$, then $\frac{\partial f}{\partial x_j} \in C^0(\Omega)$, and as both f and $\frac{\partial f}{\partial x_j}$ are locally integrable they are distributions and one must check that the derivative of the distribution associated to f (which should have been denoted $f\,dx$) is the distribution associated to $\frac{\partial f}{\partial x_j}$, and this means that one should check that for every $\varphi \in C_c^\infty(\Omega)$ one has $\int_\Omega \big(\frac{\partial f}{\partial x_j} \varphi + f \frac{\partial \varphi}{\partial x_j} \big)\,dx = 0$, but this is $\int_\Omega \frac{\partial (f\,\varphi)}{\partial x_j}\,dx$, and because $f\,\varphi$ has compact support, one can invoke FUBINI's theorem and one may start by integrating in x_j, and then in the other variables; one has to integrate on an open set O of R a function with compact support, and O could be made of a countably infinite number of open intervals, but only a finite number of intervals have to be taken into account, and for each of these intervals one integrates the derivative of a C^1 function vanishing near the ends of the interval and the integral is then 0.

The derivative of the HEAVISIDE function (which 1 for $x > 0$ and 0 for $x < 0$) is δ_0, the DIRAC mass at 0. Indeed, if D denotes $\frac{d}{dx}$, one has $\langle D\,H, \varphi \rangle = -\langle H, D\,\varphi \rangle = -\int_0^\infty D\,\varphi\,dx = \varphi(0)$ for all $\varphi \in C_c^\infty(R)$.

Let $u = -1 + 2H$, which is the sign function, so that $D\,u = 2\delta_0$; noticing that $u^3 = u$ and $u^2 = 1$, one discovers the following "paradox", that $D(u^3) = 2\delta_0$ but $3u^2 D\,u = 6\delta_0$. Of course one would have been in trouble with checking if $D(u^2)$, which is 0, coincides with $2u\,D\,u$, because the multiplication of u by δ_0 is not defined; one can actually multiply any RADON measure by a continuous function, but u is not continuous.

At this point one should remember that products are not always defined, and this will be our next topic.

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05. Wednesday January 26, 2000.

With the notion of distributions, it is now easy to give the definition of what SOBOLEV spaces are.

Definition: For a nonnegative integer m, for $1 \le p \le \infty$ and for an open set $\Omega \subset R^N$, the SOBOLEV space $W^{m,p}(\Omega)$ is the space of (equivalence classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for all derivations D^α of length $|\alpha| \le m$. It is a normed space equipped with the norm $||u|| = \sum_{|\alpha| \le m} ||D^\alpha u||_p$, or the equivalent norm $||u||_{m,p} = \big(\int_\Omega \big(\sum_{|\alpha|\le m} |D^\alpha u|^p\big)\,dx\big)^{1/p}$ for $1 \le p < \infty$ and $||u||_{m,\infty} = \max_{|\alpha|\le m} ||D^\alpha u||_\infty$ for $p = \infty$.■

Proposition: i) For $1 \le p \le \infty$ and $m \ge 0$ the SOBOLEV space $W^{m,p}(\Omega)$ is a BANACH space.

ii) For $p = 2$, $W^{m,2}(\Omega)$ is a HILBERT space, for the scalar product $((u,v)) = \int_\Omega \big(\sum_{|\alpha|\le m} D^\alpha u\, \overline{D^\alpha v}\,dx\big)$. The space $W^{m,2}(\Omega)$ is often denoted $H^m(\Omega)$, a notation also used for HARDY spaces, for which the notation $\mathcal{H}^q$ will be used (for $0 < q \le \infty$).

Proof: Let u_n be a CAUCHY[1] sequence in $W^{m,p}(\Omega)$, i.e. for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for $n, n' \ge n(\varepsilon)$ one has $||u_n - u_{n'}|| \le \varepsilon$. This implies that for eachg multi-index α with $|\alpha| \le m$ one has $||D^\alpha u_n - D^\alpha u_{n'}||_p \le \varepsilon$, i.e. $D^\alpha u_n$ is a CAUCHY sequence in $L^p(\Omega)$, and as $L^p(\Omega)$ is complete (because one uses the LEBESGUE measure, for which the RIESZ-FISCHER[2] theorem applies), one deduces that $D^\alpha u_n \to f_\alpha$ in $L^p(\Omega)$ strong. One must then proved that $f_\alpha = D^\alpha f_0$ and that u_n tends to f_0 in $W^{m,p}(\Omega)$. For this one uses the derivative in the sense of distributions, and one has $\langle D^\alpha f_0, \varphi\rangle = (-1)^{|\alpha|}\langle f_0, D^\alpha\varphi\rangle = (-1)^{|\alpha|}\lim_{n\to\infty}\langle f_n, D^\alpha \varphi\rangle = \lim_{n\to\infty}\langle D^\alpha f_n, \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$ and all multi-indices α, but when $|\alpha| \le m$ the last limit is $\langle f_\alpha, \varphi\rangle$, showing that $D^\alpha f_0 = f_\alpha$, so that $f_0 \in W^{m,p}(\Omega)$ and $D^\alpha u_n \to D^\alpha f_0$ in $L^p(\Omega)$ for $|\alpha| \le m$, so that by taking the limit $n' \to \infty$, and one finds that $||u_n - f_0|| \le \varepsilon$, i.e. $u_n \to f_0$ in $W^{n,p}(\Omega)$.

The proposed formula for the scalar product is indeed linear in u and antilinear in v, and for $v = u$ it gives the square of the norm.■

In the proof, one has shown some kind of continuity for the derivations on $\mathcal{D}'(\Omega)$. Indeed there exists a topology on $C_c^\infty(\Omega)$ for which the dual is $\mathcal{D}'(\Omega)$ and one uses on $\mathcal{D}'(\Omega)$ the weak topology, which is not metrizable, but it is rarely necessary to know what this topology is; nevertheless it is useful to know that a sequence T_n converges to T_∞ in that topology if and only if $\langle T_n, \varphi\rangle \to \langle T_\infty, \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$ (as the topology is not metrizable, the knowledge of converging sequences does not characterize what the topology is). Any derivation D^α is linear continuous from $\mathcal{D}'(\Omega)$ into itself, but the argument in the preceding proof only shows that it is sequentially continuous. Although it is rarely necessary to use the precise topology on $C_c^\infty(\Omega)$ or on $\mathcal{D}'(\Omega)$, it is useful to check that all the operations which one defines are sequentially continuous; a sequence φ_n converges to φ_∞ in $C_c^\infty(\Omega)$ if and only if all the φ_n have their supports in a compact set $K \subset \Omega$, and if $D^\alpha \varphi_n$ converges uniformly to $D^\alpha \varphi_\infty$ for all multi-indices α.

The next thing to define for distributions is multiplication by smooth functions.

Laurent SCHWARTZ has shown that it is not possible to define product of distributions in an associative way, and more precisely one has $(pv\frac{1}{x}\,x)\delta_0 = 1\,\delta_0 = \delta_0$, while $pv\frac{1}{x}(x\,\delta_0) = pv\frac{1}{x}\,0 = 0$. However, some physicists write formulas like $\delta(x)\,\delta(x) - \frac{1}{x}\,\frac{1}{x} = \frac{C}{x^2}$, but it is not clear what such a formula means.

Already we know that $\delta(x)$ should be written δ_0 and is a RADON measure but not a function, and as $\frac{1}{x}$ and $\frac{1}{x^2}$ are not locally integrable functions because of the singularities at 0, making distributions out of them requires some care. CAUCHY had already defined the notion of principal value, and by analogy Laurent SCHWARTZ defined a distribution denoted by $pv\frac{1}{x}$ and called the principal value of $\frac{1}{x}$, by $\langle pv\frac{1}{x}, \varphi\rangle = \lim_{n\to\infty}\int_{|x|\ge\frac{1}{n}} \frac{\varphi(x)}{x}\,dx$; this does define a distribution of order ≤ 1, because if $support(\varphi) \subset [-a,+a]$, then $\int_{|x|\ge\frac{1}{n}} \frac{\varphi(x)}{x}\,dx = \int_{\frac{1}{n}\le|x|\le a} \frac{\varphi(x)-\varphi(0)}{x}\,dx \to \int_{|x|\le a} \frac{\varphi(x)-\varphi(0)}{x}\,dx$, which exists because $|\varphi(x) - \varphi(0)| \le$

[1] Augustin Louis CAUCHY, French mathematician, 1789-1857. He worked in Paris. The concept of a CAUCHY sequence was first introduced a few years before him by BOLZANO.

Bernhard Placidus Johann Nepomuk BOLZANO, Czech mathematician, 1781-1848. He worked in Prague.

[2] Ernst Sigismund FISCHER, Austrian-born mathematician, 1875-1954. He worked in Cologne, Germany.

$|x|\,||D\,\varphi||_\infty$; that this distribution is not a RADON measure can be seen by either constructing a continuous function for which the definition gives $+\infty$, or by constructing a sequence of functions $\varphi_k \in C_c^\infty(R^N)$ which stay uniformly bounded, keep their support in a fixed compact set and for which $\langle pv\frac{1}{x}, \varphi_k\rangle \to +\infty$; taking φ_k nonnegative with support in $[0,1]$ and $\varphi_k(x) = 1$ for $\frac{1}{k} \le x \le 1-\frac{1}{k}$, one has $\langle pv\frac{1}{x}, \varphi_k\rangle \ge \int_{\frac{1}{k}}^{1-\frac{1}{k}} \frac{dx}{x}$. HADAMARD defined more general finite parts of $\frac{1}{x^k}$, and Laurent SCHWARTZ defined by analogy a distribution which can be considered the finite part of $\frac{1}{x^k}$.

One defines the product of a distribution by a C^∞ function, or more generally of a distribution of order $\le m$ by a C^m function as follows. As a consequence, one has $x\,pv\frac{1}{x} = 1$ and $x\,\delta_0 = 0$, so that $x(pv\frac{1}{x} + C\,\delta_0) = 1$ for all C, but $pv\frac{1}{x}$ can be shown to be the only solution T of $x\,T = 0$ which is odd.

Definition: If $T \in \mathcal{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$, then $\psi\,T$ (or $T\,\psi$) is the distribution defined by $\langle \psi\,T, \varphi\rangle = \langle T, \psi\,\varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$.■

Of course, one must check first that $\psi\,T$ is a distribution, and this follows from LEIBNIZ's formula.

LEIBNIZ's formula in one dimension states that $(f\,g)^{(n)} = \sum_{m=0}^n \binom{n}{m} f^{(m)} g^{(n-m)}$, where $h^{(k)}$ denotes the derivative of order k of h, and the binomial coefficient $\binom{n}{m}$ is $\frac{n!}{m!(n-m)!}$; it is easily proved by induction, starting from $(f\,g)' = f'g + f\,g'$. Writing an extension of LEIBNIZ's formula to the N-dimensional case is simplified by using a notation for multi-indices; $\alpha!$ means $\alpha_1! \ldots \alpha_N!$, $\binom{\alpha}{\beta}$ means $\binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_N}{\beta_N} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ and $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for $j = 1, \ldots, N$; then LEIBNIZ's formula has the same form $D^\alpha(f\,g) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta f\, D^{(\alpha-\beta)} g$, and it is easily proved by induction on N.

If $support(\varphi) \subset K \subset \Omega$, one has $||D^\alpha(\psi\,\varphi)||_\infty \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} ||D^\beta\psi||_{L^\infty(K)} ||D^{\alpha-\beta}\varphi||_\infty$, so that one has $\max_{|\alpha|\le m} ||D^\alpha(\psi\,\varphi)||_\infty \le C(K) \max_{|\alpha|\le m} ||D^\alpha\varphi||_\infty$; one also deduces that if T is a distribution of order $\le m$, then $\psi\,T$ is also a distribution of order $\le m$.

One must also check that the notation is compatible with the classical multiplication, i.e. if $f \in L^1_{loc}(\Omega)$ and T is the corresponding distribution (which should be written $f\,dx$), then the distribution S associated to $\psi\,f$ is indeed $\psi\,T$ as was just defined. This follows from the definition, as $\langle S, \varphi\rangle = \int_\Omega (\psi\,f)\varphi\,dx = \int_\Omega f(\psi\,\varphi)\,dx = \langle T, \psi\,\varphi\rangle = \langle \psi\,T, \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$.

The mapping $(\psi, T) \mapsto \psi\,T$ from $C^\infty \times \mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Omega)$ is sequentially continuous. The space $C^\infty(\Omega)$ is a FRÉCHET space and $\psi_n \to \psi_\infty$ in $C^\infty(\Omega)$ means that for every compact $K \subset \Omega$ and for every multi-index α, $D^\alpha\psi_n \to D^\alpha\psi_\infty$ uniformly on K. The topology of $\mathcal{D}'(\Omega)$ is more technical to describe but a sequence T_n converges to T_∞ in $\mathcal{D}'(\Omega)$ if and only if $\langle T_n, \varphi\rangle \to \langle T_\infty, \varphi\rangle$. Because the space $C_K^\infty(\Omega)$ of the functions in $C_c^\infty(\Omega)$ which have their support in K is a FRÉCHET space, it has BAIRE[3]'s property, from which BANACH-STEINHAUS[4] theorem follows and therefore if $T_n \to T_\infty$, then there exists a constant $C(K)$ and an integer $m(K)$ independent of n such that $|\langle T_n, \chi\rangle| \le C(K) \max_{|\alpha|\le m(K)} ||D^\alpha\chi||_\infty$ for all $\chi \in C_K^\infty(\Omega)$. Therefore $\langle(\psi_n T_n - \psi_\infty T_\infty), \varphi\rangle = \langle(\psi_n - \psi_\infty)T_n, \varphi\rangle + \langle\psi_\infty(T_n - T_\infty), \varphi\rangle$, so that one has $|\langle(\psi_n T_n - \psi_\infty T_\infty), \varphi\rangle| \le |\langle T_n, (\psi_n - \psi_\infty)\varphi\rangle| + |\langle T_n - T_\infty, \psi_\infty\varphi\rangle|$, and the first term tends to 0 because for each multi-index α, $D^\alpha\big((\psi_n - \psi_\infty)\varphi\big)$ converges uniformly to 0 by using LEIBNIZ's theorem, and the second term tends to 0 by definition.

Proposition: For $\psi \in C^\infty(\Omega), T \in \mathcal{D}'(\Omega)$ and any α, one has $D^\alpha(\psi\,T) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (D^\beta\psi)(D^{\alpha-\beta}T)$.

Proof: One proves that $\frac{\partial(\psi\,T)}{\partial x_j} = \frac{\partial\psi}{\partial x_j} T + \psi\,\frac{\partial T}{\partial x_j}$ for $j = 1, \ldots, N$, and then the formula follows by induction. For showing this, one notices that for every $\varphi \in C_c^\infty(\Omega)$ one has $\langle \frac{\partial(\psi\,T)}{\partial x_j}, \varphi\rangle = -\langle \psi\,T, \frac{\partial\varphi}{\partial x_j}\rangle = -\langle T, \psi\,\frac{\partial\varphi}{\partial x_j}\rangle = \langle T, -\frac{\partial(\psi\,\varphi)}{\partial x_j} + \varphi\frac{\partial\psi}{\partial x_j}\rangle = \langle \frac{\partial T}{\partial x_j}, \psi\,\varphi\rangle + \langle \frac{\partial\psi}{\partial x_j} T, \varphi\rangle = \langle \frac{\partial\psi}{\partial x_j} T + \psi\,\frac{\partial T}{\partial x_j}, \varphi\rangle$.■

If $\psi \in C^\infty(\Omega)$ and $a \in \Omega$, then $\psi\,\delta_a = \psi(a)\delta_a$, as $\langle \psi\,\delta_a, \varphi\rangle = \langle \delta_a, \psi\,\varphi\rangle = (\psi\,\varphi)(a) = \psi(a)\langle\delta_a, \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$. In particular $x_j\delta_0 = 0$ for $j = 1, \ldots, N$.

One has $x\,pv\frac{1}{x} = 1$, because $\langle x\,pv\frac{1}{x}, \varphi\rangle = \langle pv\frac{1}{x}, x\,\varphi\rangle = \lim_{n\to\infty} \int_{|x|\ge\frac{1}{n}} \frac{x\,\varphi(x)}{x}\,dx = \int_R \varphi(x)\,dx = \langle 1, \varphi\rangle$ for all $\varphi \in C_c^\infty(R)$.

[3] René-Louis BAIRE, French mathematician, 1874-1932. He worked in Dijon, France.

[4] Hugo Dyonizy STEINHAUS, Polish mathematician, 1887-1972. He worked in Lvov (now in Ukraine) until 1941, and in Wroclaw after 1945.

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06. Friday January 28, 2000.

Having enlarged the class of functions by introducing distributions, some partial differential equations that had been studied before may have gained new solutions which could not be considered before, and some other partial differential equations do not get new solutions. For example, the equation $x\,f = 0$ a.e. in R for $f \in L^1_{loc}(R)$ only has 0 as a solution, but for distributions $f \in \mathcal{D}'(R)$, $f = c\,\delta_0$ is a RADON measure which is a solution (and physicists do call it a DIRAC function), but it is useful to know if there are other distributions $T \in \mathcal{D}'(R)$ solutions of $x\,T = 0$.

Lemma: If $T \in \mathcal{D}'(R)$ and $x\,T = 0$, then there exists $C \in R$ such that $T = C\,\delta_0$.
Proof: If $\varphi \in C^\infty_c(R)$ satisfies $\varphi(0) = 0$, then $\varphi(x) = x\,\psi(x)$ and $\psi \in C^\infty_c(R)$; indeed the TAYLOR[1] expansion of φ near 0 is $\varphi(x) = \varphi'(0)x + \ldots + \frac{\varphi^{(n)}(0)}{n!}x^n + o(x^n)$, and therefore $\psi(x) = \varphi'(0) + \ldots + \frac{\varphi^{(n)}(0)}{n!}x^{n-1} + o(x^{n-1})$, shows that one must take $\psi(0) = \varphi'(0)$, and more generally $\psi^{(n-1)}(0) = \frac{\varphi^{(n)}(0)}{n}$ for $n \ge 1$, and as LEIBNIZ's formula gives $\varphi^{(n)}(x) = x\,\psi^{(n)}(x) + n\,\psi^{(n-1)}(x)$, the derivatives of ψ are continuous at 0. One deduces that $\langle T, \varphi\rangle = \langle T, x\,\psi\rangle = \langle x\,T, \psi\rangle = 0$. Let $\theta \in C^\infty_c(R)$ with $\theta(0) = 1$, then every function $\varphi \in C^\infty_c(R)$ may be written in the form $\varphi(x) = \varphi(0)\theta(x) + x\,\psi(x)$ for a function $\psi \in C^\infty_c(R)$, because $\varphi - \varphi(0)\theta$ vanishes at 0, and therefore $\langle T, \varphi\rangle = \langle T, \varphi(0)\theta + x\,\psi\rangle = \varphi(0)\langle T, \theta\rangle$, so that $T = C\,\delta_0$ with $C = \langle T, \theta\rangle$.∎

On the other hand if Ω is a connected open set of R^N the only distributions $T \in \mathcal{D}'(\Omega)$ which satisfy $\frac{\partial T}{\partial x_j} = 0$ for $j = 1, \ldots, N$ are the constants.

Definition: The tensor product $f_1 \otimes f_2$ of a real (or complex) function f_1 defined on a set X_1 and a real (or complex) function f_2 defined on a set X_2 is the real (or complex) function defined on $X_1 \times X_2$ by the formula $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$ for all $(x_1, x_2) \in X_1 \times X_2$.∎

Proposition: Let Ω be a connected open set of R^N. If $T \in \mathcal{D}'(\Omega)$ satisfies $\frac{\partial T}{\partial x_j} = 0$ for $j = 1, \ldots, N$, then T is a constant, i.e. there exists C such that $\langle T, \varphi\rangle = C\int_\Omega \varphi(x)\,dx$ for all $\varphi \in C^\infty_c(\Omega)$.
Proof: By a connectedness argument it is enough to show the result with Ω replaced by any open cube $\Omega_0 \subset \Omega$.

The proof is obtained by induction on the dimension N, and one starts with the case $N = 1$, Ω_0 being an interval (a, b). One notices that if $\varphi \in C^\infty_c(a, b)$ satisfies $\int_a^b \varphi(x)\,dx = 0$, then $\varphi = \frac{d\psi}{dx}$ for a function $\psi \in C^\infty_c(a, b)$, and ψ is given explicitly by $\psi(x) = \int_a^x \varphi(t)\,dt$. One chooses $\eta \in C^\infty_c(a, b)$ such that $\int_a^b \eta(x)\,dx = 1$, and then every $\varphi \in C^\infty_c(a, b)$ can be written as $\varphi = (\int_a^b \varphi(t)\,dt)\eta + \frac{d\psi}{dx}$ for a function $\psi \in C^\infty_c(a, b)$, because the integral of $\varphi - (\int_a^b \varphi(t)\,dt)\eta$ is 0. Therefore $\langle T, \varphi\rangle = \langle T, (\int_a^b \varphi(t)\,dt)\eta + \frac{d\psi}{dx}\rangle = (\int_a^b \varphi(t)\,dt)\langle T, \eta\rangle + \langle T, \frac{d\psi}{dx}\rangle = C(\int_a^b \varphi(t)\,dt) - \langle \frac{dT}{dx}, \psi\rangle = C(\int_a^b \varphi(t)\,dt)$ for all $\varphi \in C^\infty_c(a, b)$ with $C = \langle T, \eta\rangle$, and that means $T = C$.

Writing $\Omega_0 = \omega \times (a, b)$ where ω is a cube in R^{N-1}, then for $\varphi \in C^\infty_c(\omega)$ one defines $T_\varphi \in \mathcal{D}'(a, b)$ by $\langle T_\varphi, \psi\rangle = \langle T, \varphi \otimes \psi\rangle$ for $\psi \in C^\infty_c(a, b)$, and one checks immediately that this indeed defines a distribution T_φ on (a, b) because the bounds on derivatives of $\varphi \otimes \psi$ only involve a finite number of derivatives of ψ and the support of $\varphi \otimes \psi$ is the product of the supports of φ and of ψ and therefore stays in a fixed compact set when the support of ψ stays in a fixed compact set, φ being kept fixed. Then $\langle \frac{dT_\varphi}{dx_N}, \psi\rangle = -\langle T_\varphi, \frac{d\psi}{dx_N}\rangle = -\langle T, \varphi \otimes \frac{d\psi}{dx_N}\rangle = -\langle T, \frac{\partial(\varphi \otimes \psi)}{\partial x_N}\rangle = \langle \frac{\partial T}{\partial x_N}, \varphi \otimes \psi\rangle = 0$, so that T_φ is a constant C_φ, i.e. $\langle T, \varphi \otimes \psi\rangle = C_\varphi \int_a^b \psi(t)\,dt$ for every $\varphi \in C^\infty_c(\omega)$ and $\psi \in C^\infty_c(a, b)$. One uses then this formula to show that $\varphi \mapsto C_\varphi$ defines a distribution S on ω, as it is obviously linear and in order to obtain the desired bounds one chooses a function $\psi \in C^\infty_c(a, b)$ with $\int_a^b \psi(t)\,dt \ne 0$ and the bounds for S follow easily from the bounds for T, and therefore one can write $\langle T, \varphi \otimes \psi\rangle = \langle S, \varphi\rangle \int_a^b \psi(t)\,dt$ for all $\varphi \in C^\infty_c(\omega)$ and all $\psi \in C^\infty_c(a, b)$. Then for $j = 1, \ldots, N-1$ one has $0 = \langle \frac{\partial T}{\partial x_j}, \varphi \otimes \psi\rangle = -\langle T, \frac{\partial(\varphi \otimes \psi)}{\partial x_j}\rangle = -\langle T, \frac{\partial \varphi}{\partial x_j} \otimes \psi\rangle = -\langle S, \frac{\partial \varphi}{\partial x_j}\rangle \int_a^b \psi(t)\,dt = \langle \frac{\partial S}{\partial x_j}, \varphi\rangle \int_a^b \psi(t)\,dt$ and therefore $\frac{\partial S}{\partial x_j} = 0$, so that by the induction hypothesis S is a constant C_*, so that one has shown that

[1] Brook TAYLOR, English mathematician, 1685-1731.

$\langle T, \varphi \otimes \psi \rangle = C_*(\int_\omega \varphi(y)\,dy)(\int_a^b \psi(t)\,dt) = C_* \int_{\Omega_0} (\varphi \otimes \psi)(x)\,dx$, and this shows that $T = C_*$ if one uses the result that finite combinations of tensor products are dense (in the adequate topology).■

Once multiplication has been defined, and LEIBNIZ's formula has been extended, one can prove density results.

Proposition: For $1 \le p < \infty$, and any integer $m \ge 0$, the space $C_c^\infty(R^N)$ is dense in $W^{m,p}(R^N)$.
Proof: Let θ_n be a special truncating sequence, i.e. $\theta_n(x) = \theta_1\big(\frac{x}{n}\big)$, with $\theta_1 \in C_c^\infty(R^N)$, $0 \le \theta(x) \le 1$ on R^N and $\theta(x) = 1$ for $|x| \le 1$. For $u \in W^{m,p}(R^N)$, one defines $u_n = \theta_n u$, and one notices that $u_n \to u$ in $W^{m,p}(R^N)$ strong. Indeed one has $|u_n(x)| \le |u(x)|$ almost everywhere, and $u_n(x) \to u(x)$ as $n \to \infty$, and by LEBESGUE's dominated convergence theorem one deduces that $u_n \to u$ in $L^p(R^N)$ strong. Then for $|\alpha| \le m$ one has $D^\alpha u_n = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \theta_n D^{\alpha-\beta} u$, and the term for $\beta = 0$ converges to $D^\alpha u$ again by an application of LEBESGUE's dominated convergence theorem, while the terms for $|\beta| > 0$ contains derivatives of θ_n which converge uniformly to 0, so that one has $D^\alpha u_n \to D^\alpha u$ in $L^p(R^N)$ strong.

Then one approaches u_n by functions in $C_c^\infty(R^N)$ by convolution by a smoothing sequence ρ_ε for a sequence of ε converging to 0, and using a diagonal argument there is a sequence $u_n \star \rho_{\varepsilon(n)} \in C_c^\infty(R^N)$ which converges to u in $W^{m,p}(R^N)$ strong. The crucial point is to notice that for $|\alpha| \le m$ one has $D^\alpha(\rho_\varepsilon \star u_n) = \rho_\varepsilon \star D^\alpha u_n$, which converges then to $D^\alpha u_n$ in $L^p(R^N)$ strong. Indeed for any test function $\varphi \in C_c^\infty(R^N)$, one has $\langle D^\alpha(\rho_\varepsilon \star u_n), \varphi \rangle = (-1)^{|\alpha|} \langle \rho_\varepsilon \star u_n, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle u_n, \check\rho_\varepsilon \star D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle u_n, D^\alpha(\check\rho_\varepsilon \star \varphi) \rangle = \langle D^\alpha u_n, \check\rho_\varepsilon \star \varphi \rangle = \langle \rho_\varepsilon \star D^\alpha u_n, \varphi \rangle$, where $\check f$ is defined by $\check f(x) = f(-x)$.■

For $p = \infty$, the same method shows that one can approach any $u \in W^{m,\infty}(R^N)$ by a sequence $\psi_n \in C_c^\infty(R^N)$ such that for every $|\alpha| \le m$, $D^\alpha \psi_n$ converges to $D^\alpha u$ in $L^\infty(R^N)$ weak $\star$ and $L^q_{loc}(R^N)$ strong for every finite q.

If Ω is an open set of R^N, it is not true in general that $C_c^\infty(R^N)$ is dense in $W^{m,p}(\Omega)$, and one is led to the following definition.

Definition: $W_0^{m,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$.■

If the boundary $\partial\Omega$ is smooth enough, then the functions of $W_0^{m,p}(\Omega)$ are 0 on the boundary, as will be seen later. If $\partial\Omega$ is too small, then it may happen that $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$; this is related to the fact that the functions in $W^{m,p}(\Omega)$ are not necessarily continuous, as stated by the imbedding theorem of SOBOLEV, stated now and proved later.

Theorem: i) If $1 \le p < \frac{N}{m}$ then $W^{m,p}(R^N) \subset L^r(R^N)$ with $\frac{1}{r} = \frac{1}{p} - \frac{m}{N}$, but $W^{m,p}(R^N)$ is not a subspace of $L^s(R^N)$ for $s > r$.

ii) If $p = \frac{N}{m}$ then $W^{m,p}(R^N) \subset L^q(R^N)$ for every $q \in [p, \infty)$, but $W^{m,N}(R^N)$ is not a subspace of $L^\infty(R^N)$ if $p > 1$; however $W^{N,1}(R^N) \subset C_0(R^N)$.

iii) If $\frac{N}{m} < p < \infty$ then $W^{m,p}(R^N) \subset C_0(R^N)$, the space of continuous functions tending to 0 at ∞. If $\frac{N}{k} < p < \frac{N}{k-1}$ for an integer k, then $W^{k,p}(R^N) \subset C^{0,\gamma}(R^N)$, the space of HÖLDER continuous functions of order γ, with $\gamma = k - \frac{N}{p}$.■

For example if $\Omega = R^N \setminus F$, where F is a finite number of points and $p \le \frac{N}{m}$, then $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$ and coincides with $W^{m,p}(R^N)$, as will be shown later.

It is useful to know that any closed set C of R^N can be the zero set of a C^∞ function, because $R^N \setminus C$ can be written as the countable union of open balls $B(z_n, r_n)$, and if $\varphi \in C_c^\infty(R^N)$ has its support equal to the closed unit ball and is positive in the open unit ball, then one considers the series $\sum_n c_n \varphi\big(\frac{x - z_n}{r_n}\big)$ and one can choose the sequence c_n such that the series converges uniformly, as well as any of its derivatives. Therefore the zero set of a smooth function can be as irregular as one may wish (among closed sets, of course).

It is useful to know that there are open sets with thick boundary, for example if one has a numerotation of the points with rational coordinates of R^N, $z_1, \ldots, z_n, \ldots$, and for $\varepsilon > 0$ one considers $A_\varepsilon = \bigcup_n B(z_n, \varepsilon\, 2^{-n})$, then A_ε is open, has LEBESGUE measure $\le \varepsilon$ and its boundary is its complement $R^N \setminus A_\varepsilon$ which has infinite LEBESGUE measure and an empty interior as it contains no point with rational coordinates.

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07. Monday January 31, 2000.

For a continuous function u from a topological space X into a vector space, the support is the closure of the set of points $x \in X$ such that $u(x) \neq 0$. One cannot define the support of a RADON measure or a distribution in this way, and one uses a characterization of the complement of the support: it is the largest open set on which u is 0, and this leads to the following definition.

Definition: A RADON measure $\mu \in \mathcal{M}(\Omega)$ is said to be 0 on an open subset $\omega \subset \Omega$ if $\langle \mu, \varphi \rangle = 0$ for all $\varphi \in C_c(\omega)$. A distribution $T \in \mathcal{D}'(\Omega)$ is said to be 0 on an open subset $\omega \subset \Omega$ if $\langle T, \varphi \rangle = 0$ for all $\varphi \in C_c^\infty(\omega)$.■

In order to define the support of a RADON measure or a distribution, one must deduce that being 0 on a family of open sets implies being 0 on its union, and this is done by using a partition of unity.

Proposition: Let F be a closed set of R^N and let U_i, $i \in I$, be an open covering of F. Then for each $i \in I$ there exists $\theta_i \in C^\infty(R^N)$ with $support(\theta_i) \subset U_i$, $0 \leq \theta_i \leq 1$ and $\sum_{i \in I} \theta_i = 1$ on an open set V containing F, the sum being locally finite, i.e. for each $x \in \bigcup_{i \in I} U_i$ there exists an open set W_x containing x such that only a finite number of θ_j are not identically 0 on W_x.
Proof: Let $\rho_1 \in C_c^\infty(R^N)$, with $support(\rho_1) \subset \overline{B(0,1)}$, $\rho_1 \geq 0$ and $\int_{R^N} \rho_1(x)\,dx = 1$, and for $\varepsilon > 0$ let $\rho_\varepsilon(x) = \varepsilon^{-N} \rho_1\big(\frac{x}{\varepsilon}\big)$. For each $x \in F$ there exists $i(x) \in I$ such that $x \in U_{i(x)}$ and $0 < r(x) \leq 1$ such that $B\big(x, 4r(x)\big) \subset U_{i(x)}$. For $n \geq 1$, let $F_n = \{x \in F : n-1 \leq |x| \leq n\}$, and as F_n is compact and covered by the open balls $B\big(x, r(x)\big)$ for $x \in F_n$, it is covered by a finite number of them, with centers $y \in G_n$, G_n being a finite subset of F_n. One chooses $\varepsilon_n < \min_{y \in G_n} r(y)$, and for $y \in G_n$ one denotes α^y the characteristic function of $B\big(y, 2r(y)\big)$ and $\beta_n^y = \rho_{\varepsilon_n} \star \alpha^y$, so that $\beta_n^y \in C_c^\infty\Big(B\big(y, 3r(y)\big)\Big) \subset C_c^\infty(U_{i(y)})$ and $\beta_n^y = 1$ on $B\big(y, r(y)\big)$, and therefore $\gamma_n = \sum_{y \in G_n} \beta_n^y \geq 1$ on the open set $V_n = \bigcup_{y \in G_n} B\big(y, r(y)\big)$ (and $\gamma_n \geq 0$ elsewhere), which contains F_n.

For $j \in I$, let η_j be the sum of all β_n^y for which $i(y) = j$; there might be an infinite number of such y, but because $n-1 \leq |y| \leq n$ and β_n^y is 0 outside $B(y,4)$, the sum is locally finite and $\eta_j \in C^\infty(U_j)$ (if F is compact, there are only a finite number of terms and therefore $\eta_j \in C_c^\infty(U_j)$ in this case). Similarly, let $\zeta = \sum_{j \in I} \eta_j$, the sum being also locally finite and equal to $\sum_n \gamma_n$, and therefore $\zeta \geq 1$ on $V = \bigcup_n V_n$. Choose $\psi \in C^\infty(R^N)$ such that $\psi = 0$ on $\overline{V}$ and $\psi > 0$ on $R^N \setminus \overline{V}$. For $j \in I$, let $\theta_j = \frac{\eta_j}{\zeta + \varphi}$, which is C^∞ because $\zeta + \varphi$ does not vanish (as $\zeta \geq 1$ and $\psi = 0$ on $\overline{V}$ and $\psi > 0$ and $\zeta \geq 0$ outside $\overline{V}$), and therefore $support(\theta_j) \subset support(\eta_j) \subset U_j$. One has $\sum_{j \in I} \theta_j = \frac{\zeta}{\zeta + \psi}$, which is 1 on V as $\psi = 0$ on V.■

Corollary: If a RADON measure $\mu \in \mathcal{M}(\Omega)$ or a distribution $T \in \mathcal{D}'(\Omega)$ is 0 on $\omega_i \subset \Omega$ for $i \in I$, then it is 0 on $\bigcup_{i \in I} \omega_i$.
Proof: Let $\omega = \bigcup_{i \in I} \omega_i$ and $\varphi \in C_c^\infty(\omega)$ with support K. There is a finite number of functions $\theta_i \in C_c^\infty(\omega_i)$ with $\sum_i \theta_i = 1$ on K and therefore $\varphi = \sum_{i \in I} \theta_i \varphi$, and as $\theta_i \varphi \in C_c^\infty(\omega_i)$ one has $\langle T, \theta_i \varphi \rangle = 0$ and by summing in i one deduces that $\langle T, \varphi \rangle = 0$. If $\psi \in C_c(\omega)$, then for a smoothing sequence ρ_n one defines $\varphi_n = \psi \star \varphi_n$, and for n large enough the support of all the φ_n stays in a fixed compact set K' of Ω; considering μ as a distribution, the preceding result shows that $\langle \mu, \varphi_n \rangle = 0$ for n large enough, but as $\varphi_n \to \psi$ uniformly on K' one has $\langle \mu, \psi \rangle = \lim_{n \to \infty} \langle \mu, \varphi_n \rangle = 0$.■

Partitions of unity will be useful for studying how functions in SOBOLEV spaces behave near the boundary of an open set.

There are properties of SOBOLEV spaces which do depend upon the smoothness of the boundary $\partial\Omega$, but for some other properties the boundary plays no role, and these properties are said to be local, and they may be expressed for larger spaces.

Definition: For an open set $\Omega \subset R^N$, an integer $m \geq 0$ and $1 \leq p \leq \infty$, the space $W_{loc}^{m,p}(\Omega)$ is the space of distributions $T \in \mathcal{D}'(\Omega)$ such that for every $\varphi \in C_c^\infty(\Omega)$ one has $\varphi T \in W^{m,p}(\Omega)$.■

One checks immediately that the space $L_{loc}^1(\Omega)$, which was described previously as the set of (equivalence clases of) LEBESGUE measurable functions T such that for any compact $K \subset \Omega$ one has $\chi_K T \in L^1(\omega)$, where

χ_K is the characteristic function of K, is indeed identical with the space described in the preceding definition, which is the space of distributions T such that $\varphi\, T \in L^1(\Omega)$ for every $\varphi \in C_c^\infty(\Omega)$.

Of course, $W^{m,p}_{loc}(\Omega)$ is not a BANACH space, but a FRÉCHET space.

Once one will have proved that for $1 \le p < N$ one has $W^{1,p}(R^N) \subset L^{p^*}(R^N)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, then one will automatically deduce that for any open set $\Omega \subset R^N$ one has $W^{1,p}(\Omega) \subset L^{p^*}_{loc}(\Omega)$. Indeed for $\varphi \in C_c^\infty(\Omega)$ and $u \in W^{1,p}(\Omega)$, the function $\varphi\, u$ also belongs to $W^{1,p}(\Omega)$ and is 0 outside the support of φ, and by extending it by being 0 outside Ω, one finds a function $\widetilde{\varphi\, u} \in W^{1,p}(R^N) \subset L^{p^*}(R^N)$, showing that one has $\varphi\, u \in L^{p^*}(\Omega)$.

Proposition: (i) If $1 \le p,q,r \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then for $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ one has $u\, v \in W^{1,r}(\Omega)$ (and $||u\, v||_{1,r} \le C||u||_{1,p}||v||_{1,q}$).

(ii) If $1 \le p,q,s < N$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - \frac{1}{N}$, then for $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ one has $u\, v \in W^{1,s}_{loc}(\Omega)$.

Proof: The first part is a consequence of applying HÖLDER inequality to the formula $\frac{\partial(u\, v)}{\partial x_j} = \frac{\partial u}{\partial x_j}\, v + u\, \frac{\partial v}{\partial x_j}$. For proving the formula, one must show that $-\int_\Omega u\, v\, \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega \big(\frac{\partial u}{\partial x_j}\, v + u\, \frac{\partial v}{\partial x_j}\big)\varphi\, dx$ for every $\varphi \in C_c^\infty(\Omega)$. One chooses $\theta \in C_c^\infty(\Omega)$ with $\theta = 1$ on the support K of φ, and the formula to be proved does not change if one replaces u by $\theta\, u$ (as the derivative of θ vanishes on the support of φ), but as $\theta\, u$ extended by 0 outside Ω is a function of $W^{1,p}(R^N)$, one may approach it by a sequence $w_n \in C_c^\infty(R^N)$, and as the formula is true for u replaced by w_n one just let n tend to ∞ and each term converges to the right quantity.

The second part is similar and consists in using the fact that on the support K of φ one has $u \in L^{p^*}(K)$ and $v \in L^{q^*}(K)$.■

A property which is true in R^N but not in every open set Ω is the fact that $W^{1,\infty}$ coincides with the space of LIPSCHITZ functions.

Proposition: $W^{1,\infty}(R^N) = Lip(R^N)$. If Ω is an open subset of R^N, then $Lip(\Omega) \subset W^{1,\infty}(\Omega)$, and $W^{1,\infty}(\Omega) \subset L^\infty(\Omega) \cap Lip_{loc}(\Omega)$, where $Lip_{loc}(\Omega)$ is the (FRÉCHET) space of locally LIPSCHITZ functions; if $u \in W^{1,\infty}(\Omega)$ and $||grad(u)||_\infty \le K$, then one has $|u(x) - u(y)| \le K\, d_\Omega(x,y)$, where d_Ω is the geodesic distance from x to y in Ω, the shortest length of a smooth path connecting x to y in Ω.

Proof: If $u \in W^{1,\infty}(R^N)$, and ρ_n is a special smoothing sequence, then $u_n = \rho_n \star u \in C^\infty(R^N)$, $||u_n||_\infty \le ||u||_\infty$ and for $j = 1,\ldots,N$ one has $\big|\big|\frac{\partial u_n}{\partial x_j}\big|\big|_\infty \le \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_\infty \le ||grad(u)||_\infty$, and as this inequality applies to any direction it implies that $|grad(u_n)| \le ||grad(u)||_\infty$ in R^N, and therefore $|u_n(x) - u_n(y)| \le |x-y|\; ||grad(u)||_\infty$ for all $x,y \in R^N$; as a subsequence of u_n converges almost everywhere to u, one deduces that $|u(x) - u(y)| \le |x-y|\; ||grad(u)||_\infty$ for almost every $x,y \in R^N$, i.e. u is LIPSCHITZ continuous with LIPSCHITZ constant $||grad(u)||_\infty$. Conversely if $u \in Lip(R^N)$ and $u_n = u \star \rho_n \in C^\infty(R^N)$, then for any $h \in R^N$ one has $u_n - \tau_{s\,h} u_n = (u - \tau_{s\,h} u) \star \rho_n$, then one deduces $||u_n - \tau_{s\,h} u_n||_\infty \le ||u - \tau_{s\,h} u||_\infty \le K\, s\, |h|$, where K is the LIPSCHITZ constant of u, and therefore after dividing by s and letting s tend to 0 one deduces that at every point the derivative of u_n in a direction h is bounded by $K\, |h|$, i.e. $||grad(u_n)||_\infty \le K$ and then letting n tend to ∞ gives $||grad(u)||_\infty \le K$.

The preceding argument is valid if $u \in Lip(\Omega)$, as $u \star \rho_n$ is well defined at a short distance from the boundary. The passage from a bound on $||grad(u)||_\infty$ to a bound on $|u(x) - u(y)|$ for $u \in C^\infty(\Omega)$ relies on the fact that the segment $[xy]$ is included in Ω, and it can be replaced by the sum of the lengths of segments for a polygonal path joining x to y and staying inside Ω, and the infimum of these quantities is the geodesic distance $d_\Omega(x,y)$.■

If Ω is the open subset of R^2 defined in polar coordinates by $-\pi < \theta < \pi$ and $r > 1$, then the function $u = \theta$ satisfies $\frac{\partial u}{\partial x} = -\frac{y}{r^2}$ and $\frac{\partial u}{\partial y} = \frac{x}{r^2}$, so that $u \in W^{1,\infty}(\Omega)$, but for $\varepsilon > 0$ small the points with Cartesian coordinates $(-2,-\varepsilon)$ and $(-2,+\varepsilon)$ are at Euclidean distance 2ε while the difference in values of θ is converging to 2π as $\varepsilon \to 0$ (the geodesic distance tends to $2\sqrt{3} + \frac{4\pi}{3}$).

21-724. SOBOLEV spaces
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SOBOLEV's imbedding theorem requires some regularity of the boundary $\partial\Omega$. For example, one has $H^1(R^2) = W^{1,2}(R^2) \subset L^p(R^2)$ for every $p \in [2,\infty)$, but the similar property does not hold for the particular open set $\Omega = \{(x,y) : 0 < x < 1, 0 < y < x^2\}$. For proving this remark, one checks for which value $\alpha \in R$ the function $u(x) = x^\alpha$ belongs to $L^p(\Omega)$ and as one needs $\int_0^1 x^{\alpha\,p+2}\,dx = \int_\Omega x^{\alpha\,p}\,dx\,dy < \infty$, one finds that the condition is $\alpha\,p + 2 > -1$; therefore $u \in H^1(\Omega)$ if and only if $2(\alpha-1)+2 > -1$, i.e. $\alpha > -\frac{1}{2}$, and $u \in L^p(\Omega)$ if and only if $\alpha\,\frac{p}{6} > -\frac{1}{2}$, i.e. $H^1(\Omega)$ is not a subset of $L^p(\Omega)$ for $p > 6$. One checks easily the limitations for other cusps on the boundary, like for $\Omega = \{(x,y) : 0 < x < 1, 0 < y < x^\gamma\}$, with $\gamma > 1$.

A part of SOBOLEV's imbedding theorem asserts that for $1 \le p < N$ one has $W^{1,p}(R^N) \subset L^{p^*}(R^N)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, or $p^* = \frac{N\,p}{N-p}$, and one deduces then that $W^{1,p}(R^N) \subset L^q(R^N)$ for every $q \in [p, p^*]$ by the following application of HÖLDER inequality.

Lemma: If $1 \le p_0 < p_\theta < p_1 \le \infty$ and $\theta \in (0,1)$ is defined by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then one has $||u||_{p_\theta} \le ||u||_{p_0}^{1-\theta} ||u||_{p_1}^{\theta}$ for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$.
Proof: One applies HÖLDER inequality $||f\,g||_1 \le ||f||_q ||g||_{q'}$ with $f = |u|^{(1-\theta)p_\theta}$ and $g = |u|^{\theta\,p_\theta}$, with $q = \frac{p_0}{(1-\theta)p_\theta}$ and $q' = \frac{p_1}{\theta\,p_\theta}$, which are conjugate exponents.∎

The preceding result is not restricted to the LEBESGUE measure, and the restriction that $p_0 \ge 1$ is not necessary (although the notation $||v||_r$ is not a norm for $0 < r < 1$).

SOBOLEV's imbedding theorem is natural if one considers the question of scaling.

Proposition: If $1 \le q < p$ or $1 \le p < N$ and $q > p^*$, there is no finite constant C such that $||u||_q \le C\,||u||_{1,p}$ for all $u \in C_c^\infty(R^N)$.
Proof: For $\lambda > 0$, one applies the inequality $||u||_q \le C\,||u||_p + C\,||grad(u)||_p$ to the function v defined by $v(x) = u\big(\frac{x}{\lambda}\big)$ for $x \in R^N$. One notices that for $1 \le r < \infty$ one has $||v||_r = \big(\int_{R^N} \big|u\big(\frac{x}{\lambda}\big)\big|^r\,dx\big)^{1/r} = \big(\int_{R^N} |u(y)|^r\,\lambda^N\,dy\big)^{1/r} = \lambda^{N/r}||u||_r$, and $||grad(v)||_r = \lambda^{-1+N/r}||grad(u)||_r$. Therefore if the inequality was true one would deduce that $\lambda^{N/q}||u||_q \le C\,\lambda^{N/p}||u||_p + C\,\lambda^{-1+N/p}||grad(u)||_p$, i.e. an inequality of the form $||u||_q \le C\,\lambda^\alpha||u||_p + C\,\lambda^\beta||grad(u)||_p$ for all $\lambda > 0$. If one had $\alpha > 0$ and $\beta > 0$, then by letting λ tend to 0 one would deduce the contradiction $||u||_q = 0$ for all $u \in C_c^\infty(R^N)$; this corresponds to the case $q < p$. Similarly if one had $\alpha < 0$ and $\beta < 0$ one would deduce the contradiction by letting λ tend to ∞; this corresponds to the case $q > p^*$.∎

If the inequality is true for $q = p^*$, then the same argument shows that one has $||u||_{p^*} \le C\,||grad(u)||_p$ for all $u \in C_c^\infty(R^N)$. However, this is not a proof that the inequality is true, as for example the inequality $||u||_\infty \le C\,||grad(u)||_N$ implies no contradiction by the preceding scaling argument, but it is not true for $N > 1$.

One reason why one cannot deduce by a scaling argument that the limiting case of the SOBOLEV imbedding theorem does not hold for $p = N$, is that in the larger context of the LORENTZ[1] spaces all the spaces $L^{p,q}(R^N)$ scale in the same way for different values of $q \in [1,\infty]$. If all the partial derivatives of u are estimated in $L^{N,1}(R^N)$ it provides a bound for the sup norm of u, while for any $q > 1$ there exist unbounded functions v with all partial derivatives in $L^{N,q}(R^N)$.

The method of SOBOLEV for proving the imbedding theorem for $W^{1,p}(R^N)$ in the case $1 \le p < N$ was based on the use of an elementary solution of the Laplacian Δ.

Definition: If $P(\xi) = \sum_\alpha a_\alpha \xi^\alpha$ is a polynomial in $\xi \in R^N$ (with constant coefficients), and $P(D)$ denotes the differential operator $P(D) = \sum_\alpha a_\alpha D^\alpha$, an elementary solution of $P(D)$ is a distribution E such that $P(D)E = \delta_0$.∎

[1] George Gunther LORENTZ, Russian-born mathematician, born in 1910. He emigrated to United States in 1953, and he works at University of Texas, Austin.

Elementary solutions are not unique, but a particular elementary solution may often be selected by using symmetry arguments, and in the case of $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, one finds radial solutions of the form $E = C_N r^{2-N}$ for $N \geq 3$, $E = C_2 \log r$ for $N = 2$ and $E = \frac{|x|}{2}$ for $N = 1$. Anticipating on the properties of convolution with distributions, one has $u = u \star \Delta\, E = \sum_{i=1}^N \frac{\partial u}{\partial x_i} \star \frac{\partial E}{\partial x_i}$, and the result would be a consequence of YOUNG's inequality if one had $\frac{\partial E}{\partial x_i} \in L^{N'}(R^N)$, but as these derivatives are of the order of r^{1-N} this fails to be the case (as $N \geq 2$), but SOBOLEV proved that YOUNG's inequality still holds if instead of a function in $L^q(R^N)$ one uses the function $r^{-N/q}$. This line of argument by SOBOLEV was improved later by Jaak PEETRE, using the theory of interpolation which he had developed in parallel with Jacques-Louis LIONS, but the particular result of interpolation in LORENTZ spaces had been obtained by O'NEIL[2], who was extending a result about nonincreasing rearrangements of HARDY and LITTLEWOOD[3].

A second method for proving SOBOLEV's imbedding theorem was developped independently by Emilio GAGLIARDO[4] and by Louis NIRENBERG, but the same idea has also been used by Olga LADYZHENSKAYA[5]. One first proves inequalities for smooth functions with compact support, and then one extends them to SOBOLEV spaces by density.

Lemma: For $f \in C_c^\infty(R)$ (and by density for $f \in W^{1,1}(R)$), one has $2|f(x)| \leq \int_R \left|\frac{df}{dx}\right| dx$ for all $x \in R$.
Proof: From $f(x) = \int_{-\infty}^x \frac{df}{dx}(y)\, dy = -\int_x^\infty \frac{df}{dx}(y)\, dy$, one deduces that $|f(x)| \leq \int_\infty^x \left|\frac{df}{dx}\right| dy$ and $|f(x)| \leq \int_x^\infty \left|\frac{df}{dx}\right| dy$, and adding these two inequalities gives the result. As $C_c^\infty(R)$ is dense in $W^{1,1}(R)$, it shows that $W^{1,1}(R) \subset C_0(R)$.■

Lemma: For $N \geq 2$, and $i = 1, \ldots, N$, let f_i be a measurable function independent of x_i, and assume that $f_i \in L^{N-1}$ (when one restricts to $x_i = 0$ for example), then $F = \prod_{i=1}^N f_i$ belongs to $L^1(R^N)$ and $||F||_1 \leq \prod_{i=1}^N ||f_i||_{N-1}$.
Proof: The case $N = 2$ is obvious as $F(x_1, x_2) = f_1(x_2) f_2(x_1)$ and $||F||_1 = ||f_1||_1 ||f_2||_1$. For $N \geq 2$ one uses an induction on N. For $i = 1, \ldots, N-1$, let $g_i = \left(\int_R |f_i|^{N-1} dx_N\right)^{1/(N-2)}$, so that g_i is independent of x_i and x_N and $g_i \in L^{N-2}$ in its $N-2$ arguments, with $||g_i||_{N-2} \leq ||f_i||_{N-1}^{(N-1)/(N-2)}$, and by the induction hypothesis one has $G = \prod_{i=1}^{N-1} g_i \in L^1$ in the arguments $x_1, \ldots, x_{N-1}$. By integrating first in x_N and using HÖLDER inequality, one has $\int_R |F|\, dx_N \leq \prod_{i=1}^{N-1} g_i^{(N-2)/(N-1)} f_N = G^{(N-2)/(N-1)} f_N$, and therefore, as the conjugate exponent of $N-1$ is $\frac{N-1}{N-2}$, one has $\int_{R^N} |F|\, dx \leq \int_{R^{N-1}} |G|^{(N-2)/(N-1)} |f_N|\, dx_1 \ldots dx_{N-1} \leq \left(\int_{R^{N-1}} |G|\, dx_1 \ldots dx_{N-1}\right)^{(N-2)/(N-1)} ||f_N||_{N-1} \leq \prod_{i=1}^N ||f_i||_{N-1}$.■

Proposition: For $a > 1$ and $1 < p < \infty$, one has

$$\left(\int_{R^N} |u|^{N\,a/(N-1)}\, dx\right)^{(N-1)/N} \leq \frac{a}{2}\left(\prod_{i=1}^N \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_p\right)^{1/N} \left(\int_{R^N} |u|^{(a-1)p'}\, dx\right)^{1/p'}$$

for all $u \in C_c^\infty(R^N)$. For $p = 1$ one has $||u||_{1^*} \leq \frac{1}{2}\left(\prod_{i=1}^N \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_1\right)^{1/N}$ for all $u \in C_c^\infty(R^N)$.
Proof: Applying the first lemma with $f = |u|^a$ for $u \in C_c^\infty(R^N)$ (because $|u|^a$ is of class C^1 and the first lemma only requires the function to belong to $W^{1,1}(R)$), one obtains $|u(x)|^a \leq \frac{a}{2} \int_R |u|^{a-1} \left|\frac{\partial u}{\partial x_i}\right| dx_i$, so if one denotes $f_i = \left(\frac{a}{2} \int_R |u|^{a-1} \left|\frac{\partial u}{\partial x_i}\right| dx_i\right)^{1/N-1}$ one may apply the second lemma and one deduces $\int_{R^N} |u(x)|^{N\,a/(N-1)}\, dx \leq ||\prod_{i=1}^N f_i||_1 \leq \prod_{i=1}^N ||f_i||_{N-1} = \left(\frac{a}{2}\right)^{N/(N-1)} \prod_{i=1}^N \left(\int_{R^N} |u|^{a-1} \left|\frac{\partial u}{\partial x_i}\right| dx\right)^{1/(N-1)}$, which by HÖLDER inequality

[2] Richard C. O'NEIL, American mathematician. He works at State University of New York, Albany.

[3] John Edensor LITTLEWOOD, English mathematician, 1885-1977. He held the newly founded Rouse BALL Professorship at Cambridge, 1928-1950.

Walter William Rouse BALL, English mathematician, 1850-1925.

[4] Emilio GAGLIARDO, Italian mathematician. He works in Pavia.

[5] Olga Aleksandrovna LADYZHENSKAYA, Russian mathematician, born in 1925. She works at Russian Academy of Sciences, St. Petersburg.

is $\leq \left(\frac{a}{2}\right)^{N/(N-1)} \prod_{i=1}^{N} \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_p^{1/(N-1)} \left(\int_{R^N} |u|^{(a-1)p'}\,dx\right)^{N/(N-1)p'}$, and taking the power $(N-1)/N$ gives the result. For $p = 1$, one takes $a = 1$ and one stops before applying HÖLDER inequality.■

SOBOLEV's imbedding theorem $W^{1,p}(R^N) \subset L^{p^*}(R^N)$ follows in the case $1 \leq p < N$ by choosing a such that $\frac{N\,a}{N-1} = (a-1)p'$, and this common value appears to be p^*.

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If one denotes $\Lambda = \Big(\prod_{i=1}^N \big|\big|\frac{\partial u}{\partial x_i}\big|\big|_p\Big)^{1/N}$, and one defines Φ by $\Phi(a) = \int_{R^N} |u|^{N\,a/(N-1)}\,dx$, then the case $p = N$ of the preceding inequality is $\Phi(a) \le \big(\frac{a\,\Lambda}{2}\big)^{N/(N-1)}\Phi(a-1)$ for all $a > 1$. One deduces that a bound on $||u||_{1,N}$, which gives a bound on $\Phi(N-1) = ||u||_N^N$ and a bound on Λ, implies a bound for $||u||_q$ for all $q > N$. More precisely, for any integer $a \ge N$, one has $\Phi(a) \le \big(\frac{a!}{(N-1)!}\big)^{N/(N-1)}\big(\frac{\Lambda}{2}\big)^{(a+1-N)N/(N-1)}\Phi(N-1)$. After taking the logarithm one finds $\limsup_{a\to\infty} \frac{1}{a}\big(\Phi(a)\big)^{(N-1)/(N\,a)} \le \frac{\Lambda}{2e}$, or $\limsup_{q\to\infty} \frac{||u||_q}{q} \le \frac{(N-1)\Lambda}{2N\,e}$, as a consequence of STIRLING's formula[1], $a! \approx \big(\frac{a}{e}\big)^a\sqrt{2\pi\,a}$.

Another property of the case $p = N$ is that for $u \in W^{1,N}(R^N)$ and ε positive and small enough (but how small depends upon u), $e^{\varepsilon\,|u|}$ is locally integrable. Indeed once one knows that $||u||_q \le K\,q$ for $q \ge q_c$ and q_c large enough (and $K > \frac{(N-1)\Lambda}{2N\,e}$), one has $\int_{R^N}\sum_{q=q_c}^\infty \frac{(\varepsilon\,|u|)^q}{q!}\,dx = \sum_{q=q_c}^\infty \int_{R^N}\frac{(\varepsilon\,|u|)^q}{q!}\,dx \le \sum_{q=q_c}^\infty \frac{(\varepsilon\,K\,q)^q}{q!}$, which is $< \infty$ if $\varepsilon\,K\,e < 1$. The same result was also obtained in a different way by Louis NIRENBERG and Fritz JOHN[2] as a property of the space BMO (Bounded Mean Oscillation[3]), which they had introduced in part for studying the limiting case $p = N$ of SOBOLEV's imbedding theorem.

For the case $p > N$, one notices that when $(a-1)p'$ and $\frac{N\,a}{N-1}$ are equal they take the negative value $-\frac{N\,p}{p-N}$, and therefore if one denotes $q_k = -\frac{N\,p}{p-N} + \alpha\,\beta^k$ with $\beta = \frac{N(p-1)}{p(N-1)} > 1$, then the choice of a giving $(a-1)p' = q_k$ gives $\frac{N\,a}{N-1} = q_{k+1}$; one chooses $\alpha = \frac{p^2}{p-N}$ so that $q_0 = p$. Using $a \le \frac{q_k}{p'} \le \frac{\alpha}{p'}\beta^k$, one finds that

$$\int_{R^N} |u|^{q_{k+1}}\,dx \le \Big(\frac{\alpha\,\Lambda\,\beta^k}{2p'}\Big)^{N/(N-1)}\Big(\int_{R^N} |u|^{q_k}\,dx\Big)^\beta.$$

This gives an estimate of $||u||_{q_k}$ for all k in terms of $||u||_{1,p}$ (which gives a bound on $||u||_p$ and on Λ) and k; as $||u||_\infty = \lim_{r\to\infty}||u||_r$ one must show that $||u||_{q_k}$ is bounded independently of k. By homogeneity of the formula, one has $q_{k+1} = \frac{N}{N-1} + q_k\beta$ and therefore if one puts $|u| = \frac{\alpha\,\Lambda}{2p'}|v|$, the formula becomes $\int_{R^N}|v|^{q_{k+1}}\,dx \le \beta^{k\,N/(N-1)}\big(\int_{R^N}|v|^{q_k}\,dx\big)^\beta$. Denoting $f(k) = \log\big(\int_{R^N}|v|^{q_k}\,dx\big)$, one deduces $f(k+1) \le A\,k + \beta\,f(k)$ with $A = \frac{N\log\beta}{N-1}$, and by induction this gives $f(k) \le A\big((k-1)+(k-2)\beta+\ldots+2\beta^{k-3}+\beta^{k-2}\big)+\beta^k f(0)$ for $k \ge 2$, and as $q_k = -\frac{N\,p}{p-N} + \alpha\,\beta^k$ one finds that $\frac{f(k)}{q_k} \to \frac{1}{\alpha}\big(f(0) + \frac{A}{(\beta-1)^2}\big)$, giving a bound for $||u||_\infty$ in terms of $||u||_{1,p}$.

A different way to obtain a bound in $L^\infty(R^N)$, following SOBOLEV's method is to replace the elementary solution E of Δ by a parametrix[4], which instead of solving $\Delta\,E = \delta_0$ satisfies $\Delta\,F = \delta_0 + \psi$ with $\psi \in C_c^\infty(R^N)$; one takes $F = \theta\,E$ with $\theta \in C_c^\infty(R^N)$ equal to 1 in a small ball around 0, and although the derivatives $\frac{\partial F}{\partial x_j}$ are $O(r^{1-N})$ near 0 as for the partial derivatives of E, F has compact support and therefore $F \in L^q(R^N)$ for $1 \le q < \frac{N}{N-1}$, in particular for $q = p'$ if $p > N$. Using $u = u \star (\Delta\,F - \psi) = \sum_{i=1}^N \frac{\partial u}{\partial x_i} \star \frac{\partial F}{\partial x_i} - u \star \psi$ one deduces that $||u||_\infty \le \sum_{i=1}^n \big|\big|\frac{\partial u}{\partial x_i}\big|\big|_p\big|\big|\frac{\partial F}{\partial x_i}\big|\big|_{p'} + ||u||_p||\psi||_{p'}$.

Whatever the way one has obtained a bound $||u||_\infty \le A\,||u||_p + B\,||grad(u)||_p$ when $p > N$, the scaling argument implies that one has $||u||_\infty \le C\,||u||^{1-\theta}||grad(u)||_p^\theta$ with $\theta = \frac{N}{p}$. Indeed applying the inequality to

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[1] James STIRLING, Scottish mathematician, 1692-1770, only improved a formula was had been obtained by DE MOIVRE.

Abraham DE MOIVRE, French mathematician, 1667-1754.

[2] Fritz JOHN, German-born mathematician, 1910-1994. He emigrated to United States in 1935 and after 1946 he worked at the COURANT Institute for Mathematical Sciences, New York University.

[3] BMO is the space of locally integrable function for which there exists C such that $\int_Q |u - u_Q|\,dx \le C\,meas(Q)$ for every cube Q, denoting by u_Q the average of u on Q, i.e. $meas(Q)u_Q = \int_Q u(x)\,dx$.

[4] The word has been coined by HADAMARD.

$v(x) = u\big(\frac{x}{\lambda}\big)$ gives $||u||_\infty \le A\,|\lambda|^{N/p}||u||_p + B\,|\lambda|^{-1+N/p}||grad(u)||_p$, and choosing the best $\lambda > 0$ gives the desired bound.

One deduces then that for $p > N$ one has $u \in C^{0,\gamma}$ with $\gamma = 1 - \frac{N}{p}$ by applying the preceding inequality to $\tau_h u - u$. Integrating $\frac{d}{dt}u(x - t\,h)$ from 0 to 1 one obtains $|u(x-h) - u(x)| \le |h| \int_0^1 |grad(u)|(x + t\,h)\,dt$, and taking the norm in $L^p(R^N)$ of both sides (and using JENSEN's inequality) one obtains $||\tau_h u - u||_p \le |h|\,||grad(u)||_p$, and as $||grad(\tau_h u - u)||_p \le 2||grad(u)||_p$ one obtains $||\tau_h u - u||_\infty \le C'\,|h|^{1-N/p}||grad(u)||_p$.

That SOBOLEV's imbedding theorem cannot be improved can also be seen by constructing counter-examples. For instance, if $\varphi \in C_c^\infty(R^N)$ is equal to 1 in a small ball around 0, then $r^\alpha\varphi \in L^p(R^N)$ is equivalent to $p\,\alpha + N - 1 > -1$, i.e. $\alpha > -\frac{N}{p}$ and $r^\alpha\varphi \in W^{1,p}(R^N)$ is equivalent to $\alpha - 1 > -\frac{N}{p}$; if $1 \le p < N$ and $q > p^*$ one can choose α such that $\alpha - 1 > -\frac{N}{p}$ but $\alpha < -\frac{N}{q}$, giving a function in $W^{1,p}(R^N)$ which does not belong to $L^q(R^N)$.

For the case $p = N$, one considers functions $|\log r|^\beta\varphi$, and one finds that $|\log r|^\beta\varphi \in W^{1,N}(R^N)$ if and only if $|\log r|^{\beta-1}r^{-1}\varphi \in L^N(R^N)$, i.e. if and only if $N(\beta - 1) > -1$, and there exists $\beta > 0$ satisfying this inequality if $N > 1$ (for $N = 1$, a preceding lemma has shown that $W^{1,1}(R) \subset C_0(R)$).

As mentioned before, SOBOLEV's imbedding theorem can be made more precise by using LORENTZ spaces, $L^{p,q}$, which increase with q, with $L^{p,p} = L^p$ and $L^{p,\infty}$ a space introduced by MARCINKIEWICZ[5], which is the space of (equivalence classes of) measurable function satisfying $\int_\omega |f|\,dx \le C\,meas(\omega)^{1/p'}$ for every measurable set ω. Jaak PEETRE showed that for $1 \le p < N$ one has $W^{1,p}(R^N) \subset L^{p,p^*}(R^N$. For $p = N$, the result of Fritz JOHN and Louis NIRENBEG using BMO was improved by Neil TRUDINGER[6], who showed that if $u \in W^{1,N}(R^N)$, then for every $C > 0$ one has $e^{C\,|u|^{N/(N-1)}} \in L^1_{loc}(R^N)$. The result was extended by Haïm BREZIS[7] and Stephen WAINGER[8] who showed that if u has all its partial derivatives in the space $L^{N,q}(R^N)$, then $e^{C\,|u|^{q'}} \in L^1_{loc}(R^N)$.

Questions about the best constants in SOBOLEV imbedding theorems have been investigated, in particular by Thierry AUBIN[9] and by Giorgio TALENTI[10]; a good class of functions for finding the optimal constants are those of the form $\frac{1}{(1+a\,r^2)^k}$.

The preceding results can be extended to functions having derivatives $\frac{\partial u}{\partial x_j} \in L^{p_j}(R^N)$, not all p_j being equal (it occurs naturally if one coordinate denotes time and the others denote space, for example). In 1978, I visited Trento and heard a talk on this subject by Alois KUFNER[11], who followed the natural approach of Emilio GAGLIARDO[12] or Louis NIRENBERG, as the method of SOBOLEV cannot be used in this case (at least, I do not see how one could use it), but I learned afterwards that it had been obtained earlier by TROISI[13]. I have obtained an extension of all these methods for the case where the partial derivatives belong to different

[5] Józef MARCINKIEWICZ, Polish mathematician, 1910-1940. He died during World War II, presumably executed by the Soviets with thousands of other Polish officers.

[6] Neil S. TRUDINGER, Australian mathematician, born in 1942. He works at Australian National University, Canberra.

[7] Haïm BREZIS, French mathematician, born in 1944. He works at University Paris VI (Pierre et Marie CURIE), Paris (and it seems at RUTGERS University, New Brunswick, NJ.).

Pierre CURIE, French physicist, 1859-1906, and his wife Marie SLODOWSKA CURIE, Polish-born physicist, 1867-1934, jointly received the NOBEL prize in Physics in 1903, and she also received the NOBEL prize in Chemistry in 1911.

[8] Stephen WAINGER, American mathematician, born in 1936. He works at University of Wisconsin, Madison.

[9] Thierry AUBIN, French mathematician. He works at University Paris VI (Pierre et Marie CURIE), Paris.

[10] Giorgio G. TALENTI, Italian mathematician. He works in Florence.

[11] Alois KUFNER, Czech mathematician. He works at Czech Academy of Sciences, Prague.

[12] Just after the talk, I met Emilio GAGLIARDO, whom I had first met the week before in Pavia, and learned that he was also teaching in Trento; he was not the least interested by the ideas that he had introduced in the past, and he continued the discussion that we had a few days before on his favorite subject, applying Mathematics to Music.

[13] Mario TROISI, Italian mathematician. He works in Salerno.

LORENTZ spaces, by a different method (the methods that have been described do not seem to be sufficient for proving such a general result).

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For $1 \leq p < \infty$, one takes the norm of the space $W^{1,p}(R^N)$ to be $\left(\int_{R^N} |u|^p|\, dx + \int_{R^N} |grad(u)|^p\, dx\right)^{1/p}$, and it is useful to notice that adding $\int_{R^N} |u|^p\, dx$ and $\int_{R^N} |grad(u)|^p\, dx$ is a strange practice, which mathematicians follow almost all the time, and which makes physicists wonder if mathematicians know what they are talking about.

The key point is a question of units. In real problems x usually denotes the space variables, which are measured in units of length (noted L), while t denotes the time variable, measured in units of time (noted T), and if one considers the wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \sum_{j=1}^N \frac{\partial^2 u}{\partial x_j^2} = 0$, c is a characteristic velocity, measured in units $L T^{-1}$, and the equation is consistent as each of the terms of the equation is measured in units $U T^{-2}$, whatever the unit U for u is (u could be a vertical displacement if one looks at small waves on the surface of a lake or a swimming pool and $N = 2$ in that case, or a pressure if one looks at propagation of sound in the atmosphere, or in the ocean, or in the ground, and $N = 3$ in that case). For non linear equations, like the BURGERS[1]'s equation $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$, the dimension of u must be that of a velocity $L T^{-1}$, but some physicists prefer to introduce a characteristic velocity c and write it $\frac{\partial u}{\partial t} + c\, u \frac{\partial u}{\partial x} = 0$ and in that case u has no dimension.

Mathematicians studying equations from Continuum Mechanics or Physics should be careful about the question of units, and as SOBOLEV spaces were originally introduced for studying solutions of partial differential equations from Continuum Physics, this question does occur naturally in studying them. The quantities $\int_{R^N} |u|^p\, dx$ and $\int_{R^N} |grad(u)|^p\, dx$ are not measured in the same units, the first term having dimension $U^p L^N$ and the second one having dimension $U^p L^{N-p}$, and it would be more natural when dealing with physical problems to use a norm like $\left(\int_{R^N} |u|^p\, dx + L_0^p \int_{R^N} |grad(u)|^p\, dx\right)^p$, where L_0 is a characteristic length, but as was already noticed when the argument of scaling was used in relation with SOBOLEV's imbedding theorem, one can start from an inequality written without paying attention to units and then deduce from it a better one which does take into account this question.

An important remark is that for some open sets Ω and some particular subspaces of $W^{1,p}(\Omega)$ one can avoid adding the terms $\int_\Omega |u|^p\, dx$ and $\int_\Omega |grad(u)|^p\, dx$ because POINCARÉ[2] inequality holds.

Definition: If $1 \leq p \leq \infty$ and Ω is a nonempty open subset of R^N, one says that POINCARÉ inequality holds for a subspace V of $W^{1,p}(\Omega)$ if there exists a constant C such that one has $||u||_p \leq C\, ||grad(u)||_p$ for all $u \in V$.■

Of course, C has then the dimension of a length, and if there is no characteristic length that one can attach to Ω, then one expects that POINCARÉ inequality does not hold.

Proposition: i) If $meas(\Omega) < \infty$ and if the constant function 1 belongs to a subspace V of $W^{1,p}(\Omega)$, then POINCARÉ inequality does not hold on V.

ii) POINCARÉ inequality does not hold on $W_0^{1,p}(\Omega)$ if Ω contains arbitrarily large balls, i.e. if there exists a sequence $r_n \to \infty$ and points $x_n \in \Omega$ such that $B(x_n, r_n) \subset \Omega$.

iii) If Ω is included in a strip of width d, i.e. there exists $\xi \in R^N$ with $|\xi| = 1$ and $\Omega \subset \{x \in R^N : \alpha < (\xi.x) < \beta\}$ and $d = \beta - \alpha$, then $||u||_p \leq C_0\, d\, ||grad(u)||_p$ for all $u \in W_0^{1,p}(\Omega)$, where C_0 is a universal constant, i.e. independent of which Ω is used. If $p = \infty$, POINCARÉ inequality holds on $W_0^{1,\infty}(\Omega)$ if and only if there exists $C < \infty$ such that for all $x \in \Omega$ one has $dist(x, \partial\Omega) \leq C$, where $dist$ is the Euclidean distance.

iv) If $meas(\Omega) < \infty$, then POINCARÉ inequality holds for $W_0^{1,p}(\Omega)$ for $1 \leq p \leq \infty$, and one has $||u||_p \leq C(p) meas(\Omega)^{1/N}\, ||grad(u)||_p$ for all $u \in W_0^{1,p}(\Omega)$.

[1] Johannes Martinus BURGERS, Dutch-born mathematician, 1895-1981. He emigrated to United States in 1955 and worked at University of Maryland, College Park.

[2] Jules Henri POINCARÉ, French mathematician, 1854-1912. He worked in Paris. There is an Institut Henri POINCARÉ (IHP), dedicated to Mathematics and Theoretical Physics, part of University Paris VI (Pierre et Marie CURIE). I have been told that this kind of inequality which is now named after him appeared in his work on tides.

v) If the injection of V into $L^p(\Omega)$ is compact, then POINCARÉ inequality holds on a subspace V of $W^{1,p}(\Omega)$ if and only if the constant function 1 does not belong to V.

Proof: i) If $meas(\Omega) < \infty$, then $1 \in W^{1,p}(\Omega)$, but as $grad(1) = 0$ one must have $C = 0$, which is incompatible with $1 \in V$.

ii) Let $\varphi \in C_c(R^N)$ with $\varphi \neq 0$ and $support(\varphi) \subset B(0,1)$, then one defines u_n by $u_n(x) = \varphi\big(\frac{x-x_n}{r_n}\big)$ belongs to $C_c^\infty(\Omega)$ and one has $||u_n||_p = r_n^{N/p}||\varphi||_p$ and $||grad(u_n)||_p = r_n^{-1+N/p}||grad(\varphi)||_p$; if the inequality was true one would have $1 \leq \frac{C}{r_n}$. Therefore if POINCARÉ inequality holds on $W_0^{1,p}(\Omega)$ it gives an upper bound for the size of balls included in Ω.

iii) One starts from the case $N = 1$, where one has shown that $\max_{x\in R}|v(x)| \leq \frac{1}{2}\int_R\big|\frac{dv}{dx}\big|\,dx$ for all $v \in C_c^\infty(R)$, so if for an interval $I = (\alpha,\beta)$ one has $u \in C_c^\infty(I)$, one deduces that $||u||_p \leq ||u||_\infty d^{1/p}$ and as $\int_I\big|\frac{du}{dx}\big|\,dx \leq \big|\big|\frac{du}{dx}\big|\big|_p d^{\frac{1}{p'}}$, one deduces that $||u||_p \leq \frac{d}{2}\big|\big|\frac{du}{dx}\big|\big|_p$ for all $u \in C_c^\infty(I)$. One deduces the case of the strip by applying the preceding inequality in an orthogonal basis whose last vector is $e_N = \xi$, so that the strip is defined by $\alpha < x_N < \beta$, and for each choice of $x' = (x_1,\ldots,x_{N-1})$ one has $\int_\alpha^\beta |u(x',x_N)|^p\,dx_N \leq 2^{-p}d^p\int_\alpha^\beta\big|\frac{\partial u}{\partial x_N}(x',x_N)|^p\,dx_N$, and one integrates then this inequality in x' in order to obtain POINCARÉ inequality in the case $1 \leq p < \infty$. In the case $p = \infty$, the condition is necessary because of ii), and it is sufficient because for each $x \in \Omega$ there exists $z \in \partial\Omega$ with $|x - z| \leq C$, and if $u \in C_c^\infty(\Omega)$ there exists y on the segment $[x,z]$ and outside the support of u so that $|u(x)| = |u(x) - u(y)| \leq C\,||grad(u)||_\infty$; then the same inequality extends to $W_0^{1,\infty}(\Omega)$.

iv) If $p = \infty$, it follows from iii). If $1 \leq p < \infty$, one chooses $q < N$ such that $1 \leq q \leq p < q^*$, and one uses SOBOLEV imbedding theorem $||u||_{q^*} \leq C\,||grad(u)||_q$ for all $u \in C_c^\infty(R^N)$, and HÖLDER inequality: $||u||_p \leq ||u||_{q^*}meas(\Omega)^\alpha$ with $\alpha = \frac{1}{p} - \frac{1}{q^*}$ and $||grad(u)||_q \leq ||grad(u)||_p meas(\Omega)^\beta$ with $\beta = \frac{1}{q} - \frac{1}{p}$, and therefore $\alpha+\beta = \frac{1}{N}$. Without the precise estimate for the constant it can also be proved by the compactness argument used in v). There is a different proof for the case $p = 2$ based on FOURIER transform, and also a proof of the compactness property using FOURIER transform, and they will be shown later.

v) The necessity that 1 should not belong to V follows from i). That this condition is sufficient is the consequence of what I call the equivalence lemma, shown later, by taking $E_1 = \overline{V}$, $A = grad$ and $E_2 = \big(L^p(\Omega)\big)^N$ and B the injection into $E_3 = L^p(\Omega)$.■

Of course, if $\Omega_1 \subset \Omega_2$ and POINCARÉ inequality holds for $W_0^{1,p}(\Omega_2)$, then it holds for $W_0^{1,p}(\Omega_1)$, because each function of $u \in W_0^{1,p}(\Omega_1)$ can be extended by 0 and gives a function $\widetilde{u} \in W_0^{1,p}(\Omega_2)$.

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11. Wednesday February 9, 2000.

Questions of equivalence of norms play an important role in the theoretical part of Numerical Analysis, because interpolation formulas or quadrature formulas are used on triangulations made with small elements and it is important to know how the errors behave in terms of the size of these elements; in 1974, after I was told about a classical result named after James BRAMBLE[1] and HILBERT[2], whose theoretical part relies on results of Jacques DENY[3] and Jacques-Louis LIONS, I developped a more general framework, which also generalizes a different type of results by Jaak PEETRE, which I had seen being used in a book of Jacques-Louis LIONS and Enrico MAGENES in order to prove the FREDHOLM[4] alternative for elliptic boundary value problems. I call my framework the equivalence lemma, and at a theoretical level it is useful in order to obtain many variants of POINCARÉ inequalities in various subspaces of SOBOLEV spaces, but it requires enough regularity of the boundary in order to satisfy the hypothesis of compactness.

Proposition: Let E_1 be a BANACH space, E_2, E_3 normed spaces (with $||\cdot||_j$ denoting the norm of E_j), and let $A \in \mathcal{L}(E_1, E_2)$, $B \in \mathcal{L}(E_1, E_3)$ such that one has

(a) $||u||_1 \approx ||A\,u||_2 + ||B\,u||_3$

(b) B is compact.

Then one has the following properties

i) The kernel of A is finite dimensional.

ii) The range of A is closed.

iii) There exists a constant C_0 such that if F is a normed space and $L \in \mathcal{L}(E_1, F)$ satisfies $L\,u = 0$ whenever $A\,u = 0$, then one has $||L\,u||_F \leq C_0 ||L||\,||A\,u||_2$ for all $u \in E_1$.

iv) If G is a normed space and $M \in \mathcal{L}(E_1, G)$ satisfies $L\,u \neq 0$ whenever $A\,u = 0$ and $u \neq 0$, then $||u||_1 \approx ||A\,u||_2 + ||M\,u||_G$.

Proof: i) On $X = ker(A)$, the closed unit ball for $||\cdot||_1$ is compact. Indeed if $||u_n||_1 \leq 1$, then $B\,u_n$ stays in a compact of E_3 by b) so a subsequence $B\,u_m$ converges in E_3 and is therefore a CAUCHY sequence in E_3, and as $A\,u_m = 0$ it is a CAUCHY sequence in E_2 and therefore a) implies that u_m is a CAUCHY sequence in E_1, which converges as E_1 is a BANACH space. By a theorem of F. RIESZ, X must be finite dimensional.

ii) As a consequence of HAHN[5]-BANACH theorem, X being finite dimensional has a topological supplement Y, i.e. $X \cap Y = \{0\}$ and there exist $\pi_X \in \mathcal{L}(E_1, X)$ and $\pi_Y \in \mathcal{L}(E_1, Y)$ such that $e = \pi_X(e) + \pi_Y(e)$ for all $e \in E_1$, and in particular Y is closed as it is the kernel of π_X, and therefore Y is a BANACH space.

One shows then that there exists $\alpha > 0$ such that $||A\,u||_2 \geq \alpha ||u||_1$ for all $u \in Y$. Indeed, if it was not true there would exist a sequence $y_n \in Y$ with $||y_n||_1 = 1$ and $A\,y_n \to 0$, and again taking a subsequence such that $B\,y_m$ converges in E_3 one finds that y_m would be a CAUCHY sequence in Y and its limit $y_\infty \in Y$ would satisfy $A\,y_\infty = 0$, i.e. $y_\infty \in X$ and therefore $y_\infty = 0$, contradicting the fact that $||y_\infty||_1 = \lim_m ||y_m||_2 = 1$.

Then if $f_n \in R(A)$ satisfies $f_n \to f_\infty$ in E_2, one has $f_n = A\,e_n = A(\pi_X e_n + \pi_Y e_n) = A\,\pi_Y e_n$; therefore if one denotes $y_n = \pi_Y e_n$, one has $\alpha ||y_n - y_m||_1 \leq ||A\,y_n - A\,y_m||_2 = ||f_n - f_m||_2$ so that y_n is a CAUCHY sequence in Y and its limit y_∞ satisfies $A\,y_\infty = f_\infty$, showing that $R(A)$ is closed.

iii) As A is a bijection from Y onto $R(A)$ it has an inverse D, and as one considers $R(A)$ equipped with the norm of E_2, $D \in \mathcal{L}\big(R(A), Y\big)$ with $||D|| \leq \frac{1}{\alpha}$ by the previously obtained inequality (it show that $R(A)$ is a BANACH space, although one has not assumed that E_2 is a BANACH space, and the closed graph theorem has not been used). With this definition of D one has $y = D\,A\,y$ for all $y \in Y$, and in particular $D\,A\,e = \pi_Y e$

[1] James H. BRAMBLE, American mathematician. He works at Texas A & M University, College Station.

[2] Stephen R. HILBERT, American mathematician. He works at Ithaca College, Ithaca, NY.

[3] Jacques DENY, French mathematician. He was my colleague in Orsay from 1975 to 1982, one of the very few who expressed his support for my (lost) fight against the methods of falsifications organized by our communist colleagues and their friends who controlled the University; he must have been born in 1918, as a meeting organized in 1983 in his honour must have been related to his being 65, the mandatory age for retirement in France.

[4] Erik Ivar FREDHOLM, Swedish mathematician, 1866-1927. He worked in Stockholm.

[5] Hans HAHN, Austrian mathematician, 1879-1934. He worked in Vienna.

for all $e \in E_1$ because $A\,e = A\,\pi_Y e$. From the hypothesis $L\,u = 0$ for $u \in X$, one has $L\,e = L\,\pi_Y e = L\,D\,A\,e$ for all $e \in E_1$, and therefore $||L\,e||_F \le ||L||\,||D||\,||A\,e||_2$, and therefore C may be taken to be the norm of D in $\mathcal{L}\big(R(A), Y\big)$.

iv) One has $||A\,e||_2 + ||M\,e||_G \le (||A|| + ||M||)||e||_1$, and if the norms were not equivalent one could find a sequence $e_n \in E_1$ with $||e_n||_1 = 1$ and $||A\,e_n||_2 + ||M\,e_n||_G \to 0$. As before, a subsequence e_m would be such that $B\,e_m$ is a CAUCHY sequence in E_3, and as $A\,e_m \to 0$ in E_2, e_m would be a CAUCHY sequence in E_1, converging to a limit e_∞, which would satisfy the contradictory properties $||e_\infty||_1 = 1$, $A\,e_\infty = 0$ and $M\,e_\infty = 0$.■

Other applications of the equivalence lemma will be encountered later, but a crucial hypothesis is the compactness assumption without which the result may be false (but it is not always false); for example, taking $1 < p < \infty$ and $E_1 = W^{1,p}(R)$, $A = \frac{d}{dx}$, $E_2 = E_3 = L^p(R)$ and B the injection of $W^{1,p}(R)$ into $L^p(R)$ (which is not compact), then the range of A is not closed, and is dense (for $p = 1$ it is closed, equal to the subspace of functions in $L^1(R)$ with integral 0).

A compactness result, attributed to RELLICH[6] and to KONDRAŠOV[7], asserts that if Ω is a bounded open set of R^N with a smooth enough boundary $\partial\Omega$, then the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, and it can be deduced from a result associated with the names of M. RIESZ, FRÉCHET and KOLMOGOROV, but it can be proved easily from the result for $W_0^{1,p}(\Omega')$ proved below, once extension properties will have been studied. For unbounded open sets, or for bounded sets with nonsmooth boundaries, the situation is not as simple, but for the case of $W_0^{1,p}(\Omega)$ the smoothness of the boundary is not important.

Most compactness theorems use in some way the basic results of ARZELÁ[8] and ASCOLI[9]: if u_n is a bounded sequence of real continuous functions on a separable compact metric space X, then for each $x \in X$ one can extract a subsequence u_m such that $u_m(x)$ converges, and by a diagonal argument this can be achieved for all x in a countable dense subspace, and this is extended to other points if one assumes that the sequence is equicontinuous at every point, a way to say that at any point y the functions are continuous in the same way, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$, depending upon y and ε but not upon n, such that $d(y,z) \le \delta$ implies $|u_n(y) - u_n(z)| \le \varepsilon$ for all n. In order to cover many applications to weak convergence or weak $\star$ convergence like the BANACH-ALAOGLU[10] theorem, one also uses a maximality argument, hidden in TIHONOV[11] theorem, that any product of compact spaces is compact.

For functions in SOBOLEV spaces, which are not necessarily continuous, but can be approached by smooth functions, one needs to control precisely the error, and some smoothness properties of the boundary will be needed if one works with $W^{1,p}(\Omega)$, but the following result is only concerned with $W_0^{1,p}(\Omega)$.

Proposition: i) If Ω is an unbounded open subset of R^N such that there exists $r_0 > 0$ and a sequence $x_n \in \Omega$ converging to infinity with $B(x_n, r_0) \subset \Omega$, then the injection of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is not compact.

ii) If $1 \le p \le \infty$ and Ω is an open set with $meas(\Omega) < \infty$, then the injection of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact.

Proof: i) Let $\varphi \in C_c(R^N)$, with $support(\varphi) \subset B(0, r_0)$, and $\varphi \ne 0$, then $u_n = \tau_{x_n}\varphi \in W_0^{1,p}(\Omega)$, $||u_n||_{1,p}$ is constant, but no subsequence converges strongly in $L^p(\Omega)$ because u_n converges to 0 in $L^p(\Omega)$ weak (weak $\star$ if $p = \infty$) and it cannot converge strongly to 0 as its norm stays constant, and not 0.

ii) One starts by the case where Ω is bounded. For a bounded sequence $u_n \in W_0^{1,p}(\Omega)$ one wants to show that it belongs to a compact set of $L^p(\Omega)$, i.e. there is a subsequence which converges strongly in $L^p(\Omega)$. For proving that property it is enough to show that for every $\varepsilon > 0$ one can find a compact set K_ε of $L^p(\Omega)$ such that each u_n is at a distance at most $C\,\varepsilon$ of K_ε, i.e. one can decompose $u_n = v_{n,\varepsilon} + w_{n,\varepsilon}$, with $||w_{n,\varepsilon}||_p \le C\,\varepsilon$ and $v_{n,\varepsilon} \in K_\varepsilon$; indeed for a subsequence one has $\limsup_{m,m' \to \infty} ||u_m - u_{m'}||_p \le 2C\,\varepsilon$, and a diagonal subsequence is therefore a CAUCHY sequence.

[6] Franz RELLICH, German mathematician. He worked in Göttingen.

[7] V. I. KONDRAŠOV, Russian mathematician.

[8] Cesare ARZELÁ, Italian mathematician, 1847-1912. He worked in Bologna.

[9] Giulio ASCOLI, Italian mathematician, 1843-1896.

[10] Leonidas ALAOGLU, American mathematician.

[11] Andreĭ Nikolaevich TIHONOV, Russian mathematician, 1906-1993. He worked at Moscow State University.

For doing that, one extends the functions u_n by 0 outside Ω (still calling them u_n instead of $\widetilde{u_n}$), so that one has a bounded sequence $u_n \in W^{1,p}(R^N)$ with support in a fixed bounded set of R^N. For a special smoothing sequence $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho_1\big(\frac{x}{\varepsilon}\big)$ (with $\rho_1 \in C_c^\infty(R^N)$, $support(\rho_1) \subset B(0,1)$, $\rho_1 \ge 0$ and $\int_{R^N} \rho_1\,dx = 1$), one takes $v_{n,\varepsilon} = u_n \star \rho_\varepsilon$. This gives $||v_{n,\varepsilon}||_\infty \le ||u_n||_p ||\rho_\varepsilon||_{p'}$ and $\big|\big|\frac{\partial v_{n,\varepsilon}}{\partial x_j}\big|\big|_\infty \le \big|\big|\frac{\partial u_n}{\partial x_j}\big|\big|_p ||\rho_\varepsilon||_{p'}$, i.e. for $\varepsilon > 0$ fixed $v_{n,\varepsilon}$ stays in a bounded set of LIPSCHITZ functions, and keep their support in a fixed compact set of R^N, and therefore by ARZELA-ASCOLI theorem, a subsequence converges uniformly on R^N, and therefore the sequence of restrictions to Ω converges strongly in $L^\infty(\Omega)$, and therefore in $L^p(\Omega)$. In order to estimate $||u_n - v_{n,\varepsilon}||_p$, one notices that $(u_n - u_n \star \rho_\varepsilon)(x) = \int_{R^N} \rho_\varepsilon(y)\big(u_n(x) - u_n(x-y)\big)\,dy$, i.e. $u_n - u_n \star \rho_\varepsilon = \int_{R^N} \rho_\varepsilon(y)(u_n - \tau_y u_n)\,dy$, and therefore $||u_n - u_n \star \rho_\varepsilon||_p \le \int_{R^N} \rho_\varepsilon(y)||u_n - \tau_y u_n||_p\,dy$, but as one has $||u_n - \tau_y u_n||_p \le |y|\,||grad(u_n)||_p$ and $\int_{R^N} |y|\rho_\varepsilon(y)\,dy = A\,\varepsilon$, one deduces $||u_n - u_n \star \rho_\varepsilon||_p \le A\,B\,\varepsilon$, where B is an upper bound for $||grad(u_n)||_p$ for all n.

If Ω is unbounded but has finite measure, one chooses $r_0 < r_1 < \infty$ such that the measure of $\Omega \setminus B(0,r_0)$ is $< \eta$, and one chooses $\theta \in C_c^\infty(R^N)$ such that $\theta = 1$ on $B(0,r_0)$ and $support(\theta) \subset B(0,r_1)$. The sequence $u_n - \theta\,u_n$ is bounded in $W^{1,p}(R^N)$ and is 0 outside $\Omega \setminus B(0,r_0)$ (one should avoid using its support, which is closed and could be very big if $\partial\Omega$ is thick).

For $p = \infty$, the maximum distance from a point of $\Omega \setminus B(0,r_0)$ to its boundary is at most $C(N)\eta^{1/N}$, and as one can take a common LIPSCHITZ constant for all the functions $u_n - \theta\,u_n$, one deduces that they are uniformly small in $L^\infty(\Omega)$, and as $\theta\,u_n$ stays in a bounded set of $W_0^{1,p}\big(\Omega \cap B(0,r_1)\big)$, it remains in a compact of $L^p\big(\Omega \cap B(0,r_1)\big)$ by applying the result for the case of bounded open sets.

For $1 \le p < \infty$, one bounds the norm of $||u_n - \theta\,u_n||_p$ by using SOBOLEV imbedding theorem, choosing $q < N$ such that $1 \le q \le p < q^* < \infty$, and as $u_n - \theta\,u_n$ is bounded in $W^{1,q}(R^N)$ and therefore in $L^{q^*}(R^N)$, one has $||u_n - \theta\,u_n||_p \le ||u_n - \theta\,u_n||_{q^*} meas\big(\Omega \setminus B(0,r_0)\big)^\alpha$ with $\alpha = \frac{1}{p} - \frac{1}{q^*} > 0$, proving the desired uniform small bound for $||u_n - \theta\,u_n||_p$.

There is a different proof if $p = 2$ which relies on FOURIER transform.■

It is then time now to start studying the many questions where the regularity of the boundary plays a role, approximation by smooth functions, compactness, extension to the whole space, traces on the boundary.

12. Friday February 11, 2000.

For approaching functions in $W^{1,p}(\Omega)$ but not in $W_0^{1,p}(\Omega)$, one needs smooth functions whose support intersects the boundary $\partial\Omega$.

Definition: $\mathcal{D}(\overline{\Omega})$ denotes the restrictions to Ω of functions in $C_c^\infty(R^N)$.■

For some nice open sets Ω, $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $1 \le p < \infty$, but this is not true for some open sets, like a disc in the plane from which one removes a closed segment $[a,b]$, the intuitive reason being that functions in $\mathcal{D}(\overline{\Omega})$ are continuous across the segment, while there are functions in $W^{1,p}(\Omega)$ who are discontinuous across it (and for giving a mathematical meaning to this idea, one will have to define a notion of trace on the boundary). The preceding example is one where the open set is not locally on one side of the boundary, and will be ruled out for the moment, but one should remember that there are applications where one must consider open sets of this type, in the study of crack propagation, or in the scattering of waves by a thin plate, for example).

Definition: i) An open set Ω of R^N is said to have a continuous boundary, if for every $z \in \partial\Omega$, there exists $r_z > 0$, an orthonormal basis $e_1, \ldots, e_N$, and a continuous function F of $x' = (x_1, \ldots, x_{N-1})$ such that $\{x \in \Omega, |x-z| < r_z\} = \{x \in R^N, |x-z| < r_z, x_N > F(x')\}$.

ii) An open set Ω of R^N is said to have a LIPSCHITZ boundary, if the same property holds with F being a LIPSCHITZ continuous function.■

Of course, $e_1, \ldots, e_N$ and F vary with the point z, and the origin of the coordinate system may also change with z. For this class of open sets, Ω is locally on only one side of the boundary.

With the preceding definition, assuming $a < b$, the open set $\{(x,y) \in R^2, x > 0, a\,x^2 < y < b\,x^2\}$ is an open set with continuous boundary if $a\,b < 0$, but not if $a\,b \ge 0$.

Proposition: Let Ω be a bounded open set with continuous boundary. If $1 \le p < \infty$, then $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

If $u \in W^{1,\infty}(\Omega)$, there exists a sequence $u_n \in \mathcal{D}(\overline{\Omega})$ such that $u_n \rightharpoonup u$ in $L^\infty(\Omega)$ weak $\star$ and $L^q(\Omega)$ strong for $1 \le q < \infty$, and for $j = 1, \ldots, N$, $\frac{\partial u_n}{\partial x_j} \rightharpoonup \frac{\partial u}{\partial x_j}$ in $L^\infty(\Omega)$ weak $\star$ and $L^q(\Omega)$ strong for $1 \le q < \infty$.

Proof: Ω being bounded, $\partial\Omega$ is compact, and as it is covered by the open balls $B(z, r_z)$ for $z \in \partial\Omega$, it is covered by a finite number of them, with centers $z_1, \ldots, z_m$. There exists $\varepsilon > 0$ such that $\bigcup_{i=1}^m B(z_i, r_{z_i})$ contains all the points at distance $\le \varepsilon$ from $\partial\Omega$, so that $\overline{\Omega}$ is covered by the open sets $B(z_1, r_{z_1}), \ldots, B(z_m, r_{z_m})$ and $U = \{x \in \Omega, dist(x, \partial\Omega) > \varepsilon\}$. Let $\theta_1, \ldots, \theta_m, \zeta$ be a partition of unity associated to this covering, so that $\theta_i \in C_c^\infty\big(B(z_i, r_{z_i})\big), i = 1, \ldots, m$, $\zeta \in C_c^\infty(U)$ and $\sum_{i=1}^m \theta_i + \zeta = 1$ on $\overline{\Omega}$, so that every $u \in W^{1,p}(\Omega)$ can be decomposed as $\sum_{i=1}^m \theta_i u + \zeta\,u$. As $\zeta\,u$ has support in U, it can be approached by functions in $C_c^\infty(\Omega)$ by smoothing by convolution. For each i, in order to approximate $v_i = \theta_i u$ by functions in $\mathcal{D}(\overline{\Omega})$, one chooses the set of orthogonal directions which gives a continuous equation for the boundary, and one uses the fact that for a rigid displacement f (i.e. $f(x) = A\,x + a$ for an orthogonal matrix A and a vector a), and $f(\omega) = \omega'$, then if $\varphi \in W^{1,p}(\omega')$ and ψ is defined by $\psi(x) = \varphi\big(f(x)\big)$ then one has $\psi \in W^{1,p}(\omega)$). One studies the case of a special domain $\Omega_F = \{x \in R^N : x_N > F(x')\}$ with F (uniformly) continuous (as one actually only uses functions which have their support in a fixed compact set, one only needs F continuous, and because it is uniformly continuous on compact sets, one may change F far away in order to make it uniformly continuous). If $v \in W^{1,p}(\Omega_F)$ and v has compact support, one wants to approach it by functions from $\mathcal{D}(\overline{\Omega_F})$, and for this one translates it down, i.e. for $h > 0$ one defines $v_h(x', x_N) = v(x', x_N + h)$, and then one truncates v_h and one regularizes the result by convolution.

For doing the truncation, one regularizes F by convolution, and because F is uniformly continuous one can obtain in this way a function $G \in C^\infty(R^{N-1})$ such that $F - \frac{h}{6} \le G \le F + \frac{h}{6}$ on R^{N-1}; one chooses $\eta \in C^\infty(R)$ such that $\eta(t) = 1$ for $t \ge \frac{-1}{3}$ and $\eta(t) = 0$ for $t \le \frac{-2}{3}$, and one truncates v_h by defining $w_h(x) = v_h(x)\eta\big(\frac{x_N - G(x')}{h}\big)$, so that $w_h(x) = v_h(x)$ if $x_N \ge F(x') - \frac{h}{6}$ because $x_N \ge G(x') - \frac{h}{3}$ and $w_h(x) = 0$ if $x_N \le F(x') - \frac{5h}{6}$ because $x_N \le G(x') - \frac{2h}{3}$. Because of the truncation one has $\frac{\partial w_h}{\partial x_j}(x', x_N) =$

$\frac{\partial v}{\partial x_j}(x', x_N+h)\eta\big(\frac{x_N-G(x')}{h}\big) - v(x', x_N+h)\eta'\big(\frac{x_N-G(x')}{h}\big)\frac{\frac{\partial G}{\partial x_j}(x')}{h}$ for $j < N$ and $\frac{\partial w_h}{\partial x_N}(x', x_N) = \frac{\partial v}{\partial x_N}(x', x_N + h)\eta\big(\frac{x_N-G(x')}{h}\big) + v(x', x_N+h)\eta'\big(\frac{x_N-G(x')}{h}\big)\frac{1}{h}$, because the partial derivatives of v_h might have another part which is supported on $x_N = F(x') - h$ but this part is killed by the term $\eta\big(\frac{x_N-G(x')}{h}\big)$. Therefore if one calls W_h the restriction of w_h to Ω_F, one finds that $\frac{\partial W_h}{\partial x_j}(x', x_N) = \frac{\partial v}{\partial x_j}(x', x_N + h)$ in Ω_F for $j \le N$, and therefore W_h converges to v in $W^{1,p}(\Omega_F)$ strong if $p < \infty$. In order to approach W_h by fonctions in $\mathcal{D}(\overline{\Omega_F})$, one approaches w_h by convolutions by smooth functions and one restricts them to Ω_F.■

There is another way to do the preceding two steps of truncation and regularization in one single step, and the idea is to do a convolution of v with a sequence of nonnegative smoothing functions $\rho_n \in C_c^\infty(R^N)$ with integral 1 whose support shrinks to $\{0\}$. However, unless F is LIPSCHITZ continuous, one must not use a special smoothing sequence. and one lets the support of ρ_n shrink to $\{0\}$ in a special way. If $K = support(\rho)$, the convolution $(v \star \rho)(x) = \int_{R^N} v(x-y)\rho(y)\,dy$ is only defined if $x - K \subset \Omega_F$, and one wants this set to contain Ω_F, or better to contain $\Omega_{F-\eta}$ for some $\eta > 0$; for doing this, one assumes that $|y'| \le \varepsilon$ for $y \in K$, and therefore $|F(x') - F(y')| \le \omega(\varepsilon)$ where ω is the modulus of uniform continuity of F, and one asks that $y \in K$ implies $y_N \le -\eta - \omega(\varepsilon)$.

This method also applies in some different situations, like for $\Omega = \{(x,y), x > 0, 0 < y < x^2\}$, which is not an open set with continuous boundary with our definition, but $\mathcal{D}(\overline{\Omega})$ is dense, and in order to show that the cusp at 0 is not a problem one notices that if one translates Ω by a vector $(-a,-b)$ with $a > 0$ and $b > a^2$, then one obtains an open set Ω' which contains $\overline{\Omega}$, giving room for translating (by small amounts) and regularizing by convolution.

Proposition: Let Ω be a bounded open set with LIPSCHITZ boundary. Then there exists a linear continuous extension P from $W^{1,p}(\Omega)$ into $W^{1,p}(R^N)$ for $1 \le p \le \infty$ (an extension is characterized by the property that $P\,u\big|_\Omega = u$ for every $u \in W^{1,p}(\Omega)$).
Proof: One constructs the extension for the dense subspace $\mathcal{D}(\overline{\Omega})$, and using a partition of unity, it is enough to construct the extension for Ω_F, where F is LIPSCHITZ continuous. One defines $P\,u(x', x_N) = u(x', x_N)$ if $x_N > F(x')$ and $P\,u(x', x_N) = u(x', 2F(x') - x_N)$ if $x_N < F(x')$. In that way $P\,u$ is continuous at the interface $\partial\Omega_F$ and one has $\frac{\partial P\,u}{\partial x_j}(x', x_N) = \frac{\partial u}{\partial x_j}(x', x_N)$ if $x_N > F(x')$ and $j \le N$, and $\frac{\partial P\,u}{\partial x_j}(x', x_N) = \frac{\partial u}{\partial x_j}(x', 2F(x') - x_N) + 2\frac{\partial F}{\partial x_j}(x'\frac{\partial u}{\partial x_N}(x', 2F(x') - x_N)$ if $x_N < F(x')$ and $j < N$, and $\frac{\partial P\,u}{\partial x_N}(x', x_N) = -\frac{\partial u}{\partial x_N}(x', 2F(x') - x_N)$ if $x_N < F(x')$, and one verifies on these formulas that P is indeed linear continuous.■

The extension constructed is the same whatever p is, but this method does not apply for showing that there exists a continuous extension from $W^{m,p}(\Omega)$ into $W^{m,p}(R^N)$ for $m \ge 2$ because higher order derivatives of F might not exist. STEIN[1] has constructed a different extension which maps $W^{m,p}(\Omega)$ into $W^{m,p}(R^N)$ for all $m \ge 0$ and all $p \in [1,\infty]$, but we shall only consider a simpler one which can be used for open sets with smooth boundary, and the idea is shown for $\Omega = R^N_+ = \{x \in R^N, x_N > 0\}$.

Lemma: There is a continuous extension from $W^{m,p}(R^N_+)$ into $W^{m,p}(R^N)$, defined by $P\,u(x', x_N) = u(x', x_N)$ if $x_N > 0$ and $P\,u(x', x_N) = \sum_{j=1}^m \alpha_j u(x', -j\,x_N)$ if $x_N < 0$, with suitable coefficients $\alpha_j, j = 1, \ldots, m$.
Proof: Using the techniques already presented, one shows that $\mathcal{D}(\overline{R^N_+})$ is dense into $W^{m,p}(R^N_+)$. In order to check that the definition defines a continuous operator, one must show that derivatives up to order $m-1$ are continuous on $x_N = 0$. As for smooth functions taking tangential derivatives (i.e. not involving $\frac{\partial}{\partial x_N}$) commutes with restricting to $x_N = 0$, it is enough to check that $\frac{\partial^k P\,u}{\partial x_N^k}$ is continuous for $k = 0, \ldots, m-1$. One finds the condition to be $\sum_{j=1}^m \alpha_j(-j)^k = 1$ for $k = 0, \ldots, m-1$, and as this linear systen has a VANDERMONDE[2] matrix, it is invertible and the coefficients $\alpha_j, j = 1, \ldots, m$ are defined in a unique way.■

The extension property does not necessarily hold for open sets which only have a HÖLDER continuous boundary of order $\theta < 1$. This can be checked in the plane for the open set $\Omega = \{(x,y), 0 < x, -x^{1/\theta} <$

[1] Elias M. STEIN, American Mathematician, born in 1931. He received the WOLF prize in 1999. He works at Princeton University, NJ.

[2] Alexandre Théophile VANDERMONDE, French mathematician, 1735-1796.

$y < x^{1/\theta}\}$, by showing that $H^1(\Omega)$ is not (continuously) imbedded in all $L^p(\Omega)$ for $2 \le p < \infty$, which would be the case if a continuous extension existed, because SOBOLEV imbedding theorem asserts that $H^1(R^2)$ is (continuously) imbedded in all $L^p(R^2)$ for $2 \le p < \infty$.

For $\varphi \in C_c^\infty(R^2)$ with $\varphi(0) = 1$ one defines ψ by $\psi(x) = x^\alpha \varphi(x)$, and one checks for what values of α the function ψ does belong to $L^p(\Omega)$ or to $H^1(\Omega)$. The function ψ belongs to $L^p(\Omega)$ if and only if $\int_0^1 x^{p\,\alpha+1/\theta}\,dx < \infty$, i.e. $p\,\alpha + \frac{1}{\theta} > -1$, and ψ belongs to $H^1(\Omega)$ if and only $2(\alpha - 1) + \frac{1}{\theta} > -1$; the (excluded) critical value $\alpha_C = \frac{1}{2} - \frac{1}{2\theta} = \frac{\theta-1}{2\theta}$ (which is < 0) corresponds to the (excluded) critical value $p_C = -\frac{1}{\alpha_C} - \frac{1}{\theta\,\alpha_C} = \frac{2(1+\theta)}{1-\theta}$, and therefore $H^1(\Omega)$ is not imbedded in $L^p(\Omega)$ for $p > p_C$.

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13. Monday February 14, 2000

For an open set with a continuous boundary, there is a notion of restriction to the boundary (called a trace) for functions of $W^{1,p}(\Omega)$, which is easily derived for the case of Ω_F, with F continuous.

Lemma: For $u \in \mathcal{D}(\overline{\Omega_F})$ and $1 \le p < \infty$, one has $\big|\big|u\big(x', F(x')\big)\big|\big|_{L^p(R^{N-1})} \le p^{1/p}\,\big|\big|\frac{\partial u}{\partial x_N}\big|\big|^{1/p}_{L^p(\Omega_F)}||u||^{(p-1)/p}_{L^p(\Omega_F)}$.
Proof: If $v \in C^\infty_c(R)$ one has $|v(0)|^p = -\int_0^\infty \frac{d(|v|^p)}{dx}\,dx \le p\int_0^\infty |v|^{p-1}|v'|\,dx \le p\,||v||^{(p-1)/p}_{L^p(R))}||v'||^{1/p}_{L^p(R))}$. One applies this inequality to $v(t) = u\big(x', t + F(x')\big)$ and then one integrates in x', using Hölder inequality.■

For $p = \infty$, the functions of $W^{1,\infty}(\Omega_F)$ are locally uniformly continuous (each function is an equivalence class and one element of the equivalence class is continuous), and the trace is just the restriction to the boundary.

Notation: The linear continuous operator of trace on the boundary, defined by extension by (uniform) continuity of the operator of restriction defined for $\mathcal{D}(\overline{\Omega})$ will be denoted γ_0.■

Notice that γ_0 is not defined as the restriction to the boundary, because the boundary has measure 0, and the restriction to a set of measure 0 is not defined for functions which are not smooth enough.

Lemma: If $1 \le p, q, r \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then for $u \in W^{1,p}(\Omega_F)$ and $v \in W^{1,q}(\Omega_F)$ one has $u\,v \in W^{1,r}(\Omega_F)$ and $\gamma_0(u\,v) = \gamma_0 u\,\gamma_0 v$.
Proof: The formula is true if $u, v \in \mathcal{D}(\overline{\Omega_F})$ and as both sides of the equality use continuous mappings on $W^{1,p}(\Omega_F) \times W^{1,q}(\Omega_F)$, the formula is true by density.■

Using a partition of unity one can then define a notion of trace on the boundary for $u \in W^{1,p}(\Omega)$ if Ω has a continuous boundary, but one should be careful that the definition does depend upon the choice of the partition of unity and the choice of local orthonormal bases. One should be aware that the usual area measure on the boundary, i.e. the $(N-1)$-dimensional Hausdorff measure, is not defined for F continuous; for F Lipschitz continuous it is $\sqrt{1 + |\nabla F(x')|^2}\,dx'$, and it has the important property to be invariant by rigid displacements (rotations and translations). Using the invariance by rotation of the $(N-1)$-dimensional Hausdorff measure, one can show that for a bounded open set with a Lipschitz boundary the trace does not depend upon the choice of the partition of unity or the choice of local orthonormal bases.

Some notion of trace can be defined for other open sets, for example some which are not even locally on one side of their boundary. For example, let Ω be the open set of R^2 defined in polar coordinates by $r < 1$ and $0 < \theta < 2\pi$, i.e. the open unit disc slit on the nonnegative x axis. One can apply the Lemma to the open subsets Ω_+ defined by $r < 1$ and $0 < \theta < \pi$ and Ω_- defined by $r < 1$ and $\pi < \theta < 2\pi$, and therefore one can define two traces on the piece of the boundary corresponding to $y = 0$, $0 < x < 1$, one from the side of Ω_+ and one from the side of Ω_-; these two traces are not necessarily the same for $u \in W^{1,p}(\Omega)$, although they are the same for functions in $\mathcal{D}(\overline{\Omega})$ (and $\mathcal{D}(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$ in this case); there is actually some kind of a compatibility condition at 0 between the traces on the two sides (for $p > 1$).

An important result is to identify $W^{1,p}_0(\Omega)$, which is by definition the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(\Omega)$, as the kernel of γ_0, and this makes use of Hardy's inequality.

Lemma: For $p > 1$ and $f \in L^p(0,\infty)$, one defines g by $g(t) = \frac{1}{t}\int_0^t f(s)\,ds$. One has $||g||_{L^p(0,\infty)} \le \frac{p}{p-1}||f||_{L^p(0,\infty)}$.
Proof: By density, it is enough to prove the result for $f \in C^\infty_c(0,\infty)$, for which g is 0 near 0 and decays in $\frac{C}{t}$ for large t (so that g does not belong to $L^1(0,\infty)$ when $f \in L^1(0,\infty)$ and $\int_0^\infty f(t)\,dt \ne 0$). One has $t\,g'(t) + g(t) = f(t)$, and one multiplies by $p\,|g|^{p-1}sign(g)$ and one integrates on $(0,\infty)$, observing that $\int_0^\infty t(|g|^p)'\,dt = -\int_0^\infty |g|^p\,dt$ because $t\,|g(t)|^p \to 0$ as $t \to \infty$ (because $p > 1$) and therefore one has $(p-1)\int_0^\infty |g|^p\,dt = -p\int_0^\infty f\,|g|^{p-1}sign(g)\,dt$, and one concludes by using Hölder inequality.■

Another proof of Hardy's inequality uses Young's inequality for convolution, noticing that $(0,\infty)$ is a multiplicative group with Haar measure $\frac{dt}{t}$, then one has $g(t) = \int_0^t f(s)\frac{s}{t}\,\frac{ds}{s}$, i.e. g is the convolution product of f with the function h defined by $h(t)=0$ for $0<t<1$ and $h(t)=\frac{1}{t}$ for $1<t<\infty$ (so that $h\in L^1\big(0,\infty;\frac{dt}{t}\big)$).

Lemma: For F continuous, and $p>1$, one has $\big|\big|\frac{u(x',x_N)-\gamma_0 u(x')}{x_N-F(x')}\big|\big|_{L_p(\Omega_F)}\le\frac{p}{p-1}\big|\big|\frac{\partial u}{\partial x_N}\big|\big|_{L_p(\Omega_F)}$ for all $u\in W^{1,p}(\Omega_F)$.
Proof: One applies HARDY's inequality to $f(t)=\frac{\partial u}{\partial x_N}(x',F(x')+t)$, one takes the power p and then one integrates in x'.■

Proposition: If F is LIPSCHITZ continuous and $p>1$, then $W_0^{1,p}(\Omega_F)$ is the subspace of $u\in W^{1,p}(\Omega_F)$ satisfying $\gamma_0 u=0$.
Proof: If $u\in W_0^{1,p}(\Omega_F)$ then there exists a sequence $\varphi_n\in C_c^\infty(\Omega_F)$ such that $\varphi_n\to u$ in $W^{1,p}(\Omega_F)$; as γ_0 is continuous from $W^{1,p}(\Omega_F)$ to $L^p(R^{N-1})$, $\gamma_0 u$ is the limit of $\gamma_0\varphi_n$, and is 0 because each φ_n is 0 near $\partial\Omega_F$ and γ_0 is the restriction to the boundary for functions in $\mathcal{D}(\overline{\Omega_F})$.

Conversely, for $u\in W^{1,p}(\Omega_F)$ satisfying $\gamma_0 u=0$, one must approach u by a sequence from $C_c^\infty(\Omega_F)$. First one truncates at ∞, i.e. one chooses $\theta\in C_c^\infty(R^N)$ such that $\theta(x)=1$ for $|x|\le 1$ and one approaches u by u_n defined by $u_n(x)=u(x)\theta\big(\frac{x}{n}\big)$, and one has $\gamma_0 u_n=0$ and u_n converges to u (and the proof uses LEBESGUE dominated convergence theorem). One may then assume that the support of u is bounded.

Then one wants to truncate near the boundary, and for this one uses the preceding Lemma, i.e $\frac{u}{x_N-F(x')}\in L^p(\Omega_F)$. Let $\eta\in C^\infty(R)$ with $\eta(t)=0$ for $t\le 1$ and $\eta(t)=1$ for $t\ge 2$. One approaches u by u_n defined by $u_n(x)=u(x)\eta\big(n\big(x_N-F(x')\big)\big)$. The sequence u_n converges to u in $L^p(\Omega_F)$ strong by LEBESGUE dominated convergence theorem (if $p=\infty$ the convergence is in $L^\infty(\Omega_F)$ weak $\star$ and L^q_{loc} strong for all $q<\infty$, of course). Similarly $\frac{\partial u_n}{\partial x_j}$ has a term $\frac{\partial u}{\partial x_j}\eta\big(n\big(x_N-F(x')\big)\big)$ which converges to $\frac{\partial u}{\partial x_j}$, but also another term $n\,u\,\eta'\big(n\big(x_N-F(x')\big)\big)w_j$ with $w_j\in L^\infty(\Omega_F)$, as it is $-\frac{\partial F}{\partial x_j}$ if $j<N$ and 1 if $j=N$. This last term tends to 0 by LEBESGUE dominated convergence theorem because one may write it $\frac{u}{x_N-F(x')}\zeta_n\big(x_N-F(x')\big)w_j$ with $\zeta_n(t)=n\,t\,\eta'(n\,t)$, and ζ_n is bounded by $\sup_{t\in(1,2)}t|\eta'(t)|$ and $\zeta_n(t)$ tends to 0 for every $t>0$. One may then assume that u has its support bounded and bounded away from the boundary.

The last part is to regularize by convolution.■

Corollary: If Ω is bounded with LIPSCHITZ boundary and $p>1$, then $W_0^{1,p}(\Omega)$ is the subspace of $u\in W^{1,p}(\Omega)$ satisfying $\gamma_0 u=0$.
Proof: One uses a partition of unity and a local change of orthonormal basis and one applies the preceding result.■

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14. Wednesday February 16, 2000

An important result is GREEN's formula, or simply integration by parts, and for this one needs the boundary to be smooth enough so that a normal is defined almost everywhere (for the $(N-1)$-dimensional HAUSDORFF measure).

Notation: If Ω is an open set with LIPSCHITZ boundary, n denotes the unit exterior normal. For the case of $\Omega_F = \{(x', x_N), x_N > F(x')\}$ for a LIPSCHITZ function F, one has $n_j\big(x', F(x')\big) = \frac{1}{\sqrt{1+|\nabla F(x')|^2}} \frac{\partial F}{\partial x_j}(x')$ for $j < N$ and $n_N\big(x', F(x')\big) = -\frac{1}{\sqrt{1+|\nabla F(x')|^2}}$.■

Lemma: If F is LIPSCHITZ continuous, and $u \in W^{1,p}(\Omega_F)$, $v \in W^{1,p'}(\Omega_F)$, then one has $\int_{\Omega_F}\big(u \frac{\partial v}{\partial x_N} + \frac{\partial u}{\partial x_N} v\big)\,dx = \int_{\partial\Omega_F} \gamma_0 u\, \gamma_0 v\, n_N\, dH^{N-1}$, i.e. $-\int_{R^{N-1}} \gamma_0 u\, \gamma_0 v\, dx'$.
Proof: For $u, v \in \mathcal{D}(\overline{\Omega_F})$ and each $x' \in R^{N-1}$, one has $\int_{F(x')}^\infty\big(u \frac{\partial v}{\partial x_N} + \frac{\partial u}{\partial x_N} v\big)\,dx_N = \int_{F(x')}^\infty \frac{\partial (u\,v)}{\partial x_N}\,dx_N = -u\big(x', F(x')\big)v\big(x', F(x')\big)$. Integrating this equality in x' shows that the formula is true for $u, v \in \mathcal{D}(\overline{\Omega_F})$, and as both sides of the equality are continuous functionals if one uses the topologies of $W^{1,p}(\Omega_F)$ and $W^{1,p'}(\Omega_F)$, the lemma is proved (in the case where $p = 1$ or $p = \infty$, one first approaches the function in $W^{1,\infty}(\Omega_F)$ and its partial derivatives in $L^\infty(\Omega_F)$ weak $\star$, and then the other function in $W^{1,1}(\Omega_F)$ strong).■

Lemma: If F is LIPSCHITZ continuous, and $u \in W^{1,p}(\Omega_F)$, $v \in W^{1,p'}(\Omega_F)$, then for every $j = 1, \ldots, N$ one has $\int_{\Omega_F}\big(u \frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial x_j} v\big)\,dx = \int_{\partial\Omega_F} \gamma_0 u\, \gamma_0 v\, n_j\, dH^{N-1}$.
Proof: The case $j = N$ has been proved. If $j < N$ and if one makes only x_j vary, the intersection with Ω_F can be an arbitrary open subset of R, i.e. a countable union of open intervals, so in order to avoid this technical difficulty one uses a new orthogonal basis, with last vector $e'_N = \frac{1}{\sqrt{1+\varepsilon^2}}(e_N + \varepsilon\, e_j)$ (where $e_1, \ldots, e_N$ is the initial basis). Of course $\varepsilon > 0$ is taken small enough, so that using y to denote coordinates in the new basis, the open set can be written as $y_N > G(y')$ with G LIPSCHITZ continuous. Therefore the preceding Lemma shows that $\int_{\Omega_G}\big(u \frac{\partial v}{\partial y_N} + \frac{\partial u}{\partial y_N} v\big)\,dy = \int_{\partial\Omega_G} \gamma_0 u\, \gamma_0 v\,(n.e'_N)\,dH^{N-1}$. One observes that $\frac{\partial u}{\partial y_N} = \frac{1}{\sqrt{1+\varepsilon^2}}\frac{\partial u}{\partial x_N} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\frac{\partial u}{\partial x_j}$ and similarly $(n.e'_N) = \frac{1}{\sqrt{1+\varepsilon^2}} n_N + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} n_j$, and therefore after multiplication by $\sqrt{1+\varepsilon^2}$, one obtains a relation of order 1 in ε; the equality for $\varepsilon = 0$ is true at it is the preceding Lemma, and therefore the equality of the coefficients of ε gives the desired relation.■

Corollary: For any bounded open set Ω with LIPSCHITZ boundary one has $\int_\Omega\big(u \frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial x_j} v\big)\,dx = \int_{\partial\Omega} \gamma_0 u\, \gamma_0 v\, n_j\, dH^{N-1}$ for $j = 1, \ldots, N$, for all $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$.
Proof: One uses a partition of unity and the fact that one has a formulation invariant by rigid displacements, for example by writing the formula $\int_\Omega\big(u(\nabla v.e) + (\nabla u.e)\,v\big)\,dx = \int_{\partial\Omega} \gamma_0 u\, \gamma_0 v\,(n.e)\,dH^{N-1}$ for all vectors e (of course, the fact that the LEBESGUE measure dx is also invariant by rigid displacements is also used).■

It should be noticed that even for a C^∞ function F, the set of x' such that $F(x') = \lambda$ can be a general closed set; actually, for any closed set $K \subset R^{N-1}$, there exists $G \in C^\infty$, with $G \ge 0$ and $\{x', G(x') = 0\} = K$. For constructing G, one notices that the complement of K is a countable union of open balls $R^{N-1} \setminus K = \bigcup_n B(M_n, r_n)$ (for example for each point $M \notin K$ and M with rational coordinates, one keeps the largest open ball centered at M which does not intersect K). One chooses a function $\varphi \in C_c^\infty(R^{N-1})$ such that $\{x, \varphi(x) \ne 0\} = B(0,1)$, and one may assume $\varphi \ge 0$. One defines G by $G(x) = \sum_n c_n \varphi\big(\frac{x - M_n}{r_n}\big)$ with all $c_n > 0$, and G satisfies the desired property if one chooses the c_n converging to 0 fast enough so that the series converges uniformly, as well as any of its derivatives (so that $D^\alpha G \in C_0(R^{N-1})$ for any multi-index α).

However, there is a result of SARD[1] which says that for most λ the set is not that bad; for example, if $F \in C^1(R)$, then except for a set of λ with measure 0, at any point x with $F(x) = \lambda$ one has $F'(x) \ne 0$, so these points are isolated.

[1] Arthur SARD, American mathematician, 1909-1980. He worked at Queens College, New York

The next question is to identify the range of γ_0, for a bounded open set Ω with LIPSCHITZ boundary for example. This was done by Emilio GAGLIRDO, but although γ_0 is surjective from $W^{1,1}(\Omega)$ onto $L^1(\partial\Omega)$, it is not so for $p > 1$, and the image of $W^{1,p}(\Omega)$ by γ_0 is not $L^p(\partial\Omega)$ for $p > 1$. Actually for $p = \infty$ it is $W^{1,\infty}(\partial\Omega)$, the space of LIPSCHITZ continuous functions on the boundary, and as p varies from 1 to ∞ one goes from traces having no derivatives in L^1 to traces having one derivative in L^∞, and one can guess that for $1 < p < \infty$ the traces have $1 - \frac{1}{p}$ derivatives in L^p, if one finds a way to express what this means. The characterization of the traces is simpler in the case $p = 2$ because one can use FOURIER transform, which will be studied for that reason.

Before doing that, one has now a simple way to prove the compactness of the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ in the case of bounded open sets with LIPSCHITZ boundary, the case of continuous boundary being left for later.

Proposition: If Ω is bounded with LIPSCHITZ boundary, then the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact.

Proof: The basic idea is that there exists a continuous extension P from $W^{1,p}(\Omega)$ to $W^{1,p}(R^N)$, and this is seen by localization, using a partition of unity. Writing $u = \sum_i \theta_i u$, each $\theta_i u$ belongs to a space $W^{1,p}(\Omega_{F_i})$ in an orthonormal basis depending upon i, for some LIPSCHITZ continuous function F_i. There is a continuous extension P_i from $W^{1,p}(\Omega_{F_i})$ to $W^{1,p}(R^N)$, and therefore there is a continuous extension P from $W^{1,p}(\Omega)$ to $W^{1,p}(R^N)$, given by $P\,u = \sum_i P_i(\theta_i u)$.

Let $\eta \in C_c^\infty(R^N)$ such that $\eta(x) = 1$ for $x \in \overline{\Omega}$, and let Ω' be a bounded open set containing $support(\eta)$. Then if a sequence u_n is bounded in $W^{1,p}(\Omega)$, the sequence of extensions $P\,u_n$ is bounded in $W^{1,p}(R^N)$, and the sequence of truncated functions $\eta(P\,u_n)$ is bounded in $W_0^{1,p}(\Omega')$, and therefore a subsequence $\eta(P\,u_m)$ belongs to a compact of $L^p(\Omega')$, and the sequence of restrictions to Ω, which is u_m, belongs to a compact of $L^p(\Omega)$.■

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15. Friday February 18, 2000

A decomposition in FOURIER series of a scalar function f in one variable consists in writing $f(x) = \sum_{n\in Z} c_n e^{2i\,\pi\, x/T}$ for some (complex) coefficients $c_n, n \in Z$. If the series converges the function f must be periodic with period T.

A decomposition in FOURIER integral of a scalar function f in one variable consists in writing $f(x) = \int_R \hat{f}(\xi) e^{2i\,\pi\, x\,\xi}\,d\xi$ for some function $\hat{f}$.

The extension to functions of N variables leads to the following definition.

Definition: For $f \in L^1(R^N)$, the FOURIER transform of f is the function $\mathcal{F}f$ or $\hat{f}$ defined on (the dual of) R^N by $\mathcal{F}f(\xi) = \int_{R^N} f(x) e^{-2i\,\pi(x.\xi)}\,dx$. One also defines $\overline{\mathcal{F}}$ by $\overline{\mathcal{F}}f(\eta) = \int_{R^N} f(y) e^{+2i\,\pi(y.\eta)}\,dy$.■

Having learned the theory from Laurent SCHWARTZ, I use his notations, but most mathematicians do not put the coefficient 2π in the integral, and some different powers of π occur then in their formulas, in the argument of the exponentials and multiplying the integrals. One should be aware of the fact that different books may use different constants.

The FOURIER transform $\mathcal{F}$ maps $L^1(R^N)$ into $C_0(R^N)$, but one important property is that it extends as an isometry from $L^2(R^N)$ to itself, with inverse $\overline{\mathcal{F}}$.

Another important property is that it transforms derivation into multiplication, or more generally convolution into product, and one checks easily the following properties.

If $f, g \in L^1(R^N)$, so that $f \star g \in L^1(R^N)$, then one has $\mathcal{F}(f \star g) = \mathcal{F}f\,\mathcal{F}g$.

If $f \in L^1(R^N)$ and $x_j f \in L^1(R^N)$, then one has $\frac{\partial(\mathcal{F}f)}{\partial\xi_j} = \mathcal{F}(-2i\pi\, x_j f)$.

If $f \in L^1(R^N)$ and $\frac{\partial f}{\partial x_j} \in L^1(R^N)$, then $\mathcal{F}\frac{\partial f}{\partial x_j} = 2i\,\pi\,\xi_j \mathcal{F}f$.

In order to define the FOURIER transform for some distributions (the initial definition extends immediately to RADON measures with finite total mass by $\mu(\xi) = \langle \mu, e^{-2i\,\pi(\cdot.\xi)}\rangle$ and gives $\mu \in C_b(R^N)$), Laurent SCHWARTZ introduced the space of rapidly decaying smooth functions $\mathcal{S}(R^N) = \{u \in C^\infty(R^N), P\,D^\alpha u \in L^\infty(R^N)$ for all polynomials P, and all multi-indices $\alpha\}$ (which is a FRÉCHET space). By iterating the preceding remarks, one checks easily that $\mathcal{F}$ maps $\mathcal{S}(R^N)$ into itself; one also finds that its inverse is $\overline{\mathcal{F}}$, and that one has PLANCHEREL[1] formula $\int_{R^N} (\mathcal{F}f) g\,dx = \int_{R^N} f(\mathcal{F}g)\,dx$ for all $f, g \in \mathcal{S}(R^N)$. This is the relation which Laurent SCHWARTZ used for defining the FOURIER transform on the dual $\mathcal{S}'(R^N)$ of $\mathcal{S}(R^N)$, called the space of temperate distributions (one cannot define the FOURIER transform of an arbitrary distribution on R^N, or even of an arbitrary smooth function whatever its growth at ∞ is, and keep all the known properties). For $T \in \mathcal{S}'(R^N)$, one defines $\mathcal{F}T \in \mathcal{S}'(R^N)$ by $\langle \mathcal{F}T, \varphi\rangle = \langle T, \mathcal{F}\varphi\rangle$ for all $\varphi \in \mathcal{S}(R^N), T \in \mathcal{S}'(R^N)$, and one finds easily that one has $\mathcal{F}\frac{\partial T}{\partial x_j} = 2i\,\pi\,\xi_j \mathcal{F}T$ and $\mathcal{F}(-2i\,\pi\,x_j T) = \frac{\partial T}{\partial \xi_j}$ for all $T \in \mathcal{S}'(R^N)$ and $j = 1, \ldots, N$, and that $\mathcal{F}$ is an isomorphism from $\mathcal{S}'(R^N)$ onto itself, with inverse $\overline{\mathcal{F}}$.

For example, computing the FOURIER transform of $u(x) = e^{-\pi\,|x|^2}$ (which belongs to $\mathcal{S}(R^N)$), one notices that $\frac{\partial u}{\partial x_j} = -2\pi\,x_j\,u$, so that one has $\frac{\partial \mathcal{F}u}{\partial \xi_j} = -2\pi\,\xi_j\,\mathcal{F}u$ for $j = 1, \ldots, N$, and therefore $\mathcal{F}u(\xi) = C\,e^{-\pi\,|\xi|^2}$; one finds $C = 1$ by using the formula for $\xi = 0$ and using $\int_R e^{-\pi\,x^2}\,dx = 1$ (the classical method for computing $\mathcal{F}u$ consists in moving a path of integration in the complex plane).

An other example is to show that $\mathcal{F}1 = \delta_0$, by noticing that $1 \in L^\infty(R^N) \subset \mathcal{S}'(R^N)$, so that $\mathcal{F}1$ exists, and from $\frac{\partial 1}{\partial x_j} = 0$ one deduces that $\xi_j \mathcal{F}1 = 0$ for $j = 1, \ldots, N$, and therefore $\mathcal{F}1 = C\,\delta_0$; one finds $C = 1$ by using $u(x) = e^{-\pi\,|x|^2}$ in the definition, so that $C = \langle \mathcal{F}1, u\rangle = \langle 1, \mathcal{F}u\rangle = 1$.

From a scaling point of view, one should remember that using L to denotes a length unit for measuring x and U to denote a unit for measuring u, then ξ scales as L^{-1} and $\mathcal{F}u$ scales as $U\,L^N$.

[1] Michel PLANCHEREL, Swiss mathematician, 1885-1967. He worked at ETH Zürich (Eidgenössische Technische Hochschule).

Because $\mathcal{F}$ is an isometry from $L^2(R^N)$ onto itself, one can identify the image by $\mathcal{F}$ of the SOBOLEV space $H^1(R^N) = W^{1,2}(R^N)$. As $\mathcal{F}\frac{\partial u}{\partial x_j} = 2i\,\pi\,\xi_j \mathcal{F}u$, one finds that $\int_{R^N} |grad(u)|^2\,dx = \int_{R^N} 4\pi^2|\xi|^2|\mathcal{F}u|^2\,d\xi$.

Definition: For a real $s \geq 0$, $H^s(R^N) = \{u \in L^2(R^N), |\xi|^s \mathcal{F}u \in L^2(R^N)\}$.
For a real $s < 0$, $H^s(R^N) = \{u \in \mathcal{S}'(R^N), (1+|\xi|^2)^{s/2}\mathcal{F}u \in L^2(R^N)\}$.■

If s is a nonnegative integer m, then this definition of $H^s(R^N)$ does give the same space as $W^{m,2}(R^N)$.

For $s < 0$, the space $H^s(R^N)$ is not a subset of $L^2(R^N)$, and in order to use $\mathcal{F}$ one starts from an element of $\mathcal{S}'(R^N)$, but one cannot define $(1+|\xi|)^s\mathcal{F}u$ because $(1+|\xi|)^s$ is not a C^∞ function.

Of course, $H^{-s}(R^N)$ is the dual of $H^s(R^N)$, but the notation for an open set Ω is different.

Definition: For an open set $\Omega \subset R^N$, and a positive integer m, one denotes $H^{-m}(\Omega)$ the dual of $H_0^m(\Omega)$ (which is the closure of $C_c^\infty(\Omega)$ in $H^m(\Omega)$).■

Proposition: $H^{-1}(\Omega) = \{T \in \mathcal{D}'(\Omega), T = f_0 - \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}, f_0, \ldots, f_N \in L^2(\Omega)\}$. If POINCARÉ inequality holds for $H_0^1(\Omega)$, then every $g_0 \in L^2(\Omega)$ can be written as $\sum_{j=1}^N \frac{\partial g_j}{\partial x_j}$, with $g_1, \ldots, g_N \in L^2(\Omega)$.
Proof: The mapping $u \mapsto \big(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\big)$ is an isometry of $H_0^1(\Omega)$ onto a closed subspace of $L^2(\Omega)^{N+1}$, so a linear continuous form L on $H_0^1(\Omega)$ is transported onto this subspace and extended to a linear continuous form on $L^2(\Omega)^{N+1}$, and therefore there exist $f_0, \ldots, f_N \in L^2(\Omega)$ such that $L(\varphi) = \int_\Omega \big(f_0\varphi + \sum_{j=1}^N f_j \frac{\partial \varphi}{\partial x_j}\big)\,dx$ for all $\varphi \in H_0^1(\Omega)$, or equivalently for all $\varphi \in C_c^\infty(\Omega)$, but this means that $L(\varphi) = \langle T, \varphi\rangle$ with $T = f_0 - \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}$, and because $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, one has $L = T$.

If POINCARÉ inequality holds for $H_0^1(\Omega)$, then the mapping $u \mapsto \big(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\big)$ is an isometry of $H_0^1(\Omega)$ onto a closed subspace of $L^2(\Omega)^N$, and every linear continuous form L on $H_0^1(\Omega)$ has the form $L(\varphi) = \sum_{j=1}^N g_j \frac{\partial \varphi}{\partial x_j}\,dx$ for all $\varphi \in H_0^1(\Omega)$, with $g_1, \ldots, g_N \in L^2(\Omega)$, and in particular if $g_0 \in L^2(\Omega)$ one can write $\int_\Omega g_0\varphi\,dx$ in this way.■

Proposition: For any $s \in R$, $C_c^\infty(R^N)$ is dense in $H^s(R^N)$.
Proof: Considering the space $\mathcal{F}H^s$ of functions in L^2 with the weight $(1+|\xi|^2)^s$, i.e. $\int_{R^N}(1+|\xi|^2)^s|u(\xi)|^2\,d\xi < \infty$. One can approach any function in $\mathcal{F}H^s$ by functions with compact support by truncation, defining u_n by $u_n(\xi) = u(\xi)$ if $|\xi| \leq n$ and $u_n(\xi) = 0$ if $|\xi| > n$, and by LEBESGUE dominated convergence, u_n converges to u in $\mathcal{F}H^s$. Any $u \in \mathcal{F}H^s$ having compact support can be approached by functions in $C_c^\infty(R^N)$ because for a smoothing sequence ρ_m one has $\rho_m \star u \to u$ in $L^2(R^N)$, and because the supports stay in a bounded set one has $\rho_m \star u \to u$ in $\mathcal{F}H^s$. Therefore $C_c^\infty(R^N)$ is dense in $\mathcal{F}H^s$, and therefore $\mathcal{S}(R^N)$ is dense in $\mathcal{F}H^s$. Using FOURIER transform one deduces that $\mathcal{S}(R^N)$ is dense in $H^s(R^N)$. Let $m \geq s$ be a nonnegative integer then one can approach any function $v \in \mathcal{S}(R^N)$ by a sequence in $C_c^\infty(R^N)$, the convergence being in $H^m(R^N)$ strong, and therefore also in $H^s(R^N)$ strong, and this is done by approaching v by $v(x)\theta\big(\frac{x}{n}\big)$ with $\theta \in C_c^\infty(R^N)$ with $\theta(x) = 1$ for $|x| \leq 1$.■

In the characterization of the traces of functions from $H^s(R^N)$ (for $s > \frac{1}{2}$), one will use the following result.

Lemma: If $u \in \mathcal{S}(R^N)$ and $v \in \mathcal{S}(R^{N-1})$ is the restriction of u on the hyperplane $x_N = 0$, i.e. $v(x') = u(x', 0)$ for $x' \in R^{N-1}$, then one has $\mathcal{F}v(\xi') = \int_R \mathcal{F}u(\xi', \xi_N)\,d\xi_N$ for $\xi' \in R^{N-1}$.
Proof: Because $\mathcal{F}\delta_0 = 1$, one has $\varphi(0) = \int_R \mathcal{F}\varphi(\xi)\,d\xi$. One uses this relation for the function $x_N \mapsto u(x', x_N)$, and then one takes the FOURIER transform in x' of both sides of the equality.■

16. Monday February 21, 2000

Proposition: For $s > \frac{1}{2}$, functions of $H^s(R^N)$ have a trace on the hyperplane $x_N = 0$, belonging to $H^{s-\frac{1}{2}}(R^{N-1})$. The mapping γ_0 is surjective from $H^s(R^N)$ onto $H^{s-\frac{1}{2}}(R^{N-1})$.

Proof: For proving the first part, it is enough to show that there exists C such that for all $u \in \mathcal{S}(R^N)$ and v defined by $v(x') = u(x', 0)$ for $x' \in R^{N-1}$ one has $||v||_{H^{s-(1/2)}(R^{N-1})} \le C\,||u||_{H^s(R^N)}$. One has $\mathcal{F}v(\xi') = \int_R \mathcal{F}u(\xi', \xi_N)\,d\xi_N$, and one has $|\mathcal{F}v(\xi')|^2 \le \big(\int_R (1+|\xi|^2)^s |\mathcal{F}u(\xi',\xi_N)|^2\,d\xi_N\big)\big(\int_R (1+|\xi|^2)^{-s}\,d\xi_N\big)$ by CAUCHY-SCHWARTZ inequality. Using the change of variable $\xi_N = t\sqrt{1+|\xi'|^2}$, one has $\int_R (1+|\xi|^2)^{-s}\,d\xi_N = (\sqrt{1+|\xi'|^2})^{1-2s}\int_R \frac{dt}{(1+t^2)^s} = C(s)(1+|\xi'|^2)^{(1/2)-s}$, and therefore one has $(1+|\xi'|^2)^{s-(1/2)}|\mathcal{F}v(\xi')|^2 \le C(s)\int_R (1+|\xi|^2)^s|\mathcal{F}u(\xi',\xi_N)|^2\,d\xi_N$, which gives the desired result by integrating in ξ', because $C(s) < \infty$ if and only if $s > \frac{1}{2}$.

In order to prove the surjectivity, one must show that if $v \in H^{s-(1/2)}(R^{N-1})$ there exists $u \in H^s(R^N)$ such that $\mathcal{F}v(\xi') = \int_R \mathcal{F}u(\xi',\xi_N)\,d\xi_N$ for almost all $\xi' \in R^{N-1}$.

One defines $\mathcal{F}u(\xi',\xi_N) = \mathcal{F}v(\xi')\varphi\big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\big)\frac{1}{\sqrt{1+|\xi'|^2}}$, with $\varphi \in C_c^\infty(R)$ and $\int_R \varphi(t)\,dt = 1$, and one must check that $u \in H^s(R^N)$. One shows that there exists a constant C such that for all $\xi' \in R^{N-1}$ one has $\int_R (1+|\xi|^2)^s|\mathcal{F}u(\xi',\xi_N)|^2\,d\xi_N = C(1+|\xi'|^2)^{s-(1/2)}\,|\mathcal{F}v(\xi')|^2$, and this amounts to $\int_R (1+|\xi|^2)^s\big|\varphi\big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\big)\big|^2\frac{1}{1+|\xi'|^2}\,d\xi_N = C(1+|\xi'|^2)^{s-(1/2)}$ for all $\xi' \in R^{N-1}$, which is proved by using the change of variable $\xi_N = t\sqrt{1+|\xi'|^2}$ and one obtains $C = \int_R (1+t^2)^s\varphi(t)^2\,dt$.■

The condition $s > \frac{1}{2}$ has not appeared in proving the surjectivity, but one should notice that the function u constructed is not only in $H^s(R^N)$ but is such that $\mathcal{F}u$ has its support in a region $|\xi_N| \le C\sqrt{1+|\xi'|^2}$.

For a function $u \in H^2(R^N)$ for example, one can define the trace $\gamma_0 u \in H^{3/2}(R^{N-1})$ but also the normal derivative $\gamma_1 u = \gamma_0\frac{\partial u}{\partial x_N} \in H^{1/2}(R^{N-1})$ (in general one takes $\gamma_1 u$ to be the normal derivative, with the normal pointing to the outside), and a more precise result is that $u \mapsto (\gamma_0 u, \gamma_1 u)$ is surjective from $H^2(R^N)$ onto $H^{3/2}(R^{N-1}) \times H^{1/2}(R^{N-1})$, and more generally one has the following surjectivity result.

Proposition: If $m + \frac{1}{2} < s < m + 1 + \frac{1}{2}$, and for $k = 0, \ldots, m$ one denotes $\gamma_k u$ the trace on $x_N = 0$ of $\frac{\partial^k u}{\partial x_N^k}$, then $u \mapsto (\gamma_0 u, \ldots, \gamma_m u)$ is surjective from $H^s(R^N)$ onto $H^{s-(1/2)}(R^{N-1}) \times \ldots \times H^{s-m-(1/2)}(R^{N-1})$.

Proof: If $u \in \mathcal{S}(R^N)$ and $v_k = \gamma_k u$, then one has $\mathcal{F}v_k(\xi') = \int_R (2i\,\pi\,\xi_N)^k\mathcal{F}u(\xi',\xi_N)\,d\xi_N$.

If $v_k \in H^{s-k-(1/2)}(R^{N-1})$, then one lifts it by taking u defined by $\mathcal{F}u(\xi',\xi_N) = \mathcal{F}v_k(\xi')\varphi\big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\big)(1+|\xi'|^2)^{-(k+1)/2}$, and one imposes $\int_R (2i\,\pi\,t)^k\varphi(t)\,dt = 1$, but also $\int_R (2i\,\pi\,t)^j\varphi(t)\,dt = 0$ for $j = 0, \ldots, k-1, k+1, \ldots, m$, so that one has $u \in H^s(R^N)$, $\gamma_k u = v$ and $\gamma_j u = 0$ for $j = 0, \ldots, k-1, k+1, \ldots, m$.■

In order to define $H^s(\Omega)$ for $\Omega \ne R^N$, it is useful to deduce a property of $H^s(R^N)$ which does not use explicitly FOURIER transform.

Lemma: For $0 < s < 1$, $u \in H^s(R^N)$ is equivalent to $u \in L^2(R^N)$ and $\int_{R^N}\int_{R^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \infty$.

Proof: For $h \in R^N$, and $\tau_h u(x) = u(x-h)$, one has $\mathcal{F}\tau_h u(\xi) = e^{2i\,\pi(h.\xi)}\mathcal{F}u(\xi)$, and therefore $\int_{R^N}|\tau_h u - u|^2\,dx = \int_{R^N}|1-e^{2i\,\pi(h.\xi)}|^2|\mathcal{F}u(\xi)|^2\,d\xi$, and as $|1-e^{2i\,\alpha}|^2 = 4\sin^2\alpha$, one deduces that $\int_{R^N}\frac{1}{|h|^{N+2s}}\big(\int_{R^N}|\tau_h u - u|^2\,dx\big)\,dh = \int_{R^N}|\mathcal{F}u(\xi)|^2\big(\int_{R^N}\frac{4\sin^2\pi(h.\xi)}{|h|^{N+2s}}\,dh\big)\,d\xi$; in order to compute $\int_{R^N}\frac{4\sin^2\pi(h.\xi)}{|h|^{N+2s}}\,dh$ for $\xi \ne 0$, one uses the invariance by rotation and the change of variable $h = |\xi|z$ and the integral is $|\xi|^{2s}\int_{R^N}\frac{4\sin^2\pi\,z_1}{|z|^{N+2s}}\,dz$, and therefore $\int_{R^N}\int_{R^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = C\int_{R^N}|\xi|^{2s}|\mathcal{F}u(\xi)|^2\,d\xi < \infty$, and $C = \int_{R^N}\frac{4\sin^2\pi\,z_1}{|z|^{N+2s}}\,dz$ is finite because $|\sin\pi\,z_1| \le \pi\,|z|$ for z near 0 and $s < 1$ and $|\sin\pi\,z_1| \le 1$ for z near ∞ and $s > 0$.■

For an open set Ω, one could define $H^s(\Omega)$ for $0 < s < 1$ in at least three different ways; a first one is to decide that $u \in H^s(\Omega)$ means $u \in L^2(\Omega)$ and $\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \infty$; a second one is to decide that $u \in H^s(\Omega)$ means that $u = U\big|_\Omega$ with $U \in H^s(R^N)$; a third one is to define $H^s(\Omega)$ by interpolation (for a bounded open set with LIPSCHITZ boundary, the three definitions give the same space with equivalent norms).

If $u \in W^{1,\infty}(R^N) = Lip(R^N)$, then the restriction of u to $x_N = 0$ is a LIPSCHITZ continuous function and conversely; if v is a LIPSCHITZ continuous function defined on a closed set A with LIPSCHITZ constant M, then one can extend it to R^N into a LIPSCHITZ continuous function u defined on R^N and having the same LIPSCHITZ constant M by $u(x) = sup_{a \in A}\big(v(a) - M\,|a - x|\big)$.

If $u \in H^1(R^N) = W^{1,2}(R^N)$, the trace v belongs to $H^{1/2}(R^{N-1})$, which was just shown to mean $v \in L^2(R^{N-1})$ and $\int_{R^{N-1}} \int_{R^{N-1}} \frac{|v(x)-v(y)|^2}{|x-y|^N}\,dx\,dy < \infty$. Actually, if $u \in W^{1,p}(R^N)$ and $1 < p < \infty$, then Emilio GAGLIARDO showed that the trace v satisfies $v \in L^p(R^{N-1})$ and $\int_{R^{N-1}} \int_{R^{N-1}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+p-2}}\,dx\,dy < \infty$, and that this characterizes the space of traces.

If $u \in W^{1,1}(R^N)$, then Emilio GAGLIARDO's characterization is that $v \in L^1(R^{N-1})$. Jaak PEETRE has shown that there is no linear continuous lifting from $L^1(R^{N-1})$ to $W^{1,1}(R^N)$[1].

[1] I have read that statement but I have not looked for the proof.

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17. Wednesday February 23, 2000

Lemma: If $u \in H^1(R^N)$ and $N \geq 3$, then $\frac{u}{|x|} \in L^2(R^N)$.

Proof: One proves that $\big|\big|\frac{u}{|x|}\big|\big|_2 \leq C||grad(u)||_2$ for all $u \in C_c^\infty(R^N)$. For any $\alpha \in R$ one has $0 \leq \int_{R^N} \sum_{j=1}^N \big|\frac{\partial u}{\partial x_j} + \frac{\alpha\, x_j\, u}{r^2}\big|^2\,dx = \int_{R^N} |grad(u)|^2\,dx + \sum_{j=1}^N \int_{R^N} 2u\,\frac{\partial u}{\partial x_j}\,\frac{\alpha\, x_j}{r^2} + \int_{R^N} \frac{\alpha^2 u^2}{r^2}\,dx$. Because $\frac{x_j}{r^2} \in W^{1,1}_{loc}(R^N)$ (for $N \geq 3$), and $\sum_{j=1}^N \frac{\partial}{\partial x_j}\big(\frac{x_j}{r^2}\big) = \frac{N-2}{r^2}$, one deduces that $\int_{R^N} |grad(u)|^2\,dx \geq \big((N-2)\alpha - \alpha^2\big)\int_{R^N} \frac{u^2}{r^2}\,dx$, and one takes $\alpha = \frac{N-2}{2}$.■

The result is obviously false for $N = 2$, even for u smooth with $u(0) \neq 0$ as $\frac{1}{r} \notin L^2_{loc}(R^2)$.

Corollary: The space of functions in $C_c^\infty(R^N)$ which are 0 in a small ball around 0 is dense in $H^1(R^N)$ for $N \geq 3$ (and also for $N = 2$).

Proof: As $C_c^\infty(R^N)$ is dense in $H^1(R^N)$ for all N, one must approach any $u \in C_c^\infty(R^N)$ by a sequence of functions which are 0 in a small ball around 0, and this is done by taking $u_n(x) = u(x)\theta(n\,x)$ with $\theta \in C^\infty(R^N)$ equal to 0 for $|x| \leq 1$ and equal to 1 for $|x| \geq 2$. One has $u_n \to u$ in $L^2(R^N)$ by LEBESGUE dominated convergence theorem, and similarly $\frac{\partial u}{\partial x_j}\theta(n\,x) \to \frac{\partial u}{\partial x_j}$ in $L^2(R^N)$, and in order to show that $u\,n\frac{\partial \theta}{\partial x_j}(n\,x) \to 0$ in $L^2(R^N)$, one also applies LEBESGUE dominated convergence theorem to $\frac{u}{|x|}f(n\,x)$ with $f(x) = |x|\frac{\partial \theta}{\partial x_j}$.■

Lemma: The space of functions in $C_c^\infty(R^2)$ which are 0 in a small ball around 0 is dense in $H^1(R^2)$.

Proof: One cannot apply the same argument used in the corollary for $N \geq 3$. One proof consists in applying HAHN-BANACH theorem, and showing that if $T \in H^{-1}(R^2) = \big(H^1(R^2)\big)'$, and $\langle T, \varphi\rangle = 0$ for all $\varphi \in C_c^\infty(R^2)$ which are 0 in a small ball around 0, then $T = 0$. Because $\langle T, \varphi\rangle = 0$ for all $\varphi \in C_c^\infty(\omega)$ for any open set ω such that $0 \notin \overline{\omega}$, one finds that the support of T can only be $\{0\}$ (if T is not 0). As will be seen, if a distribution T has support $\{0\}$, then $T = \sum_\alpha c_\alpha D^\alpha \delta_0$ (finite sum), but if some $c_\alpha \neq 0$ then $T \notin H^{-1}(R^2)$, because $\mathcal{F}T = \sum_\alpha c_\alpha (2i\,\pi\,\xi)^\alpha$ and no nonzero polynomial P satisfies $\int_{R^2} \frac{|P(\xi)|^2}{1+|\xi|^2}\,d\xi < \infty$.■

One deduces that if $\Omega = R^N \backslash F$, where F is a finite number of points and $N \geq 2$ then $H^1_0(\Omega) = H^1(\Omega) = H^1(R^N)$. This is not true for $N = 1$ as the functions in $H^1(R)$ are continuous. With some technical changes the same proofs adapt to $W^{1,p}(R^N)$ if $1 < p \leq N$. Similar ideas show that one can approach every function of $H^1(R^N)$ for $N \geq 3$ by functions in $C_c^\infty(R^N)$ which vanish in a neighbourhood of a given segment, but that is not true for $N = 2$ as the functions in $H^1(R^2)$ have traces on the segment.

Like for the limiting case of SOBOLEV imbedding theorem, there are norms which scale in the same way, but which are not comparable; for example if $N = 2$ then $||grad(u)||_2$ and $||u||_\infty$ scale in the same way but functions in $H^1(R^2)$ are not necessarily bounded, and $\big|\big|\frac{u}{r}\big|\big|_2$ also scales in the same way but $u \in H^1(R^2)$ does not imply $\frac{u}{r} \in L^2(R^2)$. However one has the following result.

Lemma: If $\Omega \subset B(0, R_0) \subset R^2$, then the exists C such that $\big|\big|\frac{u}{r\log(r/R_0)}\big|\big|_2 \leq C||grad(u)||_2$ for all $u \in H^1_0(\Omega)$.

Proof: One proves the inequality for $u \in C_c^\infty\big(B(0,R_0)\big)$ and therefore it is true for $u \in C_c^\infty(\Omega)$ and it extends then to $H^1_0(\Omega)$. For f smooth, one develops $\int_{B(0,R_0)} \sum_{j=1}^N \big|\frac{\partial u}{\partial x_j} + x_j f(r) u\big|^2\,dx \geq 0$, and one uses the integration by parts $\int_{B(0,R_0)} 2u\,\frac{\partial u}{\partial x_j} x_j f(r)\,dx = -\int_{B(0,R_0)} |u|^2\big(f(r) + x_j^2\frac{f'(r)}{r}\big)\,dx$, which is valid if $r\,f(r)$ and $r^2 f'(r)$ belong to $L^1(0, R_0 - \varepsilon)$ for every $\varepsilon > 0$, and one deduces $\int_{B(0,R_0)} |grad(u)|^2\,dx \geq \int_{B(0,R_0)} |u|^2(2f + r\,f' - r^2 f^2)\,dx$; if one takes $f = \frac{g}{r^2}$, one has $2f + r\,f' - r^2 f^2 = \frac{g'}{r} - \frac{g^2}{r^2}$, and one approaches then $g = \frac{-1}{2\log(r/R_0)}$, which corresponds to multiplying $|u|^2$ by $\frac{1}{4r^2(\log(r/R_0))^2}$.■

Because the logaritm vanishes for $|x| = R_0$, it is important to have Ω bounded, but the same argument works if Ω is unbounded and is outside a ball $B(0, R_0)$, but there is a problem with the entire space; actually, Jacques-Louis LIONS and Jacques DENY have shown that the completion of $C_c^\infty(R^2)$ for the norm $||grad(u)||_2$ is not a space of distributions on R^2.

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18. Friday February 25, 2000

In order to identify distributions with support at a point, one uses the following result.

Proposition: Let $T \in \mathcal{D}'(\Omega)$ have its support in a compact $K_0 \subset \Omega$, and assume that T is a distribution of order m. Then if $\varphi_0 \in C_c^\infty(\Omega)$ satisfies $D^\alpha \varphi_0(x) = 0$ for all $x \in K_0$ and all multi-indices α such that $|\alpha| \leq m$, one has $\langle T, \varphi_0 \rangle = 0$.
Proof: As T is assumed to be of order m, for every compact $K \subset \Omega$ there exists $C(K)$ such that $|\langle T, \varphi \rangle| \leq C(K) \sup_{x \in K, |\alpha| \leq m} |D^\alpha \varphi(x)|$ for all $\varphi \in C_c^\infty(\Omega)$ such that *support*$(\varphi) \subset K$. Let $\varepsilon_0 > 0$ be such that $\{x \in R^N, d(x, K_0) \leq \varepsilon_0\} \subset \Omega$. For $0 < \varepsilon < \varepsilon_0$, let $K_\varepsilon = \{x \in R^N, d(x, K_0) \leq \varepsilon\}$, and let χ_ε be the characteristic function of K_ε. Let $\rho_1 \in C_c^\infty(R^N)$ with *support*$(\rho_1) \subset \overline{B(0,1)}$ and $\int_{R^N} \rho_1(x)\,dx = 1$, and as usual $\rho_\delta(x) = \frac{1}{\delta^N} \rho_1\big(\frac{x}{\delta}\big)$ for $\delta > 0$.

If $3\delta < \varepsilon_0$, let $\theta_\delta = \chi_{2\delta} \star \rho_\delta$, so that $\theta_\delta \in C_c^\infty(\Omega)$, $\theta_\delta(x) = 1$ if $x \in K_\delta$ and *support*$(\theta_\delta) \subset K_{3\delta} \subset K_{\varepsilon_0}$. One has $\langle T, \varphi_0 \rangle = \langle T, \theta_\delta\, \varphi_0 \rangle$, because the difference is $\langle T, (1 - \theta_\delta)\varphi_0 \rangle$ and the support of $1 - \theta_\delta$ is included in $\Omega \setminus K_0$, i.e. the largest open set where T is 0. One proves that $\langle T, \theta_\delta\, \varphi_0 \rangle \to 0$ as $\delta \to 0$ by showing that for any multi-index α such that $|\alpha| \leq m$ one has $\sup_{x \in K_{\varepsilon_0}} |D^\alpha(\theta_\delta \varphi_0)(x)| \to 0$ as $\delta \to 0$.

One has $|D^\beta \theta_\delta(x)| \leq C\, \delta^{-|\beta|}$ for all x and $|\beta| \leq m$. Because of TAYLOR formula and the vanishing of the derivatives of φ_0 on K up to order m, one has $|D^\gamma \varphi_0(x)| \leq d(x, K_0)^{m-|\gamma|} \eta\big(d(x, K_0)\big)$ for $|\gamma| \leq m$ and $\eta(t) \to 0$ as $t \to 0$. By LEIBNIZ formula, and using the fact that $d(x, K_0) \leq 3\delta$ for $x \in$ *support*(θ_δ), one deduces that $|D^\alpha(\theta_\delta \varphi_0)(x)| \leq C\, \eta\big(d(x, K_0)\big)$.■

Corollary: If T has support at a point $a \in \Omega$, then T is a finite combination of derivatives of the DIRAC mass at a.
Proof: If $K = \overline{B(a,r)} \subset \Omega$, then T has finite order m on K, and by the preceding result, $D^\alpha \varphi_0(a) = 0$ for all $|\alpha| \leq m$ implies $\langle T, \varphi_0 \rangle = 0$. A result of Linear Algebra says that on any vector space if for linear forms $L_0, \ldots, L_p$ every u satisfying $L_1 u = \ldots = L_p u = 0$ also satisfies $L_0 u = 0$ then there are scalars $\lambda_1, \ldots, \lambda_p$ such that $L_0 = \sum_{j=1}^p \lambda_j L_j$. Therefore there are scalars λ_α for $|\alpha| \leq m$ such that $\langle T, \varphi \rangle = \sum_{|\alpha| \leq m} \lambda_\alpha D^\alpha \varphi(a)$ for all $\varphi \in C_c^\infty(\Omega)$ with *support*$(\varphi) \subset K$, i.e. $T = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \lambda_\alpha D^\alpha \delta_a$.■

Most compactness results rely on the theorems of ARZELÁ and ASCOLI, and the basic result of interest here is that if one works on a compact set K of R^N and if one has a sequence $u_n \in C(K)$ of functions which have the same modulus of uniform continuity, i.e. $|u_n(x) - u_n(y)| \leq \omega(|x - y|)$ for all $x, y \in K$ and all n, with $\omega(t) \to 0$ as $t \to 0$, then there exists a subsequence u_m which converges uniformly on K, to $u_\infty \in C(K)$ (using a diagonal argument one extracts a subsequence which converges on a countable dense set of K, and the subsequence also converges at the other points by equicontinuity, and the limit is continuous for the same reason).

For proving compactness in $L^p(\Omega)$ for $1 \leq p < \infty$, one extends the functions by 0 and one applies a compactness result in $L^p(R^N)$, usually attributed to KOLMOGOROV.

Lemma: If a sequence u_n is bounded in $L^p(R^N)$ and satisfies

i) For every $\varepsilon > 0$, there exists $R(\varepsilon)$ such that $\int_{|x| \geq R(\varepsilon)} |u_n|^p\,dx \leq \varepsilon$ for all n.

ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h| \leq \delta$ then $\int_{R^N} |u_n(x + h) - u_n(x)|^p\,dx \leq \varepsilon$.

Then there exists a subsequence u_m which converges strongly to $u_\infty \in L^p(R^N)$.

Proof: It is enough to show that for every $\alpha > 0$ one can write $u_n = v_n + w_n$ such that $||w_n||_p \leq \alpha$ and that from v_n one can extract a converging subsequence v_m; for the subsequence u_m one has then $\limsup_{m,m' \to \infty} ||u_m - u_{m'}||_p \leq 2\alpha$. Starting from the selected subsequence, one repeats then the argument with α replaced by $\frac{\alpha}{2}$, and so on, and a diagonal subsequence is a CAUCHY sequence.

Using $\theta \in C_c^\infty(R^N)$ such that $0 \leq \theta \leq 1$ and $\theta(x) = 1$ for $|x| \leq R(\varepsilon)$, one chooses $v_n = \theta\, u_n$ and $w_n = (1 - \theta) u_n$, and one notices that $||w_n||_p \leq \varepsilon$ by i), and because $\tau_h v_n - v_n = (\tau_h \theta - \theta) \tau_h u_n + \theta(\tau_h u_n - u_n)$ and $||\tau_h \theta - \theta||_\infty \leq M\,|h|$, one finds that v_n is bounded in $L^p(R^N)$, has its support in a fixed bounded set, and satisfies ii).

Assuming that the functions u_n satisfy ii) and have their support in a fixed bounded set, one uses $v_n = u_n \star \rho_\delta$ for a special smoothing sequence ρ_δ, and $w_n = (u_n - u_n \star \rho_\delta)$; one can apply ARZELA-ASCOLI to the sequence v_n, as they form a bounded sequence of LIPSCHITZ continuous functions having their support in a fixed compact set; because $w_n(x) = \int_{R^N} \rho_\delta(y)\big(u_n(x) - u_n(x-y)\big)\,dy$, one has $||w_n||_p \leq \int_{R^N} \rho_\delta(y)||u_n - \tau_y u_n||_p\,dy \leq \varepsilon$.■

If Ω is a bounded open set which is smooth enough so that there exists a continuous extension P from $W^{1,p}(\Omega)$ to $W^{1,p}(R^N)$, then the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, because for a bounded sequence $u_n \in W^{1,p}(\Omega)$ one considers the sequence $\theta\, P\, u_n$ with $\theta \in C_c^\infty(R^N)$ and $\theta = 1$ on Ω, and the preceding results show that a subsequence $\theta\, P\, u_m$ converges strongly in $L^p(R^N)$ and therefore the restriction $u_m = \theta\, P\, u_m\big|_\Omega$ converges strongly in $L^p(\Omega)$.

Proposition: If Ω is a bounded open set with a continuous boundary, then the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact.
Proof: The preceding argument does not apply, and one must find a different proof. Using a partition of unity one has to consider the case of Ω_F with F uniformly continuous, for a subsequence having support in a bounded set. One notices that in dimension 1 one has $W^{1,p}(0,\infty) \subset C_b(0,\infty)$, and therefore one has $\int_{F(x')}^{F(x')+3\varepsilon} |u(x',x_N)|^p\,dx \leq C\,\varepsilon \int_{F(x')}^\infty \big(|u(x',x_N)|^p + \big|\frac{\partial u}{\partial x_N}(x',x_N)\big|^p\big)\,dx_N$, which one may then integrate in x'. Using the uniform continuity of F one can construct $\theta \in C^\infty(R^N)$ with $0 \leq \theta \leq 1$, $\theta(x) = 0$ if $x_N < F(x') + \varepsilon$ and $\theta(x) = 1$ if $x_N > F(x') + 2\varepsilon$. One uses then $v_n = \theta\, u_n$ and $w_n = (1-\theta)u_n$.■

It is sometimes useful to know different proofs of the same result, and using FOURIER transform one can show that if Ω is an open set with finite measure then POINCARÉ inequality holds for $H_0^1(\Omega)$ and the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact.

Let $u \in H_0^1(\Omega)$, extended by 0 outside Ω, and let $A = ||u||_2$ and $B = ||grad(u)||_2$. Because $\mathcal{F}u(\xi) = \int_\Omega u(x)e^{-2i\pi\,(x.\xi)}\,dx$ one has $|\mathcal{F}u(\xi)| \leq A\, meas(\Omega)^{1/2}$ for all $\xi \in R^N$, and therefore $\int_{|\xi|\leq\rho} |\mathcal{F}u(\xi)|^2\,d\xi \leq A^2 meas(\Omega)\omega_N\rho^N$, where ω_N is the volume of the unit ball. As $\int_{|\xi|\geq\rho} |\mathcal{F}u(\xi)|^2\,d\xi \leq \int_{|\xi|\geq\rho} \frac{4\pi^2|\xi|^2}{4\pi^2\rho^2}|\mathcal{F}u(\xi)|^2\,d\xi \leq \frac{B^2}{4\pi^2\rho^2}$, one deduces by adding these two inequalities that $A^2 = \int_{R^N} |\mathcal{F}u(\xi)|^2\,d\xi \leq A^2 meas(\Omega)\omega_N\rho^N + \frac{B^2}{4\pi^2\rho^2}$ for every $\rho > 0$, and therefore the choice $meas(\Omega)\omega_N\rho^N = \frac{1}{2}$ gives $A \leq c_N meas(\Omega)^{1/N} B$ for a universal constant c_N (i.e. independent of the open set).

In order to prove that the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, one assumes that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$ weak, and one wants to prove that $u_n \to 0$ in $L^2(\Omega)$ strong. Indeed one may take $||u_n||_2 \leq A$ and $||grad(u_n)||_2 \leq B$, and because $\mathcal{F}u_n(\xi)$ is the L^2 scalar product of u_n by a fixed function in $L^2(\Omega)$, one has $\mathcal{F}u_n(\xi) \to 0$ for every $\xi \in R^N$. Because $|\mathcal{F}u(\xi)| \leq A\, meas(\Omega)^{1/2}$ for all $\xi \in R^N$ one deduces by LEBESGUE dominated convergence theorem that $\int_{|\xi|\leq\rho} |\mathcal{F}u_n(\xi)|^2\,d\xi \to 0$ for any $\rho > 0$. Because $\int_{|\xi|\geq\rho} |\mathcal{F}u_n(\xi)|^2\,d\xi \leq \frac{B^2}{4\pi^2\rho^2}$, one deduces that $\limsup_{n\to\infty} ||u_n||_2 \leq \frac{B}{2\pi\,\rho}$, and letting then $\rho \to \infty$ one has $||u_n||_2 \to 0$.■

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19. Monday February 28, 2000

The main reason why SOBOLEV spaces are important is that they are the natural functional spaces for solving the boundary value problems of Continuum Mechanics and Physics (at least up to now); they may be elliptic equations like $\Delta u = f$ for which one invokes the names of LAPLACE or POISSON, parabolic like the heat equation $\frac{\partial u}{\partial t} - \kappa \Delta u = f$ for which one invokes the name of FOURIER, or hyperbolic like the wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f$ for which one invokes the names of D'ALEMBERT[1] or D. BERNOULLI[2].

The SOBOLEV space $H^1(\Omega)$ is adapted to problems of the form $-\sum_{i,j} \frac{\partial}{\partial x_i}\big(A_{ij} \frac{\partial u}{\partial x_j}\big) = f$, written $-div\big(A\, grad(u)\big) = f$ when the matrix A (which is usually symmetric in applications) has bounded measurable coefficients and satisfies the ellipticity condition that there exists $\alpha > 0$ such that $\sum_{ij} A_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$ for all $\xi \in R^N$, for a.e. $x \in \Omega$. There are various physical interpretations possible. A first one is to consider the stationary heat equation, so that u is the temperature and $A\, grad(u)$ is the heat flux. A second one is to consider Electrostatics, which is a simplification of MAXWELL equations where there is no magnetic field and no time dependence, so that u is the electrostatic potential, $E = -grad(u)$ is the electric field, $D = A\, grad(u)$ is the polarization field, f is the density of electric charge (usually denoted ρ), and the density of electric energy is $e = \frac{1}{2}(E.D)$; using the SOBOLEV space $H^1(\Omega)$ for u corresponds to having a finite electric energy stored in Ω (A is called the permittivity tensor in this case). A third one is to consider a different simplification of MAXWELL equations with no magnetic field and no time dependence also but where one considers the electric current j (which in principle must satisfies the equation $\frac{\partial \rho}{\partial t} + div(j) = 0$, which expresses the conservation of total electric charge) is related to the electric field by OHM's[3] law $j = \sigma E$ (so A is the conductivity tensor σ in this case).

Whatever the physical intuition is, the equation $-div\big(A\, grad(u)\big) = f$, together with boundary conditions, is dealt in a mathematical way by using LAX-MILGRAM lemma (which was also discovered by Mark VISHIK), or some variant. One considers a HILBERT space V (which is usually a closed subset of $H^1(\Omega)$ containing $H^1_0(\Omega)$), and a bilinear continuous form $a(u,v)$ on $V \times V$ (which is usually $\int_\Omega \big(A\, grad(u), grad(v)\big)\, dx$), and a linear continuous for $L(v)$ on V (which is usually of the form $\int_\Omega f\, v\, dx + \int_{\partial\Omega} g\, \gamma_0 v\, dH^{N-1}$), and there is a unique solution u of the variational formulation $a(u,v) = L(v)$ for all $v \in V$ under the (sufficient) condition that the bilinear form is V-elliptic, i.e. there exists $\alpha > 0$ such that $a(u,u) \geq \alpha\, ||u||^2_V$ for all $u \in V$. V-ellipticity holds if and only if POINCARÉ inequality holds for V.

For the case of homogeneous DIRICHLET condition, i.e. $\gamma_0 u = 0$ on $\partial\Omega$, one takes $V = H^1_0(\Omega)$ and POINCARÉ inequality holds if Ω has finite measure or is included in a strip with finite width, and there exists a unique solution for $f \in H^{-1}(\Omega)$ of $a(u,v) = L(v)$ for every $v \in V$, or equivalently for every $v \in C^\infty_c(\Omega)$ by density, and that is exactly the equation. For the case of nonhomogeneous DIRICHLET condition, i.e. $\gamma_0 u = g$ on $\partial\Omega$, one asks that g belongs to $\gamma_0 H^1(\Omega)$ (which is $H^{1/2}(\partial\Omega)$ if Ω is bounded with LIPSCHITZ boundary), so that there exists $u_1 \in H^1(\Omega)$ with $\gamma_0 u_1 = g$; one looks then for a solution $u = u_1 + U$ with $U \in H^1_0(\Omega)$ satisfying the equation with f replaced by $f + div\big(A\, grad(u_1)\big)$, and there exists a unique solution for $f \in H^{-1}(\Omega)$ and $g \in \gamma_0 H^1(\Omega)$.

If $V = H^1(\Omega)$ then POINCARÉ inequality does not hold in general, but assuming that there is a solution $u \in H^1(\Omega)$ of $a(u,v) = L(v)$ for every $v \in H^1(\Omega)$ and L has the simple form $L(v) = \int_\Omega f\, v\, dx + \int_{\partial\Omega} g\, \gamma_0 v\, dH^{N-1}$ with $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, it is useful to characterize what a solution can be in this case. Taking all $v \in C^\infty_c(\Omega)$ gives the equation $-div\big(A\, grad(u)\big) = f$ in Ω. Then assuming that the coefficients a_{ij} are LIPSCHITZ continuous and that the boundary of Ω is smooth, one can show that $u \in H^2(\Omega)$ and then an integration by parts shows that one has $(A\, \gamma_0 grad(u).n) = g$ on $\partial\Omega$, the NEUMANN[4]

[1] Jean Le Rond D'ALEMBERT, French mathematician, 1717-1783.

[2] Daniel BERNOULLI, Swiss mathematician, 1700-1782. He worked in Basel.

[3] Georg Simon OHM, German mathematician, 1789-1854. He worked in Munich.

[4] Franz Ernst NEUMANN, German mathematician, 1798-1895. He worked in Königsberg (now Kaliningrad, Russia).

condition. If the boundary is not smooth enough the solution may not belong to $H^2(\Omega)$, and the coefficients may not be smooth either, and an interpretation of the boundary condition will be studied later, but there is another important point to discuss, concerning existence.

If Ω has finite measure, $1 \in H^1(\Omega)$ and, because $a(u,1) = 0$ for every $u \in V = H^1(\Omega)$, a necessary condition for the existence of a solution is that $L(1) = 0$. With the physical interpretation of a stationary heat equation it means that the total amount of heat is 0, adding the source of heat inside Ω which is $\int_\Omega f\,dx$ and the heat flux imposed on the boundary $\partial\Omega$, which is $\int_{\partial\Omega} g\,dH^{N-1}$; if this condition is not satisfied then the solution of the evolution heat equation will not converge to a limit, and it actually tends to infinity (with a sign depending upon the sign of the total heat imposed; of course, when the temperature becomes too large, the modelization by a linear equation is not very good, and in a real problem the absolute temperature cannot become negative anyway).

If the necessary condition $L(1) = 0$ is satisfied and if the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact then a solution exists, but it is not unique as one may add an arbitrary constant to the solution (an example of a FREDHOLM alternative). If one denotes by u_Ω the average of u on Ω, and by $u_{\partial\Omega}$ the average of $\gamma_0 u$ on $\partial\Omega$, then the compactness assumption implies that POINCARÉ inequality $||u||_2 \le C||grad(u)||_2$ holds for all $u \in H^1(\Omega)$ satisfying $u_\Omega = 0$, and that it also holds for all $u \in H^1(\Omega)$ satisfying $u_{\partial\Omega} = 0$. Even if the compactness condition does not hold but one of these POINCARÉ inequality is true, then there exists a solution.

Using POINCARÉ inequality for all $u \in H^1(\Omega)$ satisfying $u_\Omega = 0$, one changes V to denote the subspace of $u \in H^1(\Omega)$ such that $u_\Omega = 0$, and the bilinear form is then V-elliptic and a solution exists. One does not have $C_c^\infty(\Omega) \subset V$, but $a(u,v) = L(v)$ only for $v \in C_c^\infty(\Omega)$ satisfying $\int_\Omega v\,dx = 0$, and therefore there exists a LAGRANGE[5] multiplier λ such that $a(u,v) = L(v) + \lambda \int_\Omega v\,dx$ for all $v \in C_c^\infty(\Omega)$, so that $-div\big(A\,grad(u)\big) = f + \lambda$ in Ω; then one obtains the boundary condition and λ is such that the necessary condition must hold and it is therefore equal to 0.

Using POINCARÉ inequality for all $u \in H^1(\Omega)$ satisfying $u_{\partial\Omega} = 0$, one changes V to denote the subspace of $u \in H^1(\Omega)$ such that $u_{\partial\Omega} = 0$, and the bilinear form is then V-elliptic and a solution exists. One has now $C_c^\infty(\Omega) \subset V$, so that $-div\big(A\,grad(u)\big) = f$ in Ω; then one obtains the boundary condition and a LAGRANGE multiplier appears in the boundary condition but the necessary condition must hold and it is therefore equal to 0.

[5] Joseph-Louis LAGRANGE, Italian-born mathematician, 1736-1813. He worked in Paris.

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20. Wednesday March 1, 2000

Let Ω be a bounded open set of R^N with LIPSCHITZ boundary, and let $u \in H^1(\Omega)$ satisfy $a(u,v) = L(v)$ for all $v \in H^1(\Omega)$, with $a(\varphi,\psi) = \int_\Omega \big(\sum_{i,j} A_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_i}\big)\,dx$ for all $\varphi,\psi \in H^1(\Omega)$ and $L(\psi) = \int_\Omega f\,\psi\,dx + \int_{\partial\Omega} g\,\gamma_0\psi\,dH^{N-1}$ for all $\psi \in H^1(\Omega)$, with $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Using all $v \in C_c^\infty(\Omega)$ one deduces that $-\sum_{i,j} \frac{\partial}{\partial x_i}\big(A_{ij}\frac{\partial u}{\partial x_j}\big) = f$ in Ω, and the question is to understand what is the meaning of the boundary condition that u satisfies.

If one assumes that $A_{ij} \in W^{1,\infty}(\Omega)$ for $i,j = 1,\ldots,N$ and $u \in H^2(\Omega)$, then $A_{ij}\frac{\partial u}{\partial x_j} \in H^1(\Omega)$ for all $i,j = 1,\ldots,N$, and an integration by parts gives $\int_\Omega f\,v\,dx = a(u,v) - \int_{\partial\Omega}\big(\sum_{i,j} A_{ij}\gamma_0\frac{\partial u}{\partial x_j}\gamma_0 v\,n_i\big)\,dH^{N-1}$ for all $v \in H^1(\Omega)$, and therefore one has $\sum_{i,j} A_{ij}\gamma_0\frac{\partial u}{\partial x_j}\gamma_0 v\,n_i = g$ on $\partial\Omega$ (so this can only happen if $g \in H^{1/2}(\partial\Omega)$).

In applications, one does not always have LIPSCHITZ coefficients A_{ij}, because one often mixes different materials and there are interfaces of discontinuity for the coefficients. In applications, one does not always have smooth boundaries, and corners in the boundary put a limit for the regularity of the solution. Although it is true that the solution for $f \in L^2(\Omega)$ and $g = 0$ does satisfy $u \in H^2(\Omega)$ if Ω is convex, it is not always true independently of the size of the corners on the boundary, as the following example in the plane shows. Let Ω be the sector $0 < \theta < \theta_0$ with $\pi < \theta_0 < 2\pi$, and let $u = r^\alpha\cos(\alpha\,\theta)\varphi$ with $\varphi \in C_c^\infty(R^2)$ with $\varphi = 1$ near the origin. Because $u_0 = r^\alpha\cos(\alpha\,\theta)$ is harmonic, i.e. satisfies $\Delta\,u_0 = 0$ (because it is the real part of z^α for example), one sees that $\Delta\,u$ is 0 near the origin; the normal derivative of u on the side $\theta = 0$ is 0, and it is also 0 on the side $\theta = \theta_0$ if $\alpha\,\theta_0 = \pi$, which gives $\frac{1}{2} < \alpha < 1$, and therefore one does not have $u \in H^2(\Omega)$, which requires $\alpha > 1$. Actually, for a convex domain Ω, $u \in H^1_0(\Omega)$ and $\Delta\,u \in L^2(\Omega)$ imply $H^2(\Omega)$.

A different way to treat this problem of giving a meaning to the NEUMANN condition, is the following argument[1] of Jacques-Louis LIONS.

Definition: $H(div;\Omega) = \{u \in \big(L^2(\Omega)\big)^N, div(u) \in L^2(\Omega)\}$.■

Of course, $H(div;\Omega)$ is a HILBERT space.

One localizes by multiplying all the components of u by the same function θ, noticing that if $v_j = \theta\,u_j$ for $j = 1,\ldots,N$, then one has $div(v) = \theta\,div(u) + \big(u.grad(\theta)\big)$.

If P is an invertible matrix and $\Omega' = P\,\Omega$, one transports a scalar function φ defined on Ω to the scalar function ψ defined on Ω' by $\psi(P\,x) = \varphi(x)$ for $x \in \Omega$, and one wants to transport $u \in H(div;\Omega)$ to $v \in H(div;\Omega')$ in such a way that one has $\int_\Omega\big(u.grad(\varphi)\big)\,dx = \int_{\Omega'}\big(v.grad(\psi)\big)\,dx'$, but as $(grad(\psi)(P\,x).P\,y) = (grad(\varphi)(x).y)$ one has $grad(\psi)(P\,x) = P^{-T}grad(\varphi)(x)$ (denoting $P^{-T} = (P^T)^{-1}$), and one asks that $\big(u(x).grad(\varphi)(x)\big) = \big(v(P\,x).grad(\psi)(P\,x)\big)|det(P)|$, which gives $v(P\,x) = |det(P)|^{-1}P\,u(x)$. If P is an orthogonal matrix then $v(P\,x) = P\,u(x)$.

Once one works on Ω_F for a LIPSCHITZ continuous function F, one proves easily that $\big(\mathcal{D}(\overline{\Omega_F})\big)^N$ is dense in $H(div;\Omega_F)$.

All this proves that $\big(\mathcal{D}(\overline{\Omega})\big)^N$ is dense in $H(div;\Omega)$. The next step is to prove that one can define the normal trace $(u.n)$; for smooth functions it means $\sum_j \gamma_0 u_j n_j$, but for $H(div;\Omega)$ the definition uses a completion argument.

Proposition: The mapping $u \mapsto (u.n) = \sum_j \gamma_0 u_j n_j$, defined from $\big(\mathcal{D}(\overline{\Omega})\big)^N$ into $L^\infty(\partial\Omega)$, extends into a linear continuous map from $H(div;\Omega)$ into $\big(\gamma_0 H^1(\Omega)\big)'$, the dual of the space of traces of functions of $H^1(\Omega)$, i.e. $H^{-1/2}(\partial\Omega)$ (as $\partial\Omega$ has no boundary, $H^{1/2}_0(\partial\Omega) = H^{1/2}(\partial\Omega)$). Moreover the mapping is surjective.

[1] I believe that he proved it while I was a student, because in the first courses that I followed he used the argument with the $H^2(\Omega)$ hypothesis, and later he started teaching the new argument with the space $H(div;\Omega)$.

Proof: For $u \in \big(\mathcal{D}(\overline{\Omega})\big)^N$ and $v \in H^1(\Omega)$ one has $\int_\Omega \big(\sum_j \big(u_j \frac{\partial v}{\partial x_j} + div(u)\, v\big)\, dx = \int_{\partial\Omega} (\sum_j \gamma_0 u_j\, n_j) \gamma_0 v\, dH^{N-1}$, and as the left side of the identity is continuous on $H(div;\Omega) \times H^1(\Omega)$, so is the right side, which one writes $\langle (u.n), \gamma_0 v\rangle$ as a linear continuous form on $\gamma_0 H^1(\Omega)$; notice that if one starts from an element of $\gamma_0 H^1(\Omega)$ it does not matter which v one chooses which has this element as its trace, as the left side will give the same value whatever the choice is.

In order to show surjectivity, one takes $g \in \big(\gamma_0 H^1(\Omega)\big)'$ and one solves $\int_\Omega \big(grad(u_*).grad(v)\big)\, dx + \int_\Omega u_* v\, dx = \langle g, \gamma_0 v\rangle$ for all $v \in H^1(\Omega)$, which has a unique solution $u_* \in H^1(\Omega)$, which satisfies $-\Delta\, u_* + u_* = 0$ in Ω and therefore $\xi_* = grad(u_*)$ belongs to $H(div;\Omega)$, satisfies $div(\xi_*) = u_*$, and the precise variational formulation says that $(\xi_*.n) = g$.■

The example of R^2 with $u_1 = f_1(x_1) f_2(x_2)$ and $u_2 = g_1(x_1) g_2(x_2)$ shows that one has $u \in H(div; R^2)$ if $f_1, g_2 \in H^1(R)$ and $f_2, g_1 \in L^2(R)$, and therefore u_1 can be discontinuous along the line $x_2 = 0$ while u_2 must be continuous, and $(u.n) = -u_2$ if $\Omega = R^2_+$.

In a problem of Electrostatics, the potential u is in $H^1(\Omega)$ and has a trace on the boundary; more generally, on any interface u takes the same value on both sides of the interface. The polarization field D satisfies $div(D) = \rho$ and therefore $D \in H(div;\Omega)$ if $\rho \in L^2(\Omega)$, and the normal component of D is continuous at any interface (if it does not support a nonzero charge). For the electric field E, it is the tangential component of E which is continuous, and its value is the tangential derivative of the trace of u; one can actually define the space $H(curl;\Omega) = \big\{E \in \big(L^2(\Omega)\big)^N, \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \in L^2(\Omega) \text{ for all } i,j = 1,\ldots,N\big\}$, and prove an analogous theorem, that the tangential trace is defined.

21-724. Interpolation spaces
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01. Wednesday March 8, 2000.

Although the term interpolation space only appeared much later, the subject has its origin in questions studied by M. RIESZ, and then by THORIN[1], and also by MARCINKIEWICZ; they might have been motivated by studying the properties of the HILBERT transform.

A holomorphic function in an open set of the complex plane is a complex valued function which has a derivative in the complex sense, i.e. $\frac{f(z)-f(z_0)}{z-z_0}$ has a limit as z tends to z_0, and if $z = x + i\,y$ and $f(z) = P(x,y) + i\,Q(x,y)$ it leads to the CAUCHY-RIEMANN equations $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ and $\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$, so that both P and Q are harmonic, i.e. satisfy $\Delta\,P = \Delta\,Q = 0$ where the Laplacian Δ is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If one works in the upper half plane $y > 0$, and one imposes the real part of f on the boundary, then P is determined (if the given trace is nice enough) and then the partial derivatives of Q are known, so that Q is defined up to addition of an arbitrary real constant. In this way one is led to study the following transform named after HILBERT, $H\,u = \frac{1}{\pi} pv\frac{1}{x} \star u$, i.e. $H\,u(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_{|y-x|>\varepsilon} \frac{u(y)}{x-y}\,dy$, which relates the real part to the imaginary part on the boundary. Using FOURIER transform one can show that H is a surjective isometry from $L^2(R)$ into itself and $H^2 = -I$. This was done a long time before Laurent SCHWARTZ extended FOURIER transform to some distributions, but his proof is as follows: one notices that $pv\frac{1}{x} \in \mathcal{S}'(R)$ as the sum of a distribution with compact support and a bounded function, and because $x\,pv\frac{1}{x} = 1$, one deduces that $\frac{d}{d\xi}\big(\mathcal{F}(pv\frac{1}{x})\big) = -2i\pi\,\delta_0$, i.e. $\mathcal{F}(pv\frac{1}{x}) = -i\pi\,sign(\xi) + C$, and one deduces $C = 0$ from the fact that $pv\frac{1}{x}$ is odd so that its FOURIER transform must be odd (of course he had defined in a natural way what it means to be even or odd for a distribution). Then one has $\mathcal{F}(H\,u)(\xi) = -i\,sign(\xi)\mathcal{F}u(\xi)$ and therefore $||H\,u||_2 = ||u||_2$ and $H^2 = -I$.

I think that it was M. RIESZ who showed that the HILBERT transform is continuous from $L^p(R)$ into itself for $1 < p < \infty$, but the result is not true for $p = 1$ or for $p = \infty$ (one usually replaces L^1 by the HARDY space $\mathcal{H}^1$, and L^∞ by BMO). I suppose that it was in relation with the properties of the HILBERT transform that M. RIESZ proved the following "interpolation" result in 1926, in the case $p_\theta \le q_\theta$; this restriction was removed by THORIN, in 1938.

Proposition: If $1 \le p_0, p_1, q_0, q_1 \le \infty$, and a linear map A is continuous from $L^{p_0}(\Omega)$ into $L^{q_0}(\Omega')$ and from $L^{p_1}(\Omega)$ into $L^{q_1}(\Omega')$ then for $0 \le \theta \le 1$ it is continuous from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Moreover one has $||A||_\theta \le ||A||_0^{1-\theta}||A||_1^\theta$, where $||A||_\theta$ denotes the norm of A as a mapping from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$.■

If the HILBERT transform was mapping $L^1(R)$ into itself, then by this interpolation result if would map $L^p(R)$ into itself for $1 \le p \le 2$ and by transposition for $2 \le p \le \infty$, but it does not map $L^1(R)$ into $L^1(R)$. However there exists a constant C such that if $u \in L^1(R)$ one has $meas\{x : |H\,u(x)| > t\} \le \frac{C\,||u||_1}{t}$ for all $t > 0$, and from that result, the continuity in $L^2(R)$ and the symmetry of the HILBERT transform one can deduce that it maps $L^p(R)$ into itself for $1 < p < \infty$.

THORIN's prooof used a property of the modulus of holomorphic functions, the three lines theorem (a variant of HADAMARD three circles theorem), stating that if $f(z)$ is holomorphic in the strip $0 < \Re z = x < 1$, continuous on the closed strip $0 \le x \le 1$ and such that $|f(i\,y)| \le M_0$ and $|f(1 + i\,y)| \le M_1$ for all $y \in R$, then one has $|f(\theta + i\,y)| \le M_0^{1-\theta}M_1^{\theta+i\,y}$ for all $\theta \in (0,1)$ and all $y \in R$.

Generalizing the idea of THORIN, a complex interpolation method was developed by Alberto CALDERÓN, by Jacques-Louis LIONS and by M. KREIN[2].

If $f \in L^p(\Omega)$, then HÖLDER inequality gives $\int_E |f|\,dx \le ||f||_p meas(E)^{1/p'}$ for all measurable subsets E of Ω, and MARCINKIEWICZ introduced a space sometimes called weak L^p (which one should not mistake with L^p equipped with the weak topology), and denoted $L^{p,\infty}$ in the scale of LORENTZ spaces, which is the space of (equivalence classes of) measurable functions g for which there exists C such that $\int_E |g|\,dx \le C\,meas(E)^{1/p'}$ for all measurable subsets E of Ω; it contains $L^p(\Omega)$ but if $\Omega \subset R^N$ and $1 \le p < \infty$ it also contains functions

[1] G. Olof THORIN, Swedish mathematician; he was a student of Marcel RIESZ in Lund, Sweden.
[2] Mark Grigorievich KREIN, Ukrainian mathematician, 1907–1989. He worked in Kiev.

like $\frac{1}{|x|^{N/p}}$. In 1939, MARCINKIEWICZ published the following result, as a note without proof, and proofs were written later by Mischa COTLAR[3] and by Antoni ZYGMUND.

Proposition: If $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and a linear map A is continuous from $L^{p_0}(\Omega)$ into $L^{q_0,\infty}(\Omega')$ and from $L^{p_1}(\Omega)$ into $L^{q_1,\infty}(\Omega')$ then for $0 \leq \theta \leq 1$ it is continuous from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$ under the condition that $p_\theta \leq q_\theta$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.■

The results of RIESZ, THORIN, and MARCINKIEWICZ, were generalized as theories of Interpolation, and the main contributors were Nachman ARONSZAJN[4], Alberto CALDERÓN, Emilio GAGLIARDO, KREIN, Jacques-Louis LIONS and Jaak PEETRE, but similar techniques have been used by specialists of Harmonic Analysis, like STEIN. Part of the motivation of Jacques-Louis LIONS was the question of identifying traces of SOBOLEV spaces and their variants.

Definition: Let E_0 and E_1 be normed spaces, continuously imbedded into a topological vector space $\mathcal{E}$ so that $E_0 \cap E_1$ and $E_0 + E_1$ are defined.

An intermediate space between E_0 and E_1 is any normed space E such that $E_0 \cap E_1 \subset E \subset E_0 + E_1$ (with continuous imbeddings).

An interpolation space between E_0 and E_1 is any intermediate space E such that every linear mapping from $E_0 + E_1$ into itself which is continuous from E_0 into itself and from E_1 into itself is automatically continuous from E into itself. It is said to be of exponent θ (with $0 \leq \theta \leq 1$), if there exists a constant C such that one has $||A||_{\mathcal{L}(E,E)} \leq C\,||A||_{\mathcal{L}(E_0,E_0)}^{1-\theta}||A||_{\mathcal{L}(E_1,E_1)}^{\theta}$ for all $A \in \mathcal{L}(E_0, E_0) \cap \mathcal{L}(E_1, E_1)$.■

One is interested in general methods (or functors) which construct interpolation spaces from two arbitrary normed spaces (or BANACH spaces, or HILBERT spaces).

For two BANACH spaces E_0, E_1, the complex method consists in looking at the space of real analytic functions f with values in $E_0 + E_1$, defined on the open strip $0 < x < 1$, continuous on the closed strip $0 \leq x \leq 1$, and such that $f(i\,y)$ is bounded in E_0 and $f(1 + i\,y)$ is bounded in E_1, equipped with the norm $||f|| = \max\{\sup_y ||f(i\,y)||_0, \sup_y ||f(1 + i\,y)||_1\}$, and one defines $[E_0, E_1]_\theta$ for $0 < \theta < 1$ as the space of $a = f(\theta)$, with the norm $||a||_{[E_0,E_1]_\theta} = \inf_{f(\theta)=a} ||f||$. Of course such a space contains $E_0 \cap E_1$, as one can take f to be a constant function taking its value in $E_0 \cap E_1$. The interpolation property follows easily from the fact that if $A \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$, then $A\,f(x + i\,y)$ satisfies a similar property with the spaces F_0 and F_1, and therefore one has $||A\,a||_{[F_0,F_1]_\theta} \leq \max\{||A||_{\mathcal{L}(E_0,F_0)}, ||A||_{\mathcal{L}(E_1,F_1)}\}||a||_{[E_0,E_1]_\theta}$, and one may actually replace $\max\{||A||_{\mathcal{L}(E_0,F_0)}, ||A||_{\mathcal{L}(E_1,F_1)}\}$ by $||A||_{\mathcal{L}(E_0,F_0)}^{1-\theta}||A||_{\mathcal{L}(E_1,F_1)}^{\theta}$.

At least for the case of Jacques-Louis LIONS, one motivation for introducing interpolation spaces was the question of traces for variants of SOBOLEV spaces. For example, if $\Omega = R^N_+ = \{x \in R^N, x_N > 0\}$, and one wants to describe the trace on the boundary of a function $u \in W^{1,p}(\Omega)$, one may consider that $u \in L^p\big(0, \infty; W^{1,p}(R^{N-1})\big)$ and that $\frac{du}{dx_N} \in L^p\big(0, \infty; L^p(R^{N-1})\big)$, and he introduced a more general framework, which Jaak PEETRE also did independently so that it gave a joint article, where they considered (strongly measurable) fonctions defined on $(0, \infty)$ with values in $E_0 + E_1$ and such that $t^{\alpha_0}u \in L^{p_0}(0, \infty; E_0)$ and $t^{\alpha_1}\frac{du}{dt} \in L^{p_1}(0, \infty; E_1)$, and looked for the space spanned by $u(0)$ for a special range of parameters where $u(0)$ is automatically defined. It seems that this is a four parameters family of spaces, but changing t into t^β shows that three parameters are enough; it was Jaak PEETRE who finally showed that the family actually depends only upon two parameters, and after simplification it led to the K-method and the J-method that we are going to study.

[3] Mischa COTLAR, Argentinian-born mathematician, 1913. He works at Central University of Venezuela, Caracas.

[4] Nachman ARONSZAJN, Polish-born mathematician, 1907-1980. He emigrated to United States in 1948, and he worked in Laurence, Kansas, where I visited him during my first visit to US, in 1971.

21-724. Interpolation spaces

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02. Friday March 10, 2000.

Definition: Let E_0 and E_1 be two normed spaces, continuously imbedded into a topological vector space $\mathcal{E}$ so that one can define $E_0 \cap E_1$, equippped with the norm $||a||_{E_0 \cap E_1} = \max\{||a||_0, ||a||_1\}$, and $E_0 + E_1$, equippped with the norm $||a||_{E_0+E_1} = \inf_{a=a_0+a_1} ||a_0||_0 + ||a_1||_1$.

For $a \in E_0 + E_1$ and $t > 0$ one defines $K(t,a) = \inf_{a=a_0+a_1} ||a_0||_0 + t\,||a_1||_1$.

For $0 < \theta < 1$ and $1 \le p \le \infty$, or for $\theta = 0, 1$ with $p = \infty$, one defines $(E_0, E_1)_{\theta,p}$ as the space of $a \in E_0 + E_1$ such that $t^{-\theta} K(t,a) \in L^p\big(0,\infty; \frac{dt}{t}\big)$, equipped with the norm $||a||_{(E_0,E_1)_{\theta,p}} = ||t^{-\theta}K(t,a)||_{L^p(0,\infty;dt/t)}$.■

An idea of Emilio GAGLIARDO is to consider a plane with coordinates x_0, x_1 and for a given $a \in E_0 + E_1$ to mark all the points such that there exists a decomposition $a = a_0 + a_1$ with $||a_0||_0 \le x_0$ and $||a_1||_1 \le x_1$. This set is convex because if $a = b_0 + b_1$ with $||b_0||_0 \le y_0$ and $||b_1||_1 \le y_1$, then for $0 < \eta < 1$ one has $a = c_0 + c_1$ with $c_0 = (1-\eta)a_0 + \eta\, b_0$ and $c_1 = (1-\eta)a_1 + \eta\, b_1$ and the triangle inequality gives $||c_0||_0 \le (1-\eta)x_0 + \eta\, y_0$ and $||c_1||_1 \le (1-\eta)x_1 + \eta\, y_1$. Using the function $t \mapsto K(t,a)$ is one way of describing the boundary of this convex set.

For $t > 0$, $a \mapsto K(t,a)$ is a norm equivalent to the norm on $E_0 + E_1$. $K(t,a)$ is nondecreasing in t and $\frac{K(t,a)}{t}$ is nonincreasing in t, and moreover $K(t,a)$ is concave in t, as an infimum of affine functions, and therefore continuous. One can give a definition of the space involving a sum instead of an integral: on an interval $e^n \le t \le e^{n+1}$ one has $K(e^n,a) \le K(t,a) \le e\,K(e^n,a)$ for $n \in Z$, and as the measure of (e^n, e^{n+1}) for the measure $\frac{dt}{t}$ is 1, one sees that $a \in (E_0,E_1)_{\theta,p}$ if and only if $e^{-n\,\theta}K(e^n,a) \in l^p(Z)$, and $||e^{-n\,\theta}K(e^n,a)||_{l^p(Z)}$ is an equivalent norm on $(E_0,E_1)_{\theta,p}$.

Lemma: If $0 < \theta < 1$ and $1 \le p < q \le \infty$, one has $(E_0,E_1)_{\theta,p} \subset (E_0,E_1)_{\theta,q}$ (with continuous imbedding).
Proof: Using the definition using sums instead of integrals, one notices that l^p is increasing with p.

Another way to prove the same result is to notice that if $1 \le p < \infty$, and $t_0 > 0$ one has $K(t,a) \ge K(t_0,a)$ for $t > t_0$ and therefore $||a||^p_{(E_0,E_1)_{\theta,p}} \ge K(t_0,a)^p \int_{t_0}^\infty t^{-\theta\,p}\,\frac{dt}{t} = K(t_0,a)^p \frac{t_0^{-\theta\,p}}{\theta\,p}$, giving $t_0^{-\theta}K(t_0,a) \le C\,||a||_{(E_0,E_1)_{\theta,p}}$, and therefore $||t^{-\theta}K(t,a)||_{L^\infty(0,\infty,dt/t)} \le C\,||a||_{(E_0,E_1)_{\theta,p}}$, and by HÖLDER inequality one obtains $||a||_{(E_0,E_1)_{\theta,q}} = ||t^{-\theta}K(t,a)||_{L^q(0,\infty,dt/t)} \le C'\,||a||_{(E_0,E_1)_{\theta,p}}$ for $p \le q \le \infty$.■

Because for $a \in E_0 + E_1$ one has $K(t,a) \ge \min\{1,t\}||a||_{E_0+E_1}$, one sees that if $(E_0,E_1)_{\theta,p}$ is not reduced to 0 one must have $t^{-\theta}\min\{1,t\} \in L^p\big(0,\infty; \frac{dt}{t}\big)$, and therefore the space $(E_0,E_1)_{\theta,p}$ is reduced to 0 if $\theta < 0$ or if $\theta > 1$, and also in the cases $\theta = 0$ or $\theta = 1$, if $p < \infty$.

Because for $a \in E_0 \cap E_1$ one has the decompositions $a = a + 0$ and $a = 0 + a$, one finds that $K(t,a) \le \min\{1,t\}||a||_{E_0\cap E_1}$, and therefore for all the pairs (θ,p) which are considered one has $E_0 \cap E_1 \subset (E_0,E_1)_{\theta,p}$ (with continuous imbedding).

Although it will be important to characterize as much as possible what these interpolation spaces are in each context, the interpolation property comes automatically.

Proposition: If A is linear from $E_0 + E_1$ into $F_0 + F_1$ and maps E_0 into F_0 with $||A\,x||_{F_0} \le M_0||x||_{E_0}$ for all $x \in E_0$ and maps E_1 into F_1 with $||A\,x||_{F_1} \le M_1||x||_{E_1}$ for all $x \in E_1$, then A is linear continuous from $(E_0,E_1)_{\theta,p}$ into $(F_0,F_1)_{\theta,p}$ for all θ, p, and for $0 < \theta < 1$ one has $||A\,a||_{(F_0,F_1)_{\theta,p}} \le M_0^{1-\theta}M_1^\theta||a||_{(E_0,E_1)_{\theta,p}}$ for all $a \in (E_0,E_1)_{\theta,p}$.
Proof: For each decomposition $a = a_0 + a_1$ with $a_0 \in E_0$ and $a_1 \in E_1$, one has $A\,a = A\,a_0 + A\,a_1$, and $A\,a_0 \in F_0$ with $||A\,a_0||_{F_0} \le M_0||a_0||_{E_0}$ and $A\,a_1 \in F_1$ with $||A\,a_1||_{F_1} \le M_1||a_1||_{E_1}$. One deduces that $K(t,A\,a) \le ||A\,a_0||_{F_0} + t\,||A\,a_1||_{F_1} \le M_0||a_0||_{E_0} + t\,M_1||a_1||_{E_1} = M_0\big(||a_0||_{E_0} + \frac{t\,M_1}{M_0}||a_1||_{E_1}\big)$, and therefore one has $K(t,A\,a) \le M_0 K\big(\frac{t\,M_1}{M_0}, a\big)$. Then using $s = \frac{t\,M_1}{M_0}$ one deduces that $t^{-\theta}K(t,A\,a) \le M_0^{1-\theta}M_1^\theta s^{-\theta}K(s,a)$ and as $\frac{dt}{t} = \frac{ds}{s}$ one finds that $||A\,a||_{(F_0,F_1)_{\theta,p}} = ||t^{-\theta}K(t,A\,a)||_{L^p(0,\infty,dt/t)} \le M_0^{1-\theta}M_1^\theta||s^{-\theta}K(s,a)||_{L^p(0,\infty,dt/t)} = M_0^{1-\theta}M_1^\theta||a||_{(E_0,E_1)_{\theta,p}}$ for all $a \in (E_0,E_1)_{\theta,p}$.■

An important example is the case $E_0 = L^1(\Omega)$, $E_1 = L^\infty(\Omega)$, for which the corresponding interpolation spaces are the LORENTZ spaces[1], for $1 < p < \infty$ and $1 \le q \le \infty$ one denotes $L^{p,q}(\Omega) = \big(L^1(\Omega), L^\infty(\Omega)\big)_{1/p',q}$,

[1] LORENTZ had introduced these spaces before the Interpolation theories were developed.

and one will find that $L^{p,p}(\Omega) = L^p(\Omega)$ (with equivalent norms). For a function $f \in L^1(\Omega) + L^\infty(\Omega)$ one can calculate explicitly $K(t,f)$, and the formula makes use of the nonincreasing rearrangement of f.

For a measurable scalar function f on Ω such that for every $\lambda > 0$ one has $meas\{x \in \Omega, |f(x)| > \lambda\} < \infty$, one can define the nonincreasing rearrangement of f, denoted f^* (and extensively used by HARDY and LITTLEWOOD). It is the only (real) nonincreasing function defined on $\big(0, meas(\Omega)\big)$ which is equimeasurable to $|f|$, and it can be defined by $\lambda \in \big[f(t_+), f(t_-)\big]$ if and only if $meas\{x \in \Omega, |f(x)| > \lambda\} \le t \le meas\{x \in \Omega, |f(x)| \ge \lambda\}$. When necessary, one extends $f^*(t)$ to be 0 for $t > meas(\Omega)$. One basic property if that for any piecewise continuous function Φ defined on $[0,\infty)$ one has $\int_\Omega \Phi(|f(x)|)\,dx = \int_0^{meas(\Omega)} \Phi\big(f^*(t)\big)\,dt$.

Lemma: If $E_0 = L^1(\Omega)$ and $E_1 = L^\infty(\Omega)$ then for any function $f \in L^1(\Omega) + L^\infty(\Omega)$ one has $K(t,f) = \int_0^t f^*(s)\,ds$ for all $t > 0$ (extending f^* by 0 for $t > meas(\Omega)$).
Proof: If one decomposes $f = f_0 + f_1$ with $f_0 \in L^1(\Omega)$ and $||f_1||_{L^\infty(\Omega)} \le \lambda$ (and $\lambda > 0$), then the infimum of $||f_0||_{L^1(\Omega)}$ is obtained by taking $f_1(x) = f(x)$ whenever $|f(x)| \le \lambda$, and $f_1(x) = \lambda \frac{f(x)}{|f(x)|}$ whenever $|f(x)| > \lambda$, and this shows that $K(t,f) = \inf_{\lambda>0}\big(\int_{|f(x)|>\lambda}(|f(x)|-\lambda)\,dx + t\,\lambda\big) = \inf_{\lambda>0}\big(\int_{f^*(s)>\lambda}(f^*(s)-\lambda)\,ds + t\,\lambda\big)$. The infimum is attained for any λ in the interval $[f^*(t_+), f^*(t_-)]$ and is $\int_0^t f^*(s)\,ds$ (one extends f by 0 outside Ω and f^* by 0 for $t > meas(\Omega)$). Indeed let τ be such that $\lambda \in [f^*(\tau_+), f^*(\tau_-)]$, then $\int_{f^*(s)>\lambda}(f^*(s) - \lambda)\,ds + t\,\lambda = \int_0^\tau f^*(s)\,ds + \lambda(t-\tau)$, and it is enough to check that $\int_t^\tau f^*(s)\,ds + \lambda(t-\tau) \ge 0$ for all $\tau > 0$; this is a consequence of $f^*(s) \ge \lambda$ for $s < t$ and $f^*(s) \le \lambda$ for $s > t$.■

In order to compare two definitions of LORENTZ spaces, we shall use HARDY inequality.

Lemma: Let $1 \le q \le \infty$ and $\alpha < 1$. If $t^\alpha \varphi \in L^q\big(0,\infty,\frac{dt}{t}\big)$ and $\psi(t) = \frac{1}{t}\int_0^t \varphi(s)\,ds$ then one has $t^\alpha \psi \in L^q\big(0,\infty,\frac{dt}{t}\big)$ and $||\psi||_{L^q(0,\infty,dt/t)} \le \frac{1}{1-\alpha}||\varphi||_{L^q(0,\infty,dt/t)}$.
Proof: The case $q = \infty$ is obvious, because $|\varphi(t)| \le M\,t^{-\alpha}$ for all $t > 0$ implies $|\psi(t)| \le \frac{M\,t^{-\alpha}}{1-\alpha}$ for all $t > 0$.

For $1 \le q < \infty$, one uses the fact that $C_c(0,\infty)$ is dense in the space of φ such that $t^\alpha\varphi \in L^q\big(0,\infty,\frac{dt}{t}\big)$, so that one may assume that $\varphi \in C_c(0,\infty)$, in which case ψ vanishes near 0 and behaves as $\frac{C}{t}$ for t large. As ψ is of class C^1 and $t\,\psi'(t) + \psi(t) = \varphi(t)$, one multiplies by $t^{\alpha\,q}|\psi|^{q-2}\psi$ and one integrates against $\frac{dt}{t}$; one finds $\int_0^\infty t\,\psi' t^{\alpha\,q}|\psi|^{q-2}\psi\frac{dt}{t} = \frac{1}{q}\int_0^\infty t^{\alpha\,q}d|\psi|^q = -\alpha\int_0^\infty t^{\alpha\,q}|\psi|^q\,\frac{dt}{t}$, because $t^\alpha\psi(t)$ tends to 0 at ∞. This shows that $(1-\alpha)\int_0^\infty |t^\alpha\psi|^q\,\frac{dt}{t} = \int_0^\infty |t^\alpha\psi|^{q-2}t^\alpha\psi\,t^\alpha\varphi\,\frac{dt}{t}$, and HÖLDER inequality implies $(1-\alpha)||t^\alpha\psi|| \le ||t^\alpha\varphi||$, where the norm is that of $L^q\big(0,\infty,\frac{dt}{t}\big)$.■

Proposition: For $1 < p < \infty$ and $1 \le q \le \infty$ one has $L^{p,q}(\Omega) = \big(L^1(\Omega), L^\infty(\Omega)\big)_{1/p',q} = \big\{f \in L^1(\Omega) + L^\infty(\Omega), t^{1/p}f^*(t) \in L^q\big(0,\infty,\frac{dt}{t}\big)\big\}$, and $||t^{1/p}f^*||_{L^q(0,\infty,dt/t)}$ is an equivalent norm (and therefore $L^{p,p}(\Omega) = L^p(\Omega)$ with an equivalent norm). $L^{p,\infty}(\Omega)$ is the weak L^p space of MARCINKIEWICZ (with an equivalent norm).
Proof: The definition of the interpolation space would have $t^{-\theta}K(t,f) \in L^q\big(0,\infty,\frac{dt}{t}\big)$, with $\theta = \frac{1}{p'}$, and as $K(t,f) = \int_0^t f^*(s)\,ds \ge t\,f^*(t)$ because f^* is nonincreasing, it implies $t^{-\theta}K(t,f) \ge t^{1-\theta}f^*(t) = t^{1/p}f^*(t)$. Conversely, if $t^{1/p}f^* \in L^q\big(0,\infty,\frac{dt}{t}\big)$, then HARDY inequality implies $t^{1/p}\frac{1}{t}\int_0^t f^*(s)\,ds \in L^q\big(0,\infty,\frac{dt}{t}\big)$, because $\alpha = \frac{1}{p} < 1$, and $t^{1/p}\frac{1}{t}\int_0^t f^*(s)\,ds = t^{-\theta}K(t,f)$.

The definition of the weak L^p space of MARCINKIEWICZ is that there exists M such that for every measurable subset ω of Ω one has $\int_\omega |f|\,dx \le M\,meas(\omega)^{1/p'}$. The statement is then the consequence of the fact that for $t > 0$ one has $\sup_{meas(\omega)=t}\int_\omega |f|\,dx = \int_0^t f^*(s)\,ds$, and this is seen by choosing $\lambda \in [f^*(t_+), f^*(t_-)]$ and defining $\omega_0 = \{x, |f(x)| > \lambda\}$ and $\omega_1 = \{x : |f(x)| \ge \lambda$, so that $meas(\omega_0) \le t \le meas(\omega_1)$ (and $|f(x)| = t$ on $\omega_1 \setminus \omega_0$). If ω is not a subset of ω_1, one increases the integral of $|f|$ by replacing the part of ω which is not in ω_1 by a part of the same measure in $\omega_1 \setminus \omega$; if ω is a subset of ω_1 but does not contain ω_0, one increases the integral of $|f|$ by replacing a part of ω which is not in ω_0 by a corresponding part of the same measure in $\omega_0 \setminus \omega$, so that finally the subsets of measure t for which the integral of $|f|$ is maximum must contain ω_0 and be contained in ω_1.

03. Monday March 13, 2000.

We want to interpolate now between SOBOLEV spaces in order to define spaces like $H^s(\Omega)$ when s is not an integer. In the case where $\Omega = R^N$, where the definition is based on FOURIER transform, they are indeed interpolation spaces with the particular choice $p = 2$, and for $p \neq 2$ they belong to a larger family of spaces named after Oleg BESOV[1].

For $s \in R$, let F_s denote the space of (equivalence classes of) measurable functions v such that $(1 + 4\pi^2|\xi|^2|)^{s/2} v \in L^2(R^N)$, then the FOURIER transform $\mathcal{F}$ is an isometry from $H^s(R^N)$ onto F_s and the inverse FOURIER transform $\overline{\mathcal{F}}$ is an isometry from F_s onto $H^s(R^N)$. Therefore the interpolation property implies that $\mathcal{F}$ maps continuously $\big(H^\alpha(R^N), H^\beta(R^N)\big)_{\theta,p}$ into $(F_\alpha, F_\beta)_{\theta,p}$ and $\overline{\mathcal{F}}$ maps $(F_\alpha, F_\beta)_{\theta,p}$ into $\big(H^\alpha(R^N), H^\beta(R^N)\big)_{\theta,p}$ and therefore $\big(H^\alpha(R^N), H^\beta(R^N)\big)_{\theta,p}$ coincides with the (temperate) distributions whose FOURIER transform belongs to $(F_\alpha, F_\beta)_{\theta,p}$ (and one deduces in the same way that it is an isometry if one uses the corresponding norms).

Identifying interpolation spaces between SOBOLEV spaces $H^s(R^N)$ is then the same question than interpolating between some L^2 spaces with weights, and this new question can be settled easily in a more general setting.

Proposition: For a (measurable) positive function w on Ω, let $E(w) = \{u : \int_\Omega |u(x)|^2 w(x)\,dx < \infty\}$ with $||u||_w = \big(\int_\Omega |u(x)|^2 w(x)\,dx\big)^{1/2}$. If w_0, w_1 are two such functions, then for $0 < \theta < 1$ one has $\big(E(w_0), E(w_1)\big)_{\theta,2} = E(w_\theta)$ with proportional norms, where $w_\theta = w_0^{1-\theta} w_1^\theta$.
Proof: One uses a variant of the K functional adapted to L^2 spaces, namely $K_2(t,a) = \inf_{a=a_0+a_1}\big(||a_0||_0^2 + t^2||a_1||_1^2\big)^{1/2}$, and one checks immediately that $K_2(t,a) \le K(t,a) \le \sqrt{2}\,K_2(t,a)$ for all $a \in E_0 + E_1$, whatever the normed spaces E_0, E_1 of the abstract theory are.

For $E_0 = E(w_0)$ and $E_1 = E(w_1)$, for any $a \in E_0 + E_1$ and $t > 0$ one can calculate explicitly $K_2(t,a)$. Indeed $K_2(t,a)^2 = \inf_{a=a_0+a_1} \int_\Omega\big(|a_0(x)|^2 w_0(x) + t^2|a_1(x)|^2 w_1(x)\big)\,dx$, and one is led to choose for $a_0(x)$ the value λ which minimizes $|\lambda|^2 w_0(x) + t^2|a(x) - \lambda|^2 w_1(x)$, and as λ is characterized by $\lambda\, w_0(x) - t^2(a(x) - \lambda) w_1(x) = 0$, one finds $a_0(x) = \frac{t^2 w_1(x)}{w_0(x)+t^2 w_1(x)}\, a(x)$ and $a_1(x) = \frac{w_0(x)}{w_0(x)+t^2 w_1(x)}\, a(x)$ (which are measurable), and this optimal choice gives $|a_0(x)|^2 w_0(x) + t^2|a_1(x)|^2 w_1(x) = \frac{t^2 w_0(x) w_1(x)}{w_0(x)+t^2 w_1(x)}\, |a(x)|^2$, and therefore $K_2(t,a) = \Big(\int_\Omega \frac{t^2 w_0(x) w_1(x)}{w_0(x)+t^2 w_1(x)}\, |a(x)|^2\,dx\Big)^{1/2}$.

For $0 < \theta < 1$ one has $||t^{-\theta} K_2(t,a)||^2_{L^2(0,\infty,dt/t)} = \int_0^\infty \int_\Omega t^{-2\theta} \frac{t^2 w_0(x) w_1(x)}{w_0(x)+t^2 w_1(x)}\, |a(x)|^2\,dx\,\frac{dt}{t}$, which one computes by integrating in t first, by FUBINI's theorem. One makes the change of variable $t^2 = \frac{w_0(x)}{w_1(x)} s^2$, so that $\frac{dt}{t} = \frac{ds}{s}$ and one finds $\int_0^\infty t^{-2\theta} \frac{t^2 w_0(x) w_1(x)}{w_0(x)+t^2 w_1(x)}\,\frac{dt}{t} = w_0(x)^{1-\theta} w_1(x)^\theta \int_0^\infty \frac{t^{1-2\theta}}{1+t^2}\,dt$, and therefore one finds $||t^{-\theta} K_2(t,a)||_{L^2(0,\infty,dt/t)} = C\,\big(\int_\Omega |a(x)|^2 w_\theta(x)\,dx\big)^2$ with $C^2 = \int_0^\infty \frac{t^{1-2\theta}}{1+t^2}\,dt$ (i.e. $C^2 = \frac{\pi}{2\sin(\pi\,\theta)}$).■

Using the Theory of Interpolation, one can improve the HAUDORFF-YOUNG inequality[2], which asserts that the FOURIER transform maps $L^p(R^N)$ into $L^{p'}(R^N)$ if $1 \le p \le 2$, and this improvement uses LORENTZ spaces. Indeed $\mathcal{F}f(\xi) = \int_{R^N} f(x) e^{-2i\,\pi(x.\xi)}\,dx$ gives immediately $||\mathcal{F}f||_\infty \le ||f||_1$, where $||\cdot||_p$ denotes the L^p norm; on the other hand $||\mathcal{F}f||_2 = ||f||_2$ and therefore the interpolation property asserts that the FOURIER transform maps $\big(L^1(R^N), L^2(R^N)\big)_{\theta,p}$ into $\big(L^\infty(R^N), L^2(R^N)\big)_{\theta,p}$. The important reiteration theorem (of Jacques-Louis LIONS and Jaak PEETRE) will show that these spaces are in the family of LORENTZ spaces, and the result will then be that for $1 < p < 2$ and $1 \le q \le \infty$ the FOURIER transform maps $L^{p,q}(R^N)$ into $L^{p',q}(R^N)$, and in particular maps $L^p(R^N)$ into $L^{p',p}(R^N)$, which is a subspace of $L^{p'}(R^N)$ because $p < p'$.

[1] Oleg V. BESOV, Russian mathematician. He works at the STEKLOV Institute of Mathematics of the Russian Academy of Sciences, Moscow.

Vladimir Andreevich STEKLOV, Russian mathematician, 1864–1926.

[2] YOUNG had proved the result when p' is an even integer, and HAUSDORFF had proved the general case.

Results concerning convolution can also be improved using the Theory of Interpolation and LORENTZ spaces, and in particular the SOBOLEV imbedding theorem can be improved, as noticed by Jaak PEETRE. The classical result is that for $1 \le p < N$ one has $W^{1,p}(R^N) \subset L^{p^*}(R^N)$ with $p^* = \frac{N\,p}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, and this will be improved for $1 < p < N$ into $W^{1,p}(R^N) \subset L^{p^*,p}(R^N)$, which is included in $L^{p^*}(R^N)$. The original proof of SOBOLEV used a convolution formula $u = \sum_j \frac{\partial u}{\partial x_j} \star \frac{\partial E}{\partial x_j}$ for an elementary solution E of Δ, and as one can take $E = \frac{C_N}{|x|^{N-2}}$ for $N \ge 3$ and $E = C_2 \log(|x|)$ for $N = 2$, one finds that $\frac{\partial E}{\partial x_j} \in L^{N',\infty}(R^N)$. Together with the reiteration theorem and a duality theorem (also of Jacques-Louis LIONS and Jaak PEETRE), which asserts[3] that $L^{N',\infty}(R^N)$ is the dual of $L^{N,1}(R^N)$, one finds that for $1 < p < N$ and $1 \le q \le \infty$, convolution of $L^{p,q}(R^N)$ by $L^{N',\infty}(R^N)$ gives a result in $L^{p^*,q}(R^N)$.

Unfortunately, this argument does not give SOBOLEV imbedding theorem for $p = 1$, or the improvement that $W^{1,1}(R^N)$ is continuously imbedded in $L^{1^*,1}(R^N)$, which is indeed true. The reason is that convolution of $L^1(R^N)$ by any LORENTZ space $L^{a,b}(R^N)$ gives a result in $L^{a,b}(R^N)$ and not better, as one can approach the DIRAC mass at 0 by a bounded sequence in $L^1(R^N)$.

This is something that one should be aware of, that different ways of using the Theory of Interpolation may lead to results in different interpolation spaces, usually differing only in the second parameter.

The usual scaling arguments, for example, are insensitive to the second parameter for the LORENTZ spaces, and cannot be used to check that a given result is optimal. For example, if $u \in L^1(R^N) + L^\infty(R^N)$ and for $\lambda \ne 0$ let U be defined by $U(x) = u(\lambda\,x)$ for $x \in R^N$, then any decomposition of $u = a_0 + a_1$ with $a_0 \in L^1(R^N)$ and $a_1 \in L^\infty(R^N)$ gives a decomposition $U = A_0 + A_1$ with $A_j(x) = a_j(\lambda\,x)$ for $x \in R^N$ and $j = 1,2$. Then one has $||A_0||_{L^1(R^N)} = |\lambda|^{-N}||a_0||_{L^1(R^N)}$ and $||A_1||_{L^\infty(R^N)} = ||a_1||_{L^\infty(R^N)}$, and therefore $K(t,U) = |\lambda|^{-N}K(t\,|\lambda|^N, u)$, from which one deduces that $||U||_{L^{p,q}(R^N)} = |\lambda|^{-N/p}||u||_{L^{p,q}(R^N)}$, so that the parameter q does not appear explicitly in the way the norm changes.

[3] The result of Jacques-Louis LIONS and Jaak PEETRE is valid for general BANACH spaces, and the particular result for LORENTZ spaces may have been known before.

21-724. Interpolation spaces
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04. Wednesday March 15, 2000.

Another interpolation method is the J-method. It is in some way a dual method compared to the K-method.

The K-method is the natural result of investigations which originated in questions of traces: if $u \in L^{p_0}(0,\infty;E_0)$ and $u' \in L^{p_1}(0,\infty;E_1)$ with $1 \le p_0, p_1 \le \infty$, then $u \in C^0([0,1];E_0+E_1)$ and therefore $u(0)$ exists and the question is to characterize the space of such values at 0 (traces). As one can change $u(t)$ in $u(t^\lambda)$ with $\lambda > 0$ and not change $u(0)$, one then discovers naturally that one can consider spaces of functions such that $t^{\alpha_0}u \in L^{p_0}(0,\infty;E_0)$ and $t^{\alpha_1}u' \in L^{p_1}(0,\infty;E_1)$, and for some set of parameters $u(0)$ exists. These ideas may have started with Emilio GAGLIARDO, and I do not know if he had first identified the traces of functions from $W^{1,p}(R^N)$ on an hyperplane before or after thinking of the general framework, but certainly Jacques-Louis LIONS and Jaak PEETRE perfected the framework, and the K-method is Jaak PEETRE's further simplification, which shows that the preceding family only depends upon two parameters.

If one wants to characterize the duals of the spaces obtained, then one finds easily that these dual spaces are naturally defined as integrals, and one considers then questions like that of identifying which are the elements $a \in E_0+E_1$ which can be written as $\int_0^\infty v(t)\,dt$ where $t^{\beta_0}v \in L^{q_0}(0,\infty;E_0)$ and $t^{\beta_1}v \in L^{q_1}(0,\infty;E_1)$, for the range of parameters where the integral is defined. Again, looking at $v(t^\lambda)$ shows that there are not really four parameters, but one important observation is that these spaces are (almost) the same than the ones defined by traces, and I do not know if Emilio GAGLIARDO had investigated such questions before the basic work of Jacques-Louis LIONS and Jaak PEETRE. The J-method is then the simplification by Jaak PEETRE of the preceding framework.

Definition: For $v \in E_0 \cap E_1$ and $t > 0$, one denotes $J(t,v) = \max\{||v||_0, t||v||_1\}$.■

The case $t = 1$ corresponds to the usual norm on $E_0 \cap E_1$, which makes both injections into E_0 or E_1 continuous and with norms at most 1. $J(t,v)$ gives then a family of equivalent norms on $E_0 \cap E_1$.

Definition: For $0 < \theta < 1$ and $1 \le p \le \infty$, or for $\theta = 0,1$ and $p = 1$, one defines $(E_0,E_1)_{\theta,p;J}$ as the space of $a \in E_0+E_1$ which can be written as $a = \int_0^\infty v(t)\,\frac{dt}{t}$ with $v(t) \in E_0 \cap E_1$ for almost all $t > 0$ and satisfying $t^{-\theta}J\big(t,v(t)\big) \in L^p\big(0,\infty;\frac{dt}{t}\big)$. One defines $||a||_{\theta,p;J} = \inf_v ||t^{-\theta}J(t,v)||_{L^p(0,\infty;dt/t)}$, the infimum being taken among all possible v whose corresponding integral gives a.■

As every $a \in E_0 \cap E_1$ can be written as $a = \int_0^\infty \varphi(t)a\,\frac{dt}{t}$ with φ having compact support in $(0,\infty)$ and satisfying $\int_0^\infty \varphi(t)\,\frac{dt}{t} = 1$, one could consider other values of θ, p, but the infimum of $||t^{-\theta}J(t,v)||_{L^p(0,\infty;dt/t)}$ would be 0 in these cases. Indeed one may replace φ by $\varphi(\lambda\,t)$ and let λ tend to ∞, and the infimum tends to 0 if $\theta < 0$ or if $\theta = 0$ and $p > 1$; similarly letting λ tend to 0 the infimum tends to 0 if $\theta > 1$ or if $\theta = 1$ and $p > 1$.

The important property is the following equivalence result, which says that apart from the extreme cases $\theta = 0,1$ where the two methods use different values of p anyway, the J-method gives the same spaces than the K-method.

Proposition: For $0 < \theta < 1$ and $1 \le p \le \infty$, the J-method gives the same spaces than the K-method, with equivalent norms.

Proof: Let $a \in (E_0,E_1)_{\theta,p;J}$, so that $a = \int_0^\infty u(s)\,\frac{ds}{s}$ with $s^{-\theta}J\big(s,u(s)\big) \in L^p\big(0,\infty,\frac{ds}{s}\big)$. As $a \mapsto K(t,a)$ is a norm, one deduces that $K(t,a) \le \int_0^\infty K\big(t,u(s)\big)\,\frac{ds}{s} \le \int_0^\infty \min\{||u(s)||_0, t\,||u(s)||_1\}\,\frac{ds}{s}$, because for $u \in E_0 \cap E_1$ one has the decompositions $u = u+0 = 0+u$ and therefore $K(t,u) \le \min\{||u||_0, t\,||u||_1\}$. Because for $u \in E_0 \cap E_1$ one has $||u||_0 \le J(s,u)$ and $||u||_1 \le \frac{1}{s}J(s,u)$, one deduces that $\min\{||u(s)||_0, t\,||u(s)||_1\} \le \min\{1,\frac{t}{s}\}J\big(s,u(s)\big)$, and therefore $t^{-\theta}K(t,a) \le \int_0^\infty \min\big\{\big(\frac{t}{s}\big)^{-\theta},\big(\frac{t}{s}\big)^{1-\theta}\big\}s^{-\theta}J\big(s,u(s)\big)\,\frac{ds}{s}$. This is a convolution product for the multiplicative group $(0,\infty)$ with the HAAR measure $\frac{dt}{t}$, of the function $t^{-\theta}J\big(t,u(t)\big)$ which belongs to $L^p\big(0,\infty;\frac{dt}{t}\big)$ and the function $\min\{t^{-\theta},t^{1-\theta}\}$, which belongs to $L^1\big(0,\infty;\frac{dt}{t}\big)$, and therefore one has $||t^{-\theta}K(t,a)||_{L^p(0,\infty;dt/t)} \le C\,||t^{-\theta}J(t,u(t))||_{L^p(0,\infty;dt/t)}$, proving that $(E_0,E_1)_{\theta,p;J} \subset (E_0,E_1)_{\theta,p}$ with continuous imbedding.

In order to prove the opposite continuous imbedding, one must start from $a \in (E_0, E_1)_{\theta,p}$ and construct $u(t) \in E_0 \cap E_1$ such that $a = \int_0^\infty u(t)\,\frac{dt}{t}$ and $t^{-\theta} J\big(t, u(t)\big) \in L^p\big(0, \infty; \frac{dt}{t}\big)$, and for that it is enough to ensure that one can construct such a u satisfying $J\big(t, u(t)\big) \le C\,K(t,a)$ for all $t > 0$. This fact is true in a slightly more general context, one chooses $u(t) = u_n$ for $e^n < t < e^{n+1}$, so that $a = \int_0^\infty u(t)\,\frac{dt}{t}$ means $a = \sum_{-\infty}^{+\infty} u_n$, and the basic construction is shown in the following Lemma. Because $a \in (E_0, E_1)_{\theta,p} \subset (E_0, E_1)_{\theta,\infty}$, one has $K(t,a) \le C\,t^\theta$ and the hypothesis of the Lemma is indeed satisfied.■

Lemma: If $a \in E_0 + E_1$ satisfies $K(t,a) \to 0$ as $t \to 0$ and $\frac{K(t,a)}{t} \to 0$ as $t \to \infty$, then for $n \in Z$ there exists $u_n \in E_0 \cap E_1$ such that the function u defined by $u(t) = u_n$ for $e^n < t < e^{n+1}$ satisfies $a = \int_0^\infty u(t)\,\frac{dt}{t}$ and $J\big(t, u(t)\big) \le C\,K(t,a)$ for all $t > 0$ (for a universal constant C).
Proof: Let $C_0 > 1$, and for each $n \in Z$ let $a = a_{0,n} + a_{1,n}$ with $a_{0,n} \in E_0$, $a_{1,n} \in E_1$ and $||a_{0,n}||_0 + e^n ||a_{1,n}||_1 \le C_0 K(e^n, a)$. In particular $||a_{0,n}||_0 \to 0$ as $n \to -\infty$ and $||a_{1,n}||_1 \to 0$ as $n \to +\infty$.

Let $u_n = a_{0,n+1} - a_{0,n} = a_{1,n} - a_{1,n+1}$, so that $u_n \in E_0 \cap E_1$, and for $i < j$ one has $u_i + \ldots + u_j = a_{0,j+1} - a_{0,i} = a - a_{1,j+1} - a_{0,i}$, which converges to a in $E_0 + E_1$ as $i \to -\infty$ and $j \to +\infty$ because $a_{1,j+1} \to 0$ in E_1 as $j \to +\infty$ and $a_{0,i} \to 0$ in E_0 as $n \to -\infty$. Because $K(t,a)$ is nondecreasing in t and $\frac{K(t,a)}{t}$ is nonincreasing in t, one has $K(e^n, a) \le K(t,a) \le K(e^{n+1}, a)$ and $\frac{t}{e^{n+1}} K(e^{n+1}, a) \le K(t,a) \le \frac{t}{e^n} K(e^n, a)$ for $e^n < t < e^{n+1}$. One has $||u_n||_0 \le ||a_{0,n+1}||_0 + ||a_{0,n}||_0 \le C_0 K(e^{n+1}, a) + C_0 K(e^n, a) \le C_0(1+e) K(t,a)$, and $t\,||u_n||_1 \le t\,||a_{1,n+1}||_1 + t\,||a_{1,n}||_1 \le C_0 \frac{t}{e^{n+1}} K(e^{n+1}, a) + C_0 \frac{t}{e^n} K(e^n, a) \le C_0(1+e) K(t,a)$, and therefore $J\big(t, u(t)\big) \le C_0(1+e) K(t,a)$.■

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Definition: For $0 \leq \theta \leq 1$, one says that a normed space E is of class $\mathcal{K}(\theta)$ if $E_0 \cap E_1 \subset E \subset (E_0, E_1)_{\theta,\infty;K}$, that E is of class $\mathcal{J}(\theta)$ if $(E_0, E_1)_{\theta,1;J} \subset E \subset E_0 + E_1$, and that E is of class $\mathcal{H}(\theta)$ if $(E_0, E_1)_{\theta,1;J} \subset E \subset (E_0, E_1)_{\theta,\infty;K}$.■

Of course, for $0 < \theta < 1$ the indices J and K may be dropped as the two interpolation methods give the same spaces, but for the extreme values $\theta = 0, 1$ the only interpolation spaces that we use correspond to $p = 1$ for the J-method or $p = \infty$ for the K-method.

The reiteration theorem will state that if F_0 is of class $\mathcal{H}(\theta_0)$ and F_1 is of class $\mathcal{H}(\theta_1)$ with $\theta_0 \neq \theta_1$, then for $0 < \theta < 1$ and $1 \leq p \leq \infty$ one has $(F_0, F_1)_{\theta,p} = (E_0, E_1)_{\eta,p}$ with $\eta = (1-\theta)\theta_0 + \eta\,\theta_1$. Therefore if $F_0 = (E_0, E_1)_{\theta_0,p_0}$ and $F_1 = (E_0, E_1)_{\theta_1,p_1}$, the interpolation space $(F_0, F_1)_{\theta,p}$ is the same, whatever the precise values p_0, p_1 are, if $\theta_0 \neq \theta_1$, but the interpolation spaces do depend upon p_0, p_1 in the case $\theta_0 = \theta_1$, and in that case they may be new spaces, i.e. not included in the family indexed by θ, p.

Proposition: A normed space E is of class $\mathcal{K}(\theta)$ if and only if $E_0 \cap E_1 \subset E$ and there exists C such that $K(t, a) \leq C\, t^\theta ||a||_E$ for all $t > 0$ and all $a \in E$.

A normed space E is of class $\mathcal{J}(\theta)$ if and only if $E \subset E_0 + E_1$ and there exists C such that $||a||_E \leq C\, t^{-\theta} J(t, a)$ for all $t > 0$ and all $a \in E_0 \cap E_1$, or if and only if there exists C such that $||a||_E \leq C\, ||a||_0^{1-\theta} ||a||_1^\theta$ for all $a \in E_0 \cap E_1$.

Proof: As one must have $||a||_{(E_0,E_1)_{\theta,\infty;K}} \leq C\, ||a||_E$ for all $a \in E$, and $||a||_{(E_0,E_1)_{\theta,\infty;K}} = ||t^{-\theta} K(t, a)||_{L^\infty(0,\infty)}$, the condition is the same than $K(t, a) \leq C\, t^\theta ||a||_E$ for all $t > 0$.

One must have $||a||_E \leq C\, ||a||_{(E_0,E_1)_{\theta,1;J}}$ for all $a \in E$, and the necessary condition follows from the fact that for $a \in E_0 \cap E_1$ one has $||a||_{(E_0,E_1)_{\theta,1;J}} \leq t^{-\theta} J(t, a)$ for all $t > 0$. Indeed if $\varphi \in L^1\big(0, \infty; \frac{dt}{t}\big)$ and $\int_0^\infty \varphi(t)\, \frac{dt}{t} = 1$, then every $a \in E_0 \cap E_1$ can be written as $a = \int_0^\infty u(t)\, \frac{dt}{t}$ with $u(t) = \varphi(t) a$ for $t > 0$, and one has $||a||_{(E_0,E_1)_{\theta,1;J}} \leq \int_0^\infty t^{-\theta} |\varphi(t)| J(t, a)\, \frac{dt}{t}$. For $t_0 > 0$, one takes a sequence φ_n converging to $t_0 \delta_{t_0}$ (for example $\varphi_n(t) = n$ if $t_0 < t < \frac{n+1}{n} t_0$ and $\varphi_n(t) = 0$ otherwise), and one obtains at the limit $||a||_{(E_0,E_1)_{\theta,1;J}} \leq t_0^{-\theta} J(t_0, a)$. Having shown that $||a||_E \leq C\, t^{-\theta} J(t, a) = C \max\{||a||_0, t\, ||a||_1\}$ for all $a \in E_0 \cap E_1$ and all $t > 0$, one takes the minimum in t, which is attained for $t = \frac{||a||_0}{||a||_1}$ and therefore $||a||_E \leq C\, t^{-\theta} J(t, a)$ for all $a \in E_0 \cap E_1$ and all $t > 0$ is equivalent to $||a||_E \leq C\, ||a||_0^{1-\theta} ||a||_1^\theta$ for all $a \in E_0 \cap E_1$. Conversely, assume that there exists a constant C such that $||a||_E \leq C\, t^{-\theta} J(t, a)$ for all $a \in E_0 \cap E_1$ and all $t > 0$; for $b \in (E_0, E_1)_{\theta,1;J}$ and a decomposition $b = \int_0^\infty u(t)\, \frac{dt}{t}$ with $u(t) \in E_0 \cap E_1$ and $\int_0^\infty t^{-\theta} J\big(t, u(t)\big)\, \frac{dt}{t} < \infty$, one has $||b||_E \leq \int_0^\infty ||u(t)||_E\, \frac{dt}{t} \leq \int_0^\infty C\, t^{-\theta} J\big(t, u(t)\big)\, \frac{dt}{t}$, and taking the infimum on all decompositions of b one deduces that $||b||_E \leq C ||b||_{(E_0,E_1)_{\theta,1;J}}$.■

The preceding observation of Jacques-Louis LIONS and Jaak PEETRE is very useful, and must often be used with the reiteration theorem. For example, SOBOLEV space $H^{1/2}(R)$ is not imbedded in $L^\infty(R)$, but the slightly smaller interpolation space $\big(H^1(R), L^2(R)\big)_{1/2,1}$ is imbedded into $C_0(R)$, the space of continuous functions tending to 0 at ∞, because of the fact that $H^1(R) \subset C_0(R)$ with the precise estimate $||u||_{L^\infty(R)} \leq ||u||_{L^2(R)}^{1/2} ||u'||_{L^2(R)}^{1/2}$; then by using the reiteration theorem, one finds that for $0 < s < \frac{1}{2}$ the space $H^s(R)$ is continuously imbedded in the LORENTZ space $L^{p(s),2}$ with $\frac{1}{p(s)} = \frac{1}{2} - s$.

Therefore, one should be aware that some results which are not true for limiting cases, like SOBOLEV imbedding theorems, may be obtained by the Theory of Interpolation because the limiting case is actually true if one uses a slightly different space, and the difference does not matter that much because of the reiteration theorem.

Interverting the order of the spaces is a special case of the reiteration theorem but can be seen easily.

Lemma: If $F_0 = E_1$ and $F_1 = E_0$, then $(F_0, F_1)_{\theta,p} = (E_0, E_1)_{1-\theta,p}$.

Proof: Denoting $K_F(t, a)$ the K functional using the spaces F_0, F_1, for any decomposition $a = a_0 + a_1$ with $a_0 \in E_0$ and $a_1 \in E_1$, one has $K_F(t, a) = \inf\big(||a_1||_1 + t\, ||a_0||_0\big) = t \inf\big(||a_0||_0 + \frac{1}{t} ||a_1||_1\big) = t\, K\big(\frac{1}{t}, a\big)$, and therefore the change of variable $t = \frac{1}{s}$ gives $||t^{-\theta} K_F(t, a)||_{L^p(0,\infty;dt/t)} = ||s^{1-\theta} K(s, a)||_{L^p(0,\infty;dt/t)}$.■

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The theorem of reiteration is proved in two steps.

Proposition: If $G_0 \subset (E_0, E_1)_{\theta_0,\infty}$ and $G_1 \subset (E_0, E_1)_{\theta_1,\infty}$ with $\theta_0 \neq \theta_1$ (and continuous imbeddings), then for $0 < \theta < 1$ and $1 \le p \le \infty$ one has $(G_0, G_1)_{\theta,p} \subset (E_0, E_1)_{\eta,p}$ with $\eta = (1-\theta)\theta_0 + \theta\,\theta_1$.
Proof: One uses the K-method and the fact that for $t > 0$ one has $||t^{-\theta_0} K(t, a)||_{L^p(0,\infty;dt/t)} \le C_0\,||a||_{G_0}$ for all $a \in G_0$, and in particular one has $K(t, g_0) \le C_0 t^{\theta_0} ||g_0||_{G_0}$ for all $g_0 \in G_0$; similarly one has $K(t, g_1) \le C_1 t^{\theta_1} ||g_1||_{G_1}$ for all $g_1 \in G_1$.

For $a \in G_0 + G_1$, let $K_G(t, a) = \inf_{a=g_0+g_1} ||g_0||_{G_0} + t\,||g_1||_{G_1}$, then $K(t, a) \le K(t, g_0) + K(t, g_1) \le C_0 t^{\theta_0} ||g_0||_{G_0} + C_1 t^{\theta_1} ||g_1||_{G_1}$, and minimizing among all decompositions of a one deduces that $K(t, a) \le C_0 t^{\theta_0} K_G\big(\frac{C_1}{C_0} t^{\theta_1-\theta_0}, a\big)$ and therefore $t^{-\eta}K(t, a) \le C_0 t^{\theta_0-\eta} K_G\big(\frac{C_1}{C_0} t^{\theta_1-\theta_0}, a\big)$. As $\theta_1 \neq \theta_0$ one may use the change of variable $s = t^{\theta_1-\theta_0}$ (which gives $\frac{ds}{s} = (\theta_1 - \theta_0)\frac{dt}{t}$), and as $s^{-\theta} = t^{-\theta(\theta_1-\theta_0)} = t^{\theta_0-\eta}$ one finds that $t^{-\eta}K(t, a) \le s^{-\theta} K_G\big(\frac{C_1}{C_0} s, a\big)$, and therefore $a \in (G_0, G_1)_{\theta,p}$ implies $a \in (E_0, E_1)_{\eta,p}$.∎

Proposition: If $(E_0, E_1)_{\theta_0,1} \subset H_0$ and $(E_0, E_1)_{\theta_1,1} \subset H_1$ with $\theta_0 \neq \theta_1$ (and continuous imbeddings), then for $0 < \theta < 1$ and $1 \le p \le \infty$ one has $(E_0, E_1)_{\eta,p} \subset (H_0, H_1)_{\theta,p}$ with $\eta = (1-\theta)\theta_0 + \theta\,\theta_1$.
Proof: One uses the J-method and the fact that for $t > 0$ one has $||u||_{H_0} \le C_0\,||u||_{(E_0,E_1)_{\theta_0,1}} \le C_0\,t^{-\theta_0} J(t, u)$ and $||u||_{H_1} \le C_1\,||u||_{(E_0,E_1)_{\theta_1,1}} \le C_1\,t^{-\theta_1} J(t, u)$ for all $u \in E_0 \cap E_1$.

For $a \in (E_0, E_1)_{\eta,p}$ one has $a = \int_0^\infty u(t)\,\frac{dt}{t}$ with $u(t) \in E_0 \cap E_1$ and $t^{-\eta} J\big(t, u(t)\big) \in L^p\big(0, \infty; \frac{dt}{t}\big)$. One chooses $\lambda = \theta_1 - \theta_0$, and because $u(t) \in E_0 \cap E_1 \subset H_0 \cap H_1$, one can estimate $J_H\big(t^\lambda, u(t)\big) = \max\{||u(t)||_{H_0}, t^\lambda\,||u(t)||_{H_1}\}$, but as $||u(t)||_{H_0} \le C_0 t^{-\theta_0} J\big(t, u(t)\big)$ and $||u(t)||_{H_1} \le C_1 t^{-\theta_1} J\big(t, u(t)\big)$, one finds that $\max\{||u(t)||_{H_0}, t^\lambda\,||u(t)||_{H_1}\} \le C\,t^{-\theta_0} J\big(t, u(t)\big)$, with $C = \max\{C_0, C_1\}$. One has $(t^\lambda)^{-\theta} J_H\big(t^\lambda, u(t)\big) \le C\,t^{-\eta} J\big(t, u(t)\big)$ because $-\lambda\,\theta - \theta_0 = -(\theta_1-\theta_0)\theta - \theta_0 = -(1-\theta)\theta_0 - \theta\,\theta_1 = -\eta$. If one defines $v(t^\lambda) = u(t)$, one finds that $a = \int_0^\infty u(t)\,\frac{dt}{t} = \lambda \int_0^\infty v(t)\,\frac{dt}{t}$, and $t^{-\theta} J_H\big(t, v(t)\big) \in L^p\big(0, \infty; \frac{dt}{t}\big)$, showing that $a \in (H_0, H_1)_{\theta,p}$.∎

Corollary: If $\theta_1 \neq \theta_0$, $(E_0, E_1)_{\theta_0,1} \subset F_0 \subset (E_0, E_1)_{\theta_0,\infty}$ and $(E_0, E_1)_{\theta_1,1} \subset F_1 \subset (E_0, E_1)_{\theta_1,\infty}$, then for $0 < \theta < 1$ and $1 \le p \le \infty$ one has $(F_0, F_1)_{\theta,p} = (E_0, E_1)_{\eta,p}$ with $\eta = (1-\theta)\theta_0 + \theta\,\theta_1$, with equivalent norms.∎

As an application, let us consider the limiting case of SOBOLEV imbedding theorem in R^2, where $H^1(R^2)$ is not imbedded in $L^\infty(R^2)$ but nevertheless for $0 < s < 1$ the space $H^s(R^2) = \big(H^1(R^2), L^2(R^2)\big)_{1-s,2}$ is actually imbedded into $\big(L^\infty(R^2), L^2(R^2)\big)_{1-s,2}$, which is $L^{a(s),2}(R^2)$ with $\frac{1}{a(s)} = \frac{1-s}{2}$ by the reiteration theorem.

The result follows from the fact that $X = \big(H^2(R^2), L^2(R^2)\big)_{1/2,1} \subset L^\infty(R^2)$ (and actually $\subset \mathcal{F}L^1(R^2) \subset C_0(R^2)$, the space of continuous functions converging to 0 at ∞). As both X and $H^1(R^2)$ are of class $\mathcal{H}(1/2)$ for $E_0 = H^2(R^2)$ and $E_1 = L^2(R^2)$, one has $H^s(R^2) = \big(H^1(R^2), L^2(R^2)\big)_{1-s,2} = \big(X, L^2(R^2)\big)_{1-s,2}$ by the reiteration theorem, and therefore $\subset \big(L^\infty(R^2), L^2(R^2)\big)_{1-s,2} = L^{a(s),2}(R^2)$.

For $s > \frac{N}{2}$ one has $H^s(R^N) \subset \mathcal{F}L^1(R^N) \subset C_0(R^N)$, because $u \in H^s(R^N)$ implies $(1+|\xi|)^s \mathcal{F}u \in L^2(R^N)$ and as $(1 + |\xi|)^{-s} \in L^2(R^N)$, one deduces that $\mathcal{F}u \in L^1(R^N)$. Because $H^2(R^2) \subset L^\infty(R^2)$, one deduces that $||u||_{L^\infty(R^2)} \le C\,||D^2u||_{L^2(R^2)} + C\,||u||_{L^2(R^2)}$ for all $u \in H^2(R^2)$, and by rescaling, i.e. applying the inequality to $u(x) = v(\lambda\,x)$ one deduces that $||u||_{L^\infty(R^2)} \le C\,|\lambda|\,||D^2u||_{L^2(R^2)} + \frac{C}{|\lambda|}||u||_{L^2(R^2)}$, and taking the best λ gives $||u||_{L^\infty(R^2)} \le C'\,||D^2u||^{1/2}_{L^2(R^2)}||u||^{1/2}_{L^2(R^2)}$; this implies $||u||_{L^\infty(R^2)} \le C'\,||u||^{1/2}_{H^2(R^2)}||u||^{1/2}_{L^2(R^2)}$ which is equivalent to $\big(H^2(R^2), L^2(R^2)\big)_{1/2,1} \subset L^\infty(R^2)$ (the same scaling argument works with $L^\infty(R^2)$ replaced by $\mathcal{F}L^1(R^2)$).

Another way to obtain the same result is to let $A = ||u||_{L^2(R^2)}$, $B = ||u||_{H^2(R^2)}$, and for $\rho > 0$ to bound $\int_{|\xi|\le\rho} |\mathcal{F}u|\,d\xi$ by $A\,\big(\int_{|\xi|\le\rho} d\xi\big)^{1/2} = \sqrt{\pi}\,A\,\rho$ and $\int_{|\xi|\ge\rho} |\mathcal{F}u|\,d\xi$ by $\big(\int_{|\xi|\ge\rho} |\xi|^4 |\mathcal{F}u|^2\,d\xi\big)^{1/2}\big(\int_{|\xi|\ge\rho} |\xi|^{-4}|\,d\xi\big)^{1/2} \le \frac{C\,B}{\rho}$, and therefore $||\mathcal{F}u||_{L^1(R^2)} \le C\big(A\,\rho + \frac{B}{\rho}\big)$ and the best ρ gives $||\mathcal{F}u||_{L^1(R^2)} \le C'\sqrt{A\,B}$.

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Before a general Theory of Interpolation had been developed, for which the interpolation property is proved for linear continuous mappings, some nonlinear interpolation method had already been used, for example for proving that the maximal function maps $L^p(R^N)$ into itself for $1 < p \leq \infty$. Probably because this classical proof is well known to specialists of Harmonic Analysis, or experts in the theory of Singular Integrals, they rarely mention the Theory of Interpolation when they use this type of argument.

Definition: For $f \in L^1_{loc}(R^N)$, the maximal function $M\,f$ is defined by $M\,f(x) = \sup_{r>0} \frac{\int_{B(x,r)} |f(y)|\,dy}{|B(x,r)|}$, where $B(x,r)$ is the ball centered at x and with radius r, and $|B(x,r)|$ is its volume.■

The concept was introduced by HARDY and LITTLEWOOD, who proved the following result in dimension 1, the general case being due to WIENER[1]

Proposition: If $1 < p \leq \infty$, then $f \in L^p(R^N)$ implies $M\,f \in L^p(R^N)$, with $||M\,f||_p \leq C(p)||f||_p$ for all $f \in L^p(R^N)$ and $C(p) \to \infty$ as $p \to 1$.■

This will be proved below, but the fact that the result is not true for $p = 1$ and that $C(p)$ must tend to ∞ as p tends to 1 is seen easily by condidering for f the characteristic function of the unit ball, for which one has $M\,f(x) \geq \frac{1}{(1+|x|)^N}$ because for $r = 1 + |x|$ one has $\int_{B(x,r)} |f(y)|\,dy = |B(0,1)|$, because $B(x,r)$ contains $B(0,1)$, and $|B(x,r)| = r^N|B(0,1)|$. Therefore $M\,f$ does not belong to $L^1(R^N)$, and as $||M\,f||_p \to \infty$ as $p \to 1$, one must have $C(p) \to \infty$.

The same argument shows that if $f \in L^1(R^N)$ and $f \neq 0$, then $M\,f$ is never in $L^1(R^N)$ as it is bounded below by $\frac{C}{|x|^N}$ for $|x|$ large, and in this case one has a result involving the weak L^1 space, which has a definition analogous to that of the MARCINKIEWICZ spaces for $p > 1$, which coincide with the LORENTZ spaces $L^{p,\infty}$ for $p > 1$, but the weak L^1 space is not included in the family of interpolation spaces between $L^1(R^N)$ and $L^\infty(R^N)$, as it is not in $L^1(R^N) + L^\infty(R^N)$ (but it was probably included in the original definition of LORENTZ spaces). The proof of the result for $p = 1$ uses a covering lemma, which seems quite classical, and was probably known to either VITALI[2] or BESICOVITCH[3] who have proved more refined covering results.

Lemma: Let A be a measurable subset of R^N, covered by a family of (closed) balls $B_i = B(C_i, r_i), i \in I$, whose radii satisfy $0 < r_i \leq R_0 < \infty$ for all $i \in I$. Then for each $\varepsilon > 0$ there exists a subfamily $J \subset I$ such that the balls B_j are disjoint for $j \in J$, and $|A| \leq (3+\varepsilon)^N \sum_{j\in J} |B_j|$.
Proof: For $0 < \alpha < 1$ one chooses a first ball B_{j_1} with radius $r_{j_1} \geq (1-\alpha)\sup_{i\in I} r_i$, and one discards all the balls which intersect B_{j_1}, and one repeats the process as long as there are any balls left. In that way, one has selected a finite or infinite subfamily J such that the balls B_j are disjoint for $j \in J$ by construction. If $\sum_{j\in J} |B_j| = +\infty$ the result is proved. If $\sum_{j\in J} |B_j| < \infty$ and if the family is infinite one has $|B_{j_n}| \to 0$ and therefore $r_{j_n} \to 0$ as $n \to \infty$, and therefore all the balls have been discarded at some time because if one has $(1-\alpha)r_i > r_{j_n}$ then the ball B_i must have been discarded before the step n, or the ball B_{j_n} could not have been selected at step n; if the family is finite then all the balls have been either selected or discarded after a finite number of steps. Any ball B_i has been discarded because it intersects a selected ball B_{j_m} and therefore one has $r_{j_m} \geq (1-\alpha)r_i$, which implies that $B_i \subset B(C_{j_m}, k\,r_{j_m})$ with $k > 1 + \frac{2}{1-\alpha}$; therefore $A \subset \bigcup_{j\in J} B(C_j, k\,r_j)$, so that $|A| \leq k^N \sum_{j\in J} |B_j|$ and taking α small one can choose $k \leq 3+\varepsilon$.■

Proposition: For $f \in L^1(R^N)$ one has $meas\{x \in R^N, |M\,f(x)| \geq t\} \leq \frac{3^N||f||_1}{t}$ for every $t > 0$.

[1] Norbert G. WIENER, American mathematician, 1894-1964. He worked at MIT, Cambridge, MA (Massachusetts Institute of Technology).

[2] Giuseppe VITALI, Italian mathematician, 1875-1932. The Department of Pure and Aplied Mathematics of the University of Modena is named "Giuseppe VITALI".

[3] Abram Samoilovitch BESICOVITCH, Russian-born mathematician, 1891-1970. He held the Rouse BALL Professorship at Cambridge, 1950-1958, succeeding LITTLEWOOD, and he visited United States from 1958 to 1966.

Proof: Let $\Omega_s = \{x \in R^N, |M f(x)| > s\}$, so that for every $x \in \Omega_s$ there exists $r(x) > 0$ such that $\frac{\int_{B(x,r(x))}|f(y)|\,dy}{|B(x,r(x))|} > s$. One uses the lemma for the covering of Ω_s by all the balls $B\big(x,r(x)\big)$ with $x \in \Omega_s$, and the radii are bounded because $s\,|B(0,1)|r(x)^N = s\,\big|B\big(x,r(x)\big)\big| < \int_{B(x,r(x))}|f(y)|\,dy \le ||f||_1$. One finds a disjoint family of balls with centers $x \in X \subset \Omega_s$ such that $|\Omega_s| \le (3+\varepsilon)^N \sum_{x\in X}\big|B\big(x,r(x)\big)\big| \le (3+\varepsilon)^N \sum_{x\in X}\frac{\int_{B(x,r(x))}|f(y)|\,dy}{s} \le \frac{(3+\varepsilon)^N}{s}||f||_1$. Letting ε tend to 0 gives $|\Omega_s| \le \frac{3^N}{s}||f||_1$ and choosing $s = t-\eta$ for $\eta > 0$ and letting η tend to 0 gives the desired bound.■

Notation: $||f||_{*1}$ denotes the smallest constant $C \ge 0$ such that $meas\{x, |f(x)| \ge t\} \le \frac{C}{t}$ for all $t > 0$.■

One has $||f||_1 \le ||f||_{*1}$ for all $f \in L^1(\Omega)$. Despite the notation, $||\cdot||_{*1}$ is not a norm; one does have $||\lambda f||_{*1} = |\lambda|\,||f||_{*1}$ for all scalars λ, but the triangle inequality does not always hold. For example, if for $\Omega = (0,1)$ one takes $f(t) = \frac{1}{t}$ and $g(t) = \frac{1}{1-t}$, and $h = f+g$, so that $||f||_{*1} = ||g||_{*1} = 1$, one has $h(t) = \frac{1}{t(1-t)}$ and $h^*(t) = h\big(\frac{t}{2}\big)$ for $0 < t < 1$, and $||h||_{*1} = 4$.

Actually one always has $||f_1+f_2||_{*1} \le \big(\sqrt{||f_1||_{*1}} + \sqrt{||f_2||_{*1}}\big)^2$ for all f_1, f_2, because for $0 < s < t < \infty$ one has $\{x : |f_1(x)+f_2(x)| \ge t\} \subset \{x : |f_1(x)| \ge s\} \cup \{x : |f_2(x)| \ge t-s\}$ and therefore $meas(\{x : |f_1(x)+f_2(x)| \ge t\}) \le \frac{||f_1||_{*1}}{s} + \frac{||f_2||_{*1}}{t-s}$; taking the value of s for which the right side is minimum, i.e. $s = \alpha\sqrt{||f_1||_{*1}}$ and $t - s = \alpha\sqrt{||f_2||_{*1}}$ with α defined by $t = \alpha(\sqrt{||f_1||_{*1}} + \sqrt{||f_2||_{*1}})$ gives $meas(\{x : |f_1(x)+f_2(x)| \ge t\}) \le \frac{\sqrt{||f_1||_{*1}}+\sqrt{||f_2||_{*1}}}{\alpha} = \frac{(\sqrt{||f_1||_{*1}}+\sqrt{||f_2||_{*1}})^2}{t}$.

Notation: If f is a measurable function on Ω for which there exists C such that $meas\{x \in \Omega : |f(x)| \ge s\} \le \frac{C}{s}$ for $s > s_0$ (and $s_0 > 0$), then one defines $K_*(t,f) = \inf_{f=g+h}||g||_{*1} + t\,||h||_\infty$ for $t > 0$.■

For $f \in L^1(\Omega) + L^\infty(\Omega)$ one has $K_*(t,f) \le K(t,f)$, because $||g||_{*1} \le ||g||_1$ for every $g \in L^1(\Omega)$.

Lemma: If there exists C such that $meas\{x \in \Omega : |f(x)| \ge s\} \le \frac{C}{s}$ for $s > s_0$, then one has $t\,f^*(t) \le K_*(t,a)$ for all $t > 0$.
Proof: Because $|f_1| \le |f_2|$ a.e. in Ω implies $||f_1||_{*1} \le ||f_2||_{*1}$, one deduces that among all the decompositions $f = g+h$ with $||h||_\infty \le \lambda$, the one for which $||g||_{*1}$ is lowest corresponds to $|g| = (|f|-\lambda)_+$ (and $|h| = \min\{|f|,\lambda\}$). For $\varepsilon > 0$ there exists $\lambda \ge 0$ such that $||(|f|-\lambda)_+||_{*1} + t\,\lambda \le (1+\varepsilon)K_*(t,f)$; if $g = (|f|-\lambda)_+$, then as the nonincreasing rearrangement of g is $(f^*-\lambda)_+$, one has $t\,(f^*(t)-\lambda)_+ \le ||g||_{*1} \le (1+\varepsilon)K_*(t,f) - t\,\lambda$. If $\lambda \le f^*(t)$ it means $t\,(f^*(t)-\lambda) \le ||g||_{*1} \le (1+\varepsilon)K_*(t,f) - t\,\lambda$, while if $\lambda > f^*(t)$ it means $0 \le ||g||_{*1} \le (1+\varepsilon)K_*(t,f) - t\,\lambda$, and in both cases one has $t\,f^*(t) \le (1+\varepsilon)K_*(t,f)$, and letting ε tend to 0 gives the desired bound.■

For $f \in L^1(\Omega) + L^\infty(\Omega)$ one has then $t\,f^*(t) \le K_*(t,f) \le K(t,f)$ for all $t > 0$.

Corollary: Let $0 < \theta < 1$ and $1 \le q \le \infty$. If there exists C such that $meas\{x \in \Omega : |f(x)| \ge s\} \le \frac{C}{s}$ for $s > s_0$ and $t^{-\theta}K_*(t,f) \in L^q\big(0,\infty;\frac{dt}{t}\big)$ then $f \in L^1(\Omega) + L^\infty(\Omega)$ and $t^{-\theta}K(t,f) \in L^q\big(0,\infty;\frac{dt}{t}\big)$, i.e. $f \in L^{p,q}(\Omega)$ for $p = \frac{1}{1-\theta}$, and $||t^{-\theta}K(t,f)||_{L^q(0,\infty;dt/t)} \le \frac{1}{\theta}||t^{-\theta}K_*(t,f)||_{L^q(0,\infty;dt/t)}$.
Proof: As $K_*(t,f)$ is nondecreasing in t, $t^{-\theta}K_*(t,f) \in L^q\big(0,\infty;\frac{dt}{t}\big)$ implies $t^{-\theta}K_*(t,f) \in L^\infty\big(0,\infty;\frac{dt}{t}\big)$, i.e. $K_*(t,f) \le C\,t^\theta$ for $t > 0$, and therefore $f^*(t) \le C\,t^{\theta-1}$ for $t > 0$, and therefore $f \in L^1(\Omega) + L^\infty(\Omega)$. One has $t^{1-\theta}f^*(t) \in L^q\big(0,\infty;\frac{dt}{t}\big)$, and by HARDY inequality one deduces $t^{-\theta}K(t,f) \in L^q\big(0,\infty;\frac{dt}{t}\big)$ with the precise estimate shown.■

One can now finish the proof of the HARDY-LITTLEWOOD / WIENER result that the maximal function maps $L^p(R^N)$ into itself for $1 < p \le \infty$, and obtain the same result for LORENTZ spaces.

Proposition: For $1 < p < \infty$ and $1 \le q \le \infty$, $f \in L^{p,q}(R^N)$ implies $M f \in L^{p,q}(R^N)$ and $||M f||_{L^{p,q}(R^N)} \le \frac{3^{N/p}}{p-1}||f||_{L^{p,q}(R^N)}$.
Proof: For $f \in L^1(R^N) + L^\infty(R^N)$ one has $K_*(t, M f) \le 3^N K\big(\frac{t}{3^N}, f\big)$ for $t > 0$. Indeed the maximal function is subadditive, because $\frac{\int_{B(x,r)}|g(y)+h(y)|\,dy}{|B(x,r)|} \le \frac{\int_{B(x,r)}|g(y)|\,dy}{|B(x,r)|} + \frac{\int_{B(x,r)}|h(y)|\,dy}{|B(x,r)|} \le M g(x) + M h(x)$ a.e. $x \in R^N$ and for all $r > 0$, one deduces that $M(g+h) \le M g + M h$ a.e. in R^N. For each decomposition $f = g+h$ with $g \in L^1(R^N)$ and $h \in L^\infty(R^N)$, one has then $M f \le M g + M h$ and therefore $M f = g_0 + h_0$ with

$0 \le g_0 \le M\,g$ and $0 \le h_0 \le M\,h$ and therefore $K_*(t, M\,f) \le ||g_0||_{*1} + t\,||h_0||_\infty \le ||M\,g||_{*1} + t\,||M\,h||_\infty \le 3^N||g||_1 + t\,||h||_\infty$, and taking the infimum on all the decompositions, the right side can be made as small as $3^N \inf_{f=g+h}\big(||g||_1 + \frac{t}{3^N}\,||h||_\infty\big) = 3^N K\big(\frac{t}{3^N}, f\big)$. If $f \in L^{p,q}(R^N)$ then $t^{-\theta}K(t,f) \in L^q\big(0,\infty; \frac{dt}{t}\big)$ with $\theta = \frac{1}{p'}$, one deduces that $||t^{-\theta}K_*(t, M\,f)||_{L^q(0,\infty;dt/t)} \le 3^{N(1-\theta)}||t^{-\theta}K(t,f)||_{L^q(0,\infty;dt/t)}$, and then this implies $||t^{-\theta}K(t, M\,f)||_{L^q(0,\infty;dt/t)} \le \frac{3^{N(1-\theta)}}{\theta}||t^{-\theta}K(t,f)||_{L^q(0,\infty;dt/t)}$.■

21-724. Interpolation spaces
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Another family of nonlinear interpolation results is based on the method of traces of Jacques-Louis LIONS and Jaak PEETRE. One considers the space of (weakly) differentiable functions from $(0,\infty)$ to E_0+E_1 such that $t^{\alpha_0}u \in L^{p_0}(0,\infty;E_0)$ and $t^{\alpha_1}u' \in L^{p_1}(0,\infty;E_1)$ and for suitable values of $\alpha_0,p_0,\alpha_1,p_1$ (namely $\alpha_0+\frac{1}{p_0}>0$ and $\alpha_1+\frac{1}{p_1}<1$) these functions are automatically continuous on $[0,1]$ with values in E_0+E_1 and the space spanned by $u(0)$ is an interpolation space. Using the change of function $v(s)=u(s^\gamma)$ with $\gamma>0$ amounts to replace α_0,α_1 by β_0,β_1 defined by $\beta_0=\gamma\,\alpha_0+\frac{\gamma-1}{p_0}$ and $\beta_1=\gamma\,\alpha_1-\frac{\gamma-1}{p'_1}$, or $\beta_0+\frac{1}{p_0}=\gamma\big(\alpha_0+\frac{1}{p_0}\big)$ and $\beta_1+\frac{1}{p_1}=\gamma\big(\alpha_1+\frac{1}{p_1}\big)+1-\gamma$, and therefore the family of interpolation spaces depends upon at most three parameters, but Jaak PEETRE showed that the corresponding space is equal to $(E_0,E_1)_{\theta,p}$, and one can choose γ such that $\beta_0+\frac{1}{p_0}=\theta$ and $\beta_1+\frac{1}{p_1}=\theta$, and p is defined by $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.

This will be shown later, but assuming that the characterization has been obtained, one can deduce a few properties.

The interpolation property for a linear operator $A\in\mathcal{L}(E_0,F_0)\cap\mathcal{L}(E_1,F_1)$ follows immediately, because $v(t)=A\,u(t)$ gives $t^{\alpha_0}v\in L^{p_0}(0,\infty;F_0)$ and $t^{\alpha_1}v'\in L^{p_1}(0,\infty;F_1)$. Actually, as was noticed by Jacques-Louis LIONS, one can deduce a nonlinear interpolation theorem.

Proposition: If $E_0\subset E_1$, $F_0\subset F_1$, and A is a possibly nonlinear operator from E_1 into F_1 which is globally LIPSCHITZ continuous, i.e. $||A(u)-A(v)||_1\le M_1||u-v||_1$ for all $u,v\in E_1$, and which maps E_0 into F_0 with $||A(u)||_0\le M_0||u||_0$ for all $u\in E_0$, then for $0<\theta<1$ and $1\le p\le\infty$, A maps $(E_0,E_1)_{\theta,p}$ into $(F_0,F_1)_{\theta,p}$, and $||A(u)||_{(F_0,F_1)_{\theta,p}}\le C||u||_{(E_0,E_1)_{\theta,p}}$ for all $u\in(E_0,E_1)_{\theta,p}$.
Proof: Defining $v(t)=A\big(u(t)\big)$, one has $||v(t+h)-v(t)||_{F_1}=||A\big(u(t+h)\big)-A\big(u(t)\big)||_{F_1}\le M_1||u(t+h)-u(t)||_{E_1}$, and dividing by $|h|$ and letting h tend to 0 one deduces that $||v'(t)||_{F_1}\le M_1||u'(t)||_{E_1}$ for a.e. $t\in(0,\infty)$. Therefore, as for the linear case, one deduces that $t^{\alpha_0}v\in L^{p_0}(0,\infty;F_0)$ and $t^{\alpha_1}v'\in L^{p_1}(0,\infty;F_1)$.∎

In 1970, Jacques-Louis LIONS had asked me to consider the case where A is only HÖLDER continuous, where his idea does not work, and I noticed that his result can be proved directly by the K-method in a way which can be extended to the case of HÖLDER continuous mappings as will be shown later, and this was also noticed by Jaak PEETRE.

One just notices that for every decomposition $a=a_0+a_1$ one has $A(a)=b_0+b_1$ with $b_0=A(a_0)$ and $b_1=A(a)-A(a_0)$ and $||b_0||_0\le M_0||a_0||_0$ and $||b_1||_1=||A(a)-A(a_0)||_1\le M_1||a-a_0||_1=M_1||a_1||_1$, and therefore $K\big(t,A(a)\big)\le M_0K\big(\frac{t\,M_1}{M_0},a\big)$.

There were other interpolation theorems, for example by Jaak PEETRE or by Felix BROWDER[1] but under the assumption that the mapping is LIPSCHITZ continuous from E_0 to F_0 and from E_1 to F_1. An application considered by Jacques Louis LIONS was to interpolate the regularity of the solution of some variational inequalities, as he had done for linear (elliptic or parabolic) equations with Enrico MAGENES, but in his example the mapping considered is not LIPSCHITZ continuous from E_0 to F_0, and I suppose that it was the reason for his particular hypothesis.

The same idea of Jacques-Louis LIONS applies to a bilinear setting (and I have generalized it to a nonlinear setting).

Proposition: Let B be bilinear from $(E_0+E_1)\times(F_0+F_1)$ into G_0+G_1, satisfying the following conditions.
i) B is bilinear continuous from $E_0\times F_0$ into G_0, and $||B(e_0,f_0)||_{G_0}\le M_0||e_0||_{E_0}||f_0||_{F_0}$ for all $e_0\in E_0$ and all $f_0\in F_0$.
ii) B is bilinear continuous from $E_0\times F_1$ into G_1 and bilinear continuous from $E_1\times F_0$ into G_1, and $||B(e_0,f_1)||_{G_1}\le M_1||e_0||_{E_0}||f_1||_{F_1}$ for all $e_0\in E_0$ and all $f_1\in F_1$, and $||B(e_1,f_0)||_{G_1}\le M_1||e_1||_{E_1}||f_0||_{F_0}$ for all $e_1\in E_1$ and all $f_0\in F_0$.

[1] Felix BROWDER, American mathematician, born in 1928. He works at RUTGERS University, New Brunswick, NJ.

Then if $0 < \theta, \eta < 1$ with $\theta + \eta < 1$ and $1 \le p, q, r \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, B is bilinear continuous from $(E_0, E_1)_{\theta,p} \times (F_0, F_1)_{\eta,q}$ into $(G_0, G_1)_{\theta+\eta,r}$, and $||B(e,f)||_{(G_0,G_1)_{\theta+\eta,r}} \le C\,||e||_{(E_0,E_1)_{\theta,p}} ||f||_{(F_0,F_1)_{\eta,q}}$ for all $e \in (E_0, E_1)_{\theta,p}$ and all $f \in (F_0, F_1)_{\eta,q}$.

Proof: Any $e \in (E_0, E_1)_{\theta,p}$ can be written as $e = u(0)$ with $t^\theta ||u(t)||_{E_0} \in L^p\big(0,\infty; \frac{dt}{t}\big)$ and $t^\theta ||u'(t)||_{E_1} \in L^p\big(0,\infty; \frac{dt}{t}\big)$, and any $f \in (F_0, F_1)_{\eta,q}$ can be written as $f = v(0)$ with $t^\eta ||v(t)||_{F_0} \in L^q\big(0,\infty; \frac{dt}{t}\big)$ and $t^\eta ||v'(t)||_{F_1} \in L^q\big(0,\infty; \frac{dt}{t}\big)$. One defines $w(t) = B\big(u(t), v(t)\big)$, and $B(e,f) = w(0)$; one has $t^{\theta+\eta} ||w(t)||_{G_0} \in L^r\big(0,\infty; \frac{dt}{t}\big)$, and because $w'(t) = B\big(u(t), v'(t)\big) + B\big(u'(t), v(t)\big)$, one has $t^{\theta+\eta} ||w'(t)||_{G_1} \in L^r\big(0,\infty; \frac{dt}{t}\big)$, and therefore $B(e,f) \in (G_0, G_1)_{\theta+\eta,r}$ with corresponding bounds.■

The same result was essentially obtained by O'NEIL, who derived precise bounds for the convolution product analogous to those for the product[1].

The product corresponds to the choice $E_0 = F_0 = G_0 = L^\infty(\Omega)$ and $E_1 = F_1 = G_1 = L^1(\Omega)$ (with $M_0 = M_1 = 1$), and in this case the result states that if $1 < p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 \le a, b, c \le \infty$ and $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$, then the product is continuous from $L^{p,a}(\Omega) \times L^{q,b}(\Omega)$ into $L^{r,c}(\Omega)$ (the limiting cases will be discussed when studying the duals of interpolation spaces); as a particular case, the product is continuous from $L^p(\Omega) \times L^q(\Omega)$ into $L^r(\Omega)$, a simple consequence of HÖLDER inequality.

The convolution product corresponds to the choice $E_0 = F_0 = G_0 = L^1(R^N)$ and $E_1 = F_1 = G_1 = L^\infty(R^N)$ (with $M_0 = M_1 = 1$), and in this case the result states that if $1 < p, q, s < \infty$ with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1$ and $1 \le a, b, c \le \infty$ and $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$, then the convolution product is continuous from $L^{p,a}(R^N) \times L^{q,b}(R^N)$ into $L^{r,c}(R^N)$. As a particular case, the convolution product is continuous from $L^p(R^N) \times L^q(R^N)$ into $L^{s,1}(R^N)$, an improvement from the YOUNG inequality; one cannot take $a = p$ and $b = q$, which would give $\frac{1}{2} < c < 1$, but one may choose $a \ge p$ and $b \ge q$ such that $c = 1$ (one can actually define interpolation spaces with $0 < \theta < 1$ and $0 < p \le \infty$, but for $0 < p < 1$ they are only quasi-normed spaces).

There is another bilinear interpolation result, due to Jacques-Louis LIONS and Jaak PEETRE, with quite different assumptions.

Proposition: Let B be bilinear from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$, satisfying the following conditions.

i) B is bilinear continuous from $E_0 \times F_0$ into G_0, and $||B(e_0, f_0)||_{G_0} \le M_0 ||e_0||_{E_0} ||f_0||_{F_0}$ for all $e_0 \in E_0$ and all $f_0 \in F_0$.

ii) B is bilinear continuous from $E_1 \times F_1$ into G_1, and $||B(e_1, f_1)||_{G_1} \le M_1 ||e_1||_{E_1} ||f_1||_{F_1}$ for all $e_1 \in E_1$ and all $f_1 \in F_1$.

Then if $0 < \theta < 1$ and $1 \le p, q, r \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, B is bilinear continuous from $(E_0, E_1)_{\theta,p} \times (F_0, F_1)_{\theta,q}$ into $(G_0, G_1)_{\theta,r}$, and $||B(e,f)||_{(G_0,G_1)_{\theta,r}} \le C\,||e||_{(E_0,E_1)_{\theta,p}} ||f||_{(F_0,F_1)_{\theta,q}}$ for all $e \in (E_0, E_1)_{\theta,p}$ and all $f \in (F_0, F_1)_{\theta,q}$.■

It should be noticed that there are situations where both theorems can be used but give different results, in the second parameter (the first one is usually the same, compatible with scaling properties). For example, applying this last bilinear theorem to the product with $E_0 = F_0 = G_0 = L^\infty(\Omega)$, and $E_1 = F_1 = L^2(\Omega)$ an $G_1 = L^1(\Omega)$, one only obtains that the product maps $L^{p,a}(\Omega) \times L^{p,b}(\Omega)$ into $L^{p/2,c}(\Omega)$ with $2 < p < \infty$ and $1 \le a, b, c \le \infty$ and $\frac{1}{c} = \frac{1}{a} + \frac{1}{b} - 1$, while the first bilinear theorem gives the result in $L^{p/2,1}(\Omega)$ (but it has used a more general information, that the product of a function in $L^1(\Omega)$ and a function in $L^\infty(\Omega)$ is defined).

A similar situation arises for the so called RIESZ-THORIN theorem, which states that if a linear mapping is continous from $L^{p_0}(\Omega)$ into $L^{q_0}(\Omega')$ and from $L^{p_1}(\Omega)$ into $L^{q_1}(\Omega')$, then for $0 < \theta < 1$ it is continuous from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. RIESZ had only proved this result under the additional assumption that $p_\theta \le q_\theta$, and this condition was removed by THORIN. The K-method follows RIESZ's approach and implies that the mapping is continuous from $L^{p_\theta,p}(\Omega)$ into $L^{q_\theta,p}(\Omega')$ for any $p \in [1,\infty]$, but if one chooses $p = p_\theta$, the space $L^{q_\theta,p_\theta}(\Omega')$ is only included in $L^{q_\theta}(\Omega')$ if $p_\theta \le q_\theta$. THORIN's method corresponds to the complex interpolation method, which on this example is more precise, but for other questions has the disadvantage of having only one parameter.

[1] HARDY and LITTLEWOOD have shown that $\int_0^t (f\,g)^*\,ds \le \int_0^t f^* g^*\,ds$ for all $t > 0$.

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Talking about variants of interpolation methods, it is useful to obtain $L^p(\Omega)$ as an interpolation space between $L^1(\Omega)$ and $L^\infty(\Omega)$, but with the exact L^p norm.

Notation: For $a \in E_0 + E_1$, one defines $L^*(s,a)$ as the infimum of $||a_0||_0$ among the decompositions $a = a_0 + a_1$ with $a_0 \in E_0, a_1 \in E_1$ and $||a_1||_1 \le s$.■

In relation to the GAGLIARDO (convex) set associated to a, i.e. the set of $(x_0, x_1) \in [0,\infty) \times [0,\infty)$ such that there exists a decomposition $a = a_0 + a_1$ with $a_0 \in E_0, a_1 \in E_1$ and $||a_0||_0 \le x_0, ||a_1||_1 \le x_1$, the boundary of this set has the equation $x_0 = L^*(x_1, a)$.

Lemma: For $E_0 = L^1(\Omega)$ and $E_1 = L^\infty(\Omega)$, and $1 < p < \infty$ one has $\int_0^\infty p(p-1)s^{p-2}L^*(s,a)\,ds = \int_\Omega |a(x)|^p\,dx$ for every $a \in L^1(\Omega) + L^\infty(\Omega)$.
Proof: The optimal decomposition consists in taking $a_1(x) = a(x)$ if $|a(x)| \le s$ and $a_1(x) = \frac{s\,a(x)}{|a(x)|}$ if $|a(x)| > s$, so that $L^*(s,a) = \int_{|a(x)|\ge s}(|a(x)| - s)_+\,dx$. Then, using FUBINI's theorem, one has $\int_0^\infty p(p-1)s^{p-2}L^*(s,a)\,ds = \int_0^\infty p(p-1)s^{p-2}\big(\int_{|a(x)|\ge s}(|a(x)|-s)_+\,dx\big)\,ds = \int_\Omega\big(\int_0^{|a(x)|} p(p-1)s^{p-2}(|a(x)|-s)\,dx\big)\,ds = \int_\Omega |a(x)|^p\,dx$.■

If Φ is convex of class C^2 on R and $\Phi(0) = \Phi'(0) = 0$ then one has $\int_0^\infty \Phi''(s)L^*(s,a)\,ds = \int_\Omega \Phi(|a(x)|)\,dx$ for every $a \in L^1(\Omega) + L^\infty(\Omega)$ (one uses the TAYLOR formula with remainder, $\Phi(h) = \Phi(0) + \Phi'(0)h + \int_0^h (h-t)\Phi''(t)\,dt$). If Φ is convex with $\Phi(0) = \Phi'(0) = 0$ then Φ'' is a nonnegative measure and one must use a STIELTJES[1] integral. Therefore, the same approach can deal with ORLICZ[2] spaces.

Lemma: If A is a linear mapping from $L^1(\Omega) + L^\infty(\Omega)$ into $L^1(\Omega') + L^\infty(\Omega')$ which is continuous from $L^1(\Omega)$ into $L^1(\Omega')$ with norm M_1 and which is continuous from $L^\infty(\Omega)$ into $L^\infty(\Omega')$ with norm M_∞, then for $1 < p < \infty$ it is continuous from $L^p(\Omega)$ into $L^p(\Omega')$ with norm $\le M_1^{1/p}M_\infty^{1/p'}$.
Proof: For every decomposition $a = a_0 + a_1$ with $||a_1||_1 \le s$ one has a decomposition $A\,a = A\,a_0 + A\,a_1$ with $||A\,a||_1 \le M_\infty s$ and as $||A\,a||_0 \le M_1||a_0||_0$ one deduces that $L^*(M_\infty s, A\,a) \le M_1 L^*(s,a)$. Then $\int_0^\infty s^{p-2}L^*(s, A\,a)\,ds \le \int_0^\infty s^{p-2}M_1L^*\big(\frac{s}{M_\infty}, a\big)\,ds = M_1M_\infty^{p-1}\int_0^\infty \sigma^{p-2}L^*(\sigma,a)\,d\sigma$, i.e. $\int_{\Omega'} |A\,a(x)|^p\,dx \le M_1M_\infty^{p-1}\int_\Omega |a(x)|^p\,dx$.■

In order to describe some technical improvements concerning imbedding theorems of spaces of SOBOLEV type into LORENTZ spaces, it is useful to derive equivalent ways to check that a function belongs to a LORENTZ space $L^{p,q}(\Omega)$, with $1 < p < \infty$ and $1 \le q \le \infty$.

The definition that has been used was that $f \in L^{p,q}(\Omega)$ means that $t^{-1/p'}K(t,f) \in L^q\big(0,\infty;\frac{dt}{t}\big)$, and as $K(t,f)$ can be expressed in terms of the nonincreasing rearrangement f^* of f by $K(t,f) = \int_0^t f^*(s)\,ds$, $f \in L^{p,q}(\Omega)$ is equivalent to $t^{-1/p'}\big(\int_0^t f^*(s)\,ds\big) \in L^q\big(0,\infty;\frac{dt}{t}\big)$.

Because $t\,f^*(t) \le \int_0^t f^*(s)\,ds$, $f \in L^{p,q}(\Omega)$ implies $t^{1/p}f^*(t) \in L^q\big(0,\infty;\frac{dt}{t}\big)$, or $t^{(1/p)-(1/q)}f^*(t) \in L^q(0,\infty)$, which was the definition used by LORENTZ. It is indeed equivalent if $1 < p < \infty$ by HARDY's inequality, but this definition can also be used for $p = 1$.

The nonincreasing rearrangement is defined on $\big(0, meas(\Omega)\big)$, and one extends it by 0 in order to have it defined on $(0,\infty)$. For $n \in Z$, one chooses $a_n \in [f^*(e^n_+), f^*(e^n_-)]$, so that one has $meas\{x, |f(x)| > a_n\} \le e^n \le meas\{x, |f(x)| \ge a_n\}$ for every $n \in Z$. Then one has $f \in L^{p,q}(\Omega)$ if and only if $e^{n/p}a_n \in l^q(Z)$. Indeed for $e^n < t < e^{n+1}$ one has $f^*(e^{n+1}_-) \le f^*(t) \le f^*(e^n_+)$, and therefore $a_{n+1} \le f^*(t) \le a_n$; this implies $\alpha\,||e^{n/p}a_n||_{l^q(Z)} \le ||t^{1/p}f^*(t)||_{L^q(0,\infty;dt/t)} \le \beta\,||e^{n/p}a_n||_{l^q(Z)}$, for two positive constants α, β.

[1] Thomas Jan STIELTJES, Dutch-born mathematician, 1856-1894. He worked in Toulouse, France.
[2] Władysław ORLICZ, Polish mathematician, 1903-1990. He worked at the Polish Academy of Sciences, Poznan.

If for every $\lambda > 0$ one has $meas\{x, |f(x)| > \lambda\} < \infty$, then $f \in L^{p,q}(\Omega)$ if and only if $e^{n/p}(a_n - a_{n+1}) \in l^q(Z)$. The added condition is equivalent to $\lim_{n\to\infty} a_n = 0$. Let $b_n = a_n - a_{n+1}$, so that $a_n = b_n + b_{n+1} + \ldots$, and $e^n a_n = e^n b_n + e^{-1} e^{n+1} b_{n+1} + \ldots$, so that $e^n a_n$ is obtained from $e^n b_n$ by a convolution with c_n defined by $c_n = 0$ for $n > 0$ and $c_n = e^n$ for $n \leq 0$, and then an application of YOUNG's inequality gives the result.

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One can obtain a nonoptimal imdedding theorem $W^{1,p}(R^N) \subset L^q(R^N)$ by decomposing $u = (u - u \star \rho_\varepsilon) + u \star \rho_\varepsilon$, where ρ_ε is a special smoothing sequence. Using $||\tau_h u - u||_p \leq |h|\,||grad(u)||_p$, and $u - u \star \rho_\varepsilon = \int (u - \tau_y u)\rho_\varepsilon(y)\,dy$ one has $||u - u \star \rho_\varepsilon||_p \leq \int |y|\,|\rho_\varepsilon(y)|\,dy\,||grad(u)||_p = C\,\varepsilon\,||grad(u)||_p$; one also has $||u \star \rho_\varepsilon||_\infty \leq ||u||_p ||\rho_\varepsilon||_{p'} \leq C\,||u||_p \varepsilon^{-N/p'}$. This means that using $E_0 = L^p(R^N)$ and $E_1 = L^\infty(R^N)$, one has $K(t, u) \leq C\,\varepsilon + C\,t\,\varepsilon^{-N/p'}$ for all $\varepsilon > 0$, and taking the best ε gives $K(t, u) \leq C\,t^{p'/(N+p')}$, i.e. $u \in (E_0, E_1)_{\theta,\infty} = L^{q_\theta,\infty}(R^N)$, with $\theta > 0$ and therefore $q_\theta > p$ and choosing any $q \in (p, q_\theta)$ one has shown that there exists $q > p$ such that $||u||_q \leq A\,||u||_p + B\,||grad(u)||_p$ for all $u \in W^{1,p}(R^N)$. The precise value of q obtained is not important, as long as $q > p$, and from this nonoptimal imbedding theorem, the best known imbeddings will be derived.

The first step is the usual scaling argument; one should not add $||u||_p$ and $||grad(u)||_p$ which are not measured in the same unit, and one applies the inequality to u_λ defined by $u_\lambda(x) = u\big(\frac{x}{\lambda}\big)$.

One obtains $|\lambda|^{N/q}||u||_q \leq A\,|\lambda|^{N/p}||u||_p + B\,|\lambda|^{(N/p)-1}||grad(u)||_p$ for all $u \in W^{1,p}(R^N)$. One chooses the best λ, and one deduces that there exists $\theta \in (0, 1]$ such that $||u||_q \leq C\,||u||_p^{1-\theta}||grad(u)||_p^\theta$ for all $u \in W^{1,p}(R^N)$ (in the case $1 \leq p < N$ one finds that one must have $q \leq p^* = \frac{N\,p}{N-p}$, or a contradiction would be derived by letting λ tend to 0). The new inequality is now invariant by scaling, which means that θ is such that $\frac{N}{q} = (1-\theta)\frac{N}{p} + \theta\big(\frac{N}{p} - 1\big) = \frac{N}{p} - \theta$, i.e. $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p^*}$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ (with $p^* = \infty$ for $p = N$ and $p^* < 0$ for $p > N$).

The second step is to apply the inequality to a sequence of function $\varphi_n(u)$, but it is important to choose a sequence which is adapted to u, and it uses the levels a_n introduced previously.

One defines φ_n by $\varphi_n(0) = 0$ and $\varphi'(v) = 1$ is $a_n < |v| < a_{n-1}$ and $\varphi'(v) = 0$ otherwise. One has $\big|\big|grad\big(\varphi_n(u)\big)\big|\big| = \gamma_n \in l^p(Z)$. For any $r < \infty$ one has $\int_{R^N} |\varphi_n(u)|^r\,dx \leq |a_n - a_{n-1}|^r meas\{x, |u(x)| > a_n\} \leq |a_n - a_{n-1}|^r e^n$ and $\int_{R^N} |\varphi_n(u)|^r\,dx \geq |a_n - a_{n-1}|^r meas\{x, |u(x)| \geq a_{n-1}\} \geq |a_n - a_{n-1}|^r e^{n-1}$. One deduces that $|a_n - a_{n-1}|e^{(n-1)/q} \leq C\big(|a_n - a_{n-1}|e^{n/p}\big)^{1-\theta}\gamma_n^\theta$, and therefore $|a_n - a_{n-1}|^\theta e^{(n/q)-(n(1-\theta)/p)} \leq C\,e^{1/q}\gamma_n^\theta$, and using $\frac{1}{q} - \frac{1-\theta}{p} = \frac{\theta}{p^*}$, one deduces that $|a_n - a_{n-1}|e^{n/p^*} \leq C^{1/\theta}\,e^{1/\theta\,q}\gamma_n$, and therefore $|a_n - a_{n-1}|e^{n/p^*} \in l^p(Z)$.

In the case $1 \leq p < N$ one has $p^* = \frac{N\,p}{N-p} < \infty$, and one has shown that $u \in W^{1,p}(R^N)$ implies $u \in L^{p^*,p}(R^N)$, the improvement by Jaak PEETRE of the original result of Sergei SOBOLEV.

In the case $p = N$ one has $p^* = 0$ and therefore $|a_n - a_{n-1}| \in l^N(Z)$. Let $b_n = a_n - a_{n+1} \geq 0$, then as n tends to $-\infty$ one has $a_n = b_n + b_{n+1} + \ldots + b_{m-1} + a_m$, and an application of HÖLDER inequality gives $|a_n| \leq |a_m| + \big(|b_n|^N + \ldots + |b_{m-1}|^N\big)^{1/N}|n-m|^{1/N'}$, and therefore for every $\varepsilon > 0$ one can choose m such that for $n < m$ one has $|a_n| \leq \varepsilon\,|n-m|^{1/N'} + |a_m|$, i.e. for every $\varepsilon > 0$ there exists $C(\varepsilon)$ such that $|a_n|^{N'} \leq \varepsilon\,|n| + C(\varepsilon)$ for all $n \leq 0$.

For $\kappa > 0$ one chooses $\varepsilon < \frac{1}{\kappa}$, and one computes the integral of $e^{\kappa\,|u|^{N'}}$ on the set where $|u(x)| \geq a_0$; on the set where $a_{n+1} \leq |u(x)| < a_n$, which has a measure $\leq e^{n+1}$, one has $|u(x)| \leq a_n$ and therefore $\kappa\,|u|^{N'} \leq \kappa|a_n|^{N'} \leq \kappa\,\varepsilon\,|n| + \kappa\,C(\varepsilon)$ for all $n \leq 0$, and therefore $\int_{|u(x)|\geq a_0} e^{\kappa\,|u|^{N'}}\,dx \leq \sum_{n=\infty}^0 e^n\,e^{\kappa\,\varepsilon\,|n| + \kappa\,C(\varepsilon)} < \infty$. Therefore $e^{\kappa\,|u|^{N'}} \in L^1_{loc}(R^N)$, which is the improvement by Neil TRUDINGER of a result of Fritz JOHN and Louis NIRENBERG concerning BMO, which improved the result of Sergei SOBOLEV for the limiting case $p = N$.

For $p > N$ one has $p^* < 0$ and therefore from $|a_n - a_{n-1}|e^{n/p^*} \in l^p(Z)$, one deduces that $\sum_{n=-\infty}^0 |a_n - a_{n-1}| < \infty$, and therefore that $|a_n| \leq M$ for all $n \in Z$. Having proved that $W^{1,p}(R^N) \subset L^\infty(R^N)$, one must have $||u||_\infty \leq C\,||u||_p^{1-\theta}||grad(u)||_p^\theta$, and $\frac{1}{p} - \frac{\theta}{N} = 0$, i.e. $\theta = \frac{N}{p}$.

One applies then the result to $v = \tau_h u - u$, for which one has $||v||_p \leq |h|\,||grad(u)||_p$ and $||grad(v)||_p \leq 2||grad(u)||_p$, and one obtains the estimate $||\tau_h u - u||_\infty \leq C\,|h|^{1-(N/p)}||grad(u)||_p$, i.e. the fonctions in $W^{1,p}(R^N)$ are HÖLDER continuous with exponent $\alpha = 1 - \frac{N}{p}$, as Sergei SOBOLEV had proved.

21-724. Interpolation spaces

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11. Friday April 7, 2000.

The original method of proof of Sergei SOBOLEV consisted in writing $u = \sum_j \frac{\partial u}{\partial x_j} \star \frac{\partial E}{\partial x_j}$ for an elementary solution E of Δ, and it is not adapted to the case where the derivatives are in different spaces.

The different proof of Emilio GAGLIRDO and of Louis NIRENBERG can be used for the case where $\frac{\partial u}{\partial x_j} \in L^{p_j}(R^N)$ for $j = 1, \ldots, N$, and this was done by TROISI and by Alois KUFNER.

The case where the derivatives are in the same LORENTZ space can be proved by using the Theory of Interpolation, as was done by Jaak PEETRE, but the limiting case where $\frac{\partial u}{\partial x_j} \in L^{N,p}(R^N)$ for $j = 1, \ldots, N$ was treated by Haïm BREZIS and Stephen WAINGER by analyzing a formula of O'NEIL about the nonincreasing rearrangement of a convolution product.

As far as I know, these classical methods do not permit to treat the case where the derivatives are in different LORENTZ spaces; of course, this question is quite academic, but serves as training ground for situations which often occur where one has different informations in different directions, for example because some coordinates represent space and another one represents time.

First, it is useful to observe that SOBOLEV imbedding theorem for $p = 1$ is essentially the isoperimetric inequality. The classical isoperimetric inequality says that among measurable sets A of R^N with a given volume, the $(N-1)$-dimensional measure of the boundary ∂A is minimum when A is a sphere; equivalently, for a given measure of the boundary, the volume is maximum for a sphere.

Analytically it means that $meas(A) \leq C_0\big(meas(\partial A)\big)^{N/(N-1)}$, and it tells what the best constant C_0 is, while SOBOLEV imbedding theorem for $W^{1,1}(R^N)$ gives $\int_{R^N} |u|^{1^*}\,dx \leq C_1 ||grad(u)||_1^{N/(N-1)}$, but does not insist in identifying what the best constant C_1 is.

The relation between the two inequalities is that one can apply the last inequality to $u = \chi_A$, the characteristic function of A, which is not in $W^{1,1}(R^N)$ but has its partial derivatives $\frac{\partial \chi_A}{\partial x_j}$ which are RADON measures, and one can apply the inequality to $\chi_A \star \rho_\varepsilon$ and then let ε tend to 0; in this way one learns that $C_0 \leq C_1$. Conversely, knowing the isoperimetric inequality, one can approach a function u by a sum of characteristic functions, using $A_n = \{x, n\varepsilon \leq u(x) < (n+1)\varepsilon\}$ and deduce SOBOLEV imbedding theorem, so that $C_1 \leq C_0$ and the two inequalities are essentially the same. However, the proof of the last part involves the technical study of functions of bounded variation (denoted BV), which is clasical in one dimension, but owes to the work of Ennio DE GIORGI[1], FEDERER[2] and Wendell FLEMING[3] for the development of the N-dimensional case.

Starting from SOBOLEV imbedding theorem $W^{1,1}(R^N) \subset L^{1^*}(R^N)$, one can easily derive all the results already obtained. Using the functions φ_n adapted to u, one writes $||\varphi_n(u)||_{1^*} \leq C_0 ||\varphi'_n(u) grad(u)||_1$, and by HÖLDER inequality one has $||\varphi'_n(u) grad(u)||_1 \leq ||\varphi'_n(u) grad(u)||_p e^{n/p'}$, and one deduces the same inequality than before, $|a_{n-1} - a_n| e^{n/p^*} \in l^p(Z)$.

The case of derivatives in (different) LORENTZ spaces will be obtained by using a multiplicative variant.

Lemma: The additive version $||u||_{1^*} \leq A \sum_j \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_1$ for all $u \in W^{1,1}(R^N)$ is equivalent to the multiplicative version $||u||_{1^*} \leq N\,A \left(\prod_j \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_1\right)^{1/N}$ for all $u \in W^{1,1}(R^N)$.

Proof: One rescales with a different scaling in different directions, i.e. one applies the additive version to v defined by $v(x_1, \ldots, x_N) = u\big(\frac{x_1}{\lambda_1}, \ldots, \frac{x_N}{\lambda_N}\big)$, and one obtains $(\lambda_1 \ldots \lambda_N)^{1/1^*} ||u||_{1^*} \leq A\,\lambda_1 \ldots \lambda_N \sum_j \frac{1}{\lambda_j} \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_1$.

[1] Ennio DE GIORGI, Italian mathematician, 1928-1996. He received the WOLF prize in 1990. He worked at Scuola Normale Superiore, Pisa

[2] Herbert G. FEDERER, American mathematician, born in 1920. He works at BROWN University, Providence, RI.

Nicholas BROWN Jr., American merchant, 1769-1841.

[3] Wendell H. FLEMING, American mathematician, born in 1928. He works at BROWN University, Providence, RI.

Then one notices that if $\lambda_1 \ldots \lambda_N = \mu > 0$ is given then the minimum of $\sum_j \frac{\alpha_j}{\lambda_j}$ is attained when $\lambda_j = \beta\,\alpha_j$ for all j for some LAGRANGE multiplier β, and the constraint gives $\beta^N \alpha_1 \ldots \alpha_N = \mu$, so that the minimum is $\frac{N}{\mu}(\alpha_1 \ldots \alpha_N)^{1/N}$; one applies this remark to the case $\alpha_j = \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_1$ and one finds the multiplicative version, as the powers of μ are identical on both sides of the inequality (because the inequality is already invariant when one rescales all the coordinates in the same way).

The multiplicative version implies the additive version by the geometric-arithmetic inequality, i.e. $(a_1 \ldots a_N)^{1/N} \leq \frac{a_1 + \ldots + a_N}{N}$ for all positive numbers $a_1, \ldots, a_N$; putting $a_j = e^{z_j}$ this is just the convexity inequality for the exponential function.

Proposition: Let u satisfy $\frac{\partial u}{\partial x_j} \in L^{p_j, q_j}(R^N)$ where $1 < p_j < \infty$ and $1 \leq q_j \leq \infty$ for $j = 1, \ldots, N$. Let p_{eff} and p^*_{eff} be defined by $\frac{1}{p_{eff}} = \frac{1}{N}\sum_j \frac{1}{p_j}$ and $\frac{1}{p^*_{eff}} = \frac{1}{p_{eff}} - \frac{1}{N}$; similarly, q_{eff} and q^*_{eff} are defined by $\frac{1}{q_{eff}} = \frac{1}{N}\sum_j \frac{1}{q_j}$, and $\frac{1}{q^*_{eff}} = \frac{1}{q_{eff}} - \frac{1}{N}$. Then one has $|a_{n-1} - a_n| e^{n/p^*_{eff}} \in l^{q_{eff}}(Z)$.
Proof: Let $f_j = \frac{\partial u}{\partial x_j}$ for $j = 1, \ldots, N$. One applies the multiplicative version to $\varphi_n(u)$, and one has to estimate $||\varphi'_n(u) f_j||_1$. A classical result of HARDY and LITTLEWOOD states that for all $f \in L^1(\Omega) + L^\infty(\Omega)$ and all measurable subsets $\omega \subset \Omega$ one has $\int_\omega |f(x)|\,dx \leq \int_0^{meas(\omega)} f^*(s)\,ds$. As the measure of the points where $\varphi'_n(u) \neq 0$ is at most e^n, one deduces that $||\varphi'_n(u) f_j||_1 \leq K(e^n, f_j)$ for $j = 1, \ldots, N$. Because $e^{-n\theta_j} K(e^n, f_j) \in l^{q_j}(Z)$ with $\theta_j = \frac{1}{p'_j}$ for $j = 1, \ldots, N$, one deduces by HÖLDER inequality that $e^{-n/p'_{eff}} ||\varphi_n(u)||_{1^*} \leq N\,A\big(\prod_j e^{-n\theta_j} K(e^n, f_j)\big)^{1/N} \in l^{q_{eff}}(Z)$, and this gives the desired inequality.■

If $p_{eff} < N$ then it means $u \in L^{p_{eff}, q_{eff}}(R^N)$.

If $p_{eff} = N$ and $q_{eff} = 1$, which means that $q_j = 1$ for $j = 1, \ldots, N$, then one has $|a_{n-1} - a_n| \in l^1(Z)$ and therefore one deduces a bound for a_n, i.e. $u \in L^\infty(R^N)$; using the density of $C^\infty_c(R^N)$ in $L^{p_j, 1}(R^N)$, one deduces that $u \in C_0(R^N)$.

If $p_{eff} = N$ and $1 < q_{eff} < \infty$, then for every $\kappa > 0$ one has $e^{\kappa |u|^{q'_{eff}}} \in L^1_{loc}(R^N)$.

If $p_{eff} = N$ and $q_{eff} = \infty$, then one deduces that $|a_n| \leq \alpha\,|n| + \beta$ and therefore there exists $\varepsilon_0 > 0$ such that $e^{\varepsilon_0 |u|} \in L^1_{loc}(R^N)$. This is the case for example when all the derivatives belong to $L^{N,\infty}(R^N)$, and because $\log(|x|)$ is such a function, it is not always true that $e^{\kappa |u|} \in L^1_{loc}(R^N)$ for all $\kappa > 0$. The space of functions considered are in BMO, and Fritz JOHN and Louis NIRENBERG have shown the stronger result that for every function in BMO there exists $\varepsilon_0 > 0$ such that $e^{\varepsilon_0 |u|} \in L^1_{loc}(R^N)$.

If $p_{eff} > N$ then one has $u \in L^\infty(R^N)$. Of course, one can deduce that u is continuous, but because there are different derivatives in different directions, there is no automatic statement that u must be HÖLDER continuous.

Having different informations on derivatives in different directions is usual for parabolic equations like the heat equation. For example, let Ω be an open set of R^N, given $u_0 \in L^2(\Omega)$, one can show that there exists a unique solution u of $\frac{\partial u}{\partial t} - \Delta\,u = 0$ in $\Omega \times (0,T)$ satisfying the initial condition $u(x,0) = u_0(x)$ in Ω and the homogeneous DIRICHLET boundary condition $\gamma_0 u = 0$ on $\partial\Omega \times (0,T)$, in the sense that $u \in C\big([0,T]; L^2(\Omega)\big)$, $u \in L^2\big(0,T; H^1_0(\Omega)\big)$ and $\frac{\partial u}{\partial t} \in L^2\big(0,T; H^{-1}(\Omega)\big)$.

If $u_0 \in H^1_0(\Omega)$ then the solution also satisfies $u \in C^0\big([0,T]; H^1_0(\Omega)\big)$, $\Delta\,u, \frac{\partial u}{\partial t} \in L^2\big(0,T; L^2(\Omega)\big) = L^2\big(\Omega \times (0,T)\big)$; if the boundary is of class C^1 or if the open set Ω is convex (or if an inequality holds for the total curvature of the boundary), then one has $u \in L^2\big(0,T; H^2(\Omega)\big)$.

If u_0 belongs to an interpolation space between $H^1_0(\Omega)$ and $L^2(\Omega)$ then one has intermediate results, but this requires enough smoothness for the boundary.

As an example, consider a function $u(x,t)$ defined on $R^N \times R$ and satisfying $u, \frac{\partial u}{\partial t}, \Delta\,u \in L^2(R^{N+1})$ (this implies that $\frac{\partial u}{\partial x_j} \in L^2(R^{N+1})$ for $j = 1, \ldots, N$ by using FOURIER transform). Denoting (ξ, τ) the dual variables, the information is equivalent to $\mathcal{F}u, \tau\,\mathcal{F}u, |\xi|^2 \mathcal{F}u \in L^2(R^{N+1})$ (and therefore $\xi_j \mathcal{F}u \in L^2(R^{N+1})$ for $j = 1, \ldots, N$).

Because $(1 + |\tau| + |\xi|^2)\mathcal{F}u \in L^2(R^{N+1})$, if one shows that $\frac{1}{1+|\tau|+|\xi|^2} \in L^{p,\infty}(R^{N+1})$ for some $p \in (2,\infty)$, one deduces that $\mathcal{F}u \in L^{q,2}(R^{N+1})$ with $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$ and because $1 < q < 2$ and $\overline{\mathcal{F}}$ maps $L^1(R^{N+1})$ into $L^\infty(R^{N+1})$ and $L^2(R^{N+1})$ into itself, one finds that $u \in L^{q',2}(R^{N+1})$.

One has $\frac{1}{1+|\tau|+|\xi|^2} \in L^\infty(R^{N+1})$, and it is the behaviour at ∞ that is interesting for obtaining the smallest value of p, and therefore one checks for what value of p one has $\frac{1}{|\tau|+|\xi|^2} \in L^{p,\infty}(R^{N+1})$ and one obtains the same information for the smaller function $\frac{1}{1+|\tau|+|\xi|^2}$. One uses the homogeneity properties of the function, and for $\lambda > 0$ one computes $meas\{(\xi,\tau), \frac{1}{|\tau|+|\xi|^2} \geq \lambda\}$; by making the change of coordinates $\tau = \lambda^{-1}\tau'$ and $\xi = \lambda^{-1/2}\xi'$, it is $C\,\lambda^{-1-(N/2)}$ with $C = meas\{(\xi,\tau), \frac{1}{|\tau|+|\xi|^2} \geq 1\}$, and this corresponds to $p = 1 + \frac{N}{2} = \frac{N+2}{2}$, which gives $q = \frac{2(N+2)}{N+6}$ and $q' = \frac{2(N+2)}{N-2}$ if $N \geq 3$. For $N = 2$ one finds that $u \in L^r(R^3)$ for all $r < \infty$.

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Using FOURIER transform one can obtain an imbedding result for spaces $H^s(R^N)$ into LORENTZ spaces.

Proposition: For $s > \frac{N}{2}$ one has $H^s(R^N) \subset \mathcal{F}L^1(R^N) \subset C_0(R^N)$.

For $0 < s < \frac{N}{2}$ one has $H^s(R^N) \subset L^{p(s),2}(R^N)$ with $\frac{1}{p(s)} = \frac{1}{2} - \frac{s}{N}$.

Proof: One has $\mathcal{F}u(\xi) = (1+|\xi|^s)\mathcal{F}u(\xi)\,\frac{1}{1+|\xi|^s}$, and as $u \in H^s(R^N)$ means $(1+|\xi|^s)\mathcal{F}u(\xi) \in L^2(R^N)$, and one must check in which LORENTZ space the function $\frac{1}{1+|\xi|^s}$ is.

For $s > \frac{N}{2}$ one has $\frac{1}{1+|\xi|^s} \in L^2(R^N)$ and therefore $\mathcal{F}u \in L^1(R^N)$, and one uses the fact that $\mathcal{F}$ or $\overline{\mathcal{F}}$ map $L^1(R^N)$ into $C_0(R^N)$.

For $0 < s < \frac{N}{2}$ one has $\frac{1}{1+|\xi|^s} \le \frac{1}{|\xi|^s} \in L^{N/s,\infty}(R^N)$, and therefore $\mathcal{F}u \in L^{a(s),2}(R^N)$ with $\frac{1}{a(s)} = \frac{1}{2} + \frac{s}{N}$. Because $\mathcal{F}^{-1} = \overline{\mathcal{F}}$ maps $L^1(R^N)$ into $L^\infty(R^N)$ and $L^2(R^N)$ into itself, it maps $\big(L^1(R^N), L^2(R^N)\big)_{\theta,2}$ into $\big(L^\infty(R^N), L^2(R^N)\big)_{\theta,2}$; the first space is $L^{p(\theta),2}(R^N)$ if $\frac{1}{p(\theta)} = \frac{1-\theta}{1} + \frac{\theta}{2}$, and the last space is $L^{q(\theta),2}(R^N)$ with $\frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{\infty} = 1 - \frac{1}{p(\theta)}$, so that $q(\theta) = p(\theta)'$. Therefore $\mathcal{F}u \in L^{a(s),2}(R^N)$ implies $u \in L^{b(s),2}(R^N)$ with $b(s) = a(s)'$ i.e. $\frac{1}{b(s)} = \frac{1}{2} - \frac{s}{N}$.$\blacksquare$

For SOBOLEV spaces corresponding to $p \ne 2$, one cannot use FOURIER transform, and proofs must be obtained in a different way.

Definition: For $1 \le p \le \infty$ and $0 < s < 1$, the SOBOLEV space $W^{s,p}(R^N)$ is defined as $W^{s,p}(R^N) = \big(W^{1,p}(R^N), L^p(R^N)\big)_{1-s,p}$.

For $1 \le p,q \le \infty$, and $0 < s < 1$ one defines the BESOV space $B^{s,p}_q(R^N) = \big(W^{1,p}(R^N), L^p(R^N)\big)_{1-s,q}$.$\blacksquare$

If k is a positive integer and $k < s < k+1$, one may define the SOBOLEV space $W^{s,p}(R^N)$ or the BESOV space $B^{s,p}_q(R^N)$ in at least two ways; one way is that $u \in W^{k,p}(R^N)$ and for all multi-indices α of length $|\alpha| = k$ one has $D^\alpha u \in W^{s-k,p}(R^N)$ or $B^{s-k,p}_q(R^N)$; another way is to define it as $\big(W^{m,p}(R^N), L^p(R^N)\big)_{\theta,q}$, with $q = p$ or not, with an integer $m \ge k+1$, and with $(1-\theta)m = s$. Of course, there are a few technical questions to check in order to show that these two definitions coincide.

Before using scaling arguments for $W^{m,p}(R^N)$ it is useful to remark that an equivalent norm is $||u||_p + \sum_{|\alpha|=m} ||D^\alpha u||_p$, i.e. for $0 < |\beta| < m$ one can bound $||D^\beta u||_p$ in terms of $||u||_p$ and all the norms $||D^\alpha u||_p$ for the multi-indices of length exactly equal to m. One may start in one dimension, noticing that $(\varphi\, H)' = \delta_0 + \psi$, where H is the HEAVISIDE function, $\varphi \in C^\infty_c(R)$ is equal to 1 near 0 and therefore $\psi \in C^\infty_c(R)$; one deduces that $(\varphi\, H) \star u'' = u' + \psi' \star u$, from which one deduces that $||u'||_p \le ||\varphi\, H||_1 ||u''||_p + ||\psi'||_1\, ||u||_p$. Similarly, if $0 < k < m$, one replaces the HEAVISIDE function by the function K defined by $K(t) = \frac{t^{m-k-1}}{(m-k)!} H(t)$ for all t, so that $(\varphi\, K)^{(m-k)} = \delta_0 + \chi$ with $\chi \in C^\infty_c(R)$, and deriving k times and taking the convolution with u one finds that $||u^{(k)}||_p \le ||\varphi\, K||_1 ||u^{(m)}||_p + ||\chi^{(k)}||_1\, ||u||_p$. In order to bound $||D^\beta u||_p$, one denotes $\gamma = (0, \beta_2, \ldots, \beta_N)$, $\alpha = (m + \beta_1 - |\beta|, \beta_2, \ldots, \beta_N)$ and $v = D^\gamma u$, so that for almost all $x_2, \ldots, x_N$ one can use the one dimensional result in order to derive a bound for the norm of $D^\beta u = D_1^{\beta_1} v$ in terms of v and $D^\alpha u = D_1^{m-|\beta|} v$; then one takes the power p and one integrates in $x_2, \ldots, x_N$; one finishes by an induction argument on N.

By using a scaling argument one deduces that $||D^\beta u||_p \le C\, ||u||_p^{1-\theta} \big(\sum_{|\alpha|=m} ||D^\alpha u||_p\big)^\theta$ with $\theta = \frac{|\beta|}{m}$. This implies that if $0 < k < m$ one has $\big(W^{m,p}(R^N), L^p(R^N)\big)_{(m-k)/m,1} \subset W^{k,p}(R^N)$.

For using the reiteration theorem of Jacques-Louis LIONS and Jaak PEETRE, one also needs to check that $W^{k,p}(R^N) \subset \big(W^{m,p}(R^N), L^p(R^N)\big)_{(m-k)/m,\infty}$. For doing this, one tries the usual decomposition $u = \rho_\varepsilon \star u + (u - \rho_\varepsilon \star u)$, where ρ_ε is a special smoothing sequence; for $|\alpha| = m$ one writes $D^\alpha(\rho_\varepsilon \star u) = D^\beta \rho_\varepsilon \star D^\gamma u$ with $\alpha = \beta + \gamma$ and $|\beta| = m - k$ and $|\gamma| = k$, so that $||\rho_\varepsilon \star u||_{W^{m,p}(R^N)} \le C\, \varepsilon^{k-m}$; one has $u(x) - (\rho_\varepsilon \star u)(x) = \int_{R^N} \rho_\varepsilon(y)\big(u(x) - u(x-y)\big)\,dy$, and if $k = 1$ one just uses the fact that $||u - \tau_y u||_p \le C\,|y|\,||grad(u)||_p$, but if $k > 1$ one must be more careful, and besides the condition $\int_{R^N} \rho_1(y)\,dy = 1$ one also imposes the conditions

$\int_{R^N} y^\gamma \rho_1(y)\,dy = 0$ for all multi-indices γ with $1 \le |\gamma| \le k-1$. One uses the TAYLOR expansion with integral remainder $f(1) = f(0) + f'(0) + \ldots + \frac{f^{(k-1)}(0)}{(k-1)!} + \int_0^1 \frac{(1-t)^{k-1} f^{(k)}(t)}{(k-1)!}\,dt$ for the function f defined by $f(t) = u(x - t\,y)$, using the fact that for $1 \le |\gamma| \le k-1$ one has $\int_{R^N} \rho_\varepsilon(y) D^\gamma u(x) y^\gamma\,dy = 0$, and one deduces that $||u - \rho_\varepsilon \star u||_p \le C\,\varepsilon^k$, and as this decomposition is valid for every $\varepsilon > 0$ it proves the assertion.

Repeated applications of SOBOLEV's imbedding theorem for $W^{1,q}(R^N)$ show that if $p > \frac{N}{m}$ one has $W^{m,p}(R^N) \subset L^\infty(R^N)$, and a scaling argument gives then $||u||_\infty \le C||u||_p^{1-\theta}\big(\sum_{|\alpha|=m} ||D^\alpha u||_p\big)^\theta$ with $\theta = \frac{N}{m\,p}$, and this means that $\big(W^{m,p}(R^N), L^p(R^N)\big)_{\theta,1} \subset L^\infty(R^N)$ with $1 - \theta = \frac{N}{m\,p}$. From this, using the reiteration theorem one deduces that for $0 < s < \frac{N}{p}$ one has $W^{s,p}(R^N) \subset L^{p(s),q}(R^N)$ and $B^{s,p}_q(R^N) \subset L^{p(s),q}(R^N)$ with $\frac{1}{p(s)} = \frac{1}{p} - \frac{s}{N}$.

Another problem where an interpolation space with second parameter 1 is useful is the question of traces of H^s spaces. For $s > \frac{1}{2}$ functions in $H^s(R^N)$ have a trace on R^{N-1}, which belongs to $H^{s-(1/2)}(R^{N-1})$, and one can reiterate this argument, so that if $s > \frac{k}{2}$ functions in $H^s(R^N)$ have a trace on R^{N-k}, which belongs to $H^{s-(k/2)}(R^{N-k})$, and the continuity of functions in $H^s(R^N)$ for $s > \frac{N}{2}$ appears then as a natural question related to taking traces on subspaces. Although functions in $H^{1/2}(R^N)$ do not have traces on R^{N-1} (actually functions in $C_c^\infty(R^N)$ which vanish near R^{N-1} are dense in $H^{1/2}(R^N)$), the slightly smaller space $\big(H^1(R^N), L^2(R^N)\big)_{1/2,1}$ do have traces on R^{N-1}, and the space of traces is exactly $L^2(R^{N-1})$. That traces exist and belong to $L^2(R^{N-1})$ follows immediately from the standard estimate $||\gamma_0 u||_2 \le C\,||u||_2^{1/2}||\partial_N u||_2^{1/2}$, but I had not heard about this remark before a talk by Shmuel AGMON in 1975, where he discussed some joint work[1] with Lars HÖRMANDER where they had proved surjectivity by an argument of Functional Analysis, working explicitly on the transposed operator. One can construct directly a lifting by adapting an argument[2] which is related to the theorem of Emilio GAGLIARDO that every function in $L^1(R^{N-1})$ is the trace of a function in $W^{1,1}(R^N)$, and it is useful to describe that result first.

The idea is shown on the case of R^2, and given a function $f \in L^1(R)$ one wants to construct $u \in W^{1,1}(R^2)$ whose trace on the x axis is f. One uses a standard approximation argument used in Numerical Analysis, based on continuous piecewise affine functions.

For a (positive) mesh size h one considers the space E_h of functions in $L^1(R)$ which are affine on each interval $(k\,h, (k+1)h)$ for $k \in Z$, and continuous at the nodes $k\,h, k \in Z$; because there exists $0 < \alpha < \beta$ such that $\alpha(|g(0)| + |g(1)|) \le \int_0^1 |g(x)|\,dx \le \beta(|g(0)| + |g(1)|)$ for all affine functions on $(0,1)$, one deduces that for $g \in V_h$ one has $2\alpha\,h \sum_{k\in Z} |g(k\,h)| \le ||g||_1 = \int_R |g(x)|\,dx \le 2\beta\,h \sum_{k\in Z} |g(k\,h)|$; the important observation is that $V_h \subset W^{1,1}(R)$ and for $g \in V_h$ one has $||g'||_1 = \sum_{k\in Z} |g((k+1)h) - g(k\,h)| \le 2\sum_{k\in Z} |g(k\,h)| \le \frac{1}{\alpha\,h}||g||_1$.

A function $g \in V_h$ is lifted to a function in $W^{1,1}(R^2)$ by the explicit formula $G(x,y) = g(x)e^{-|y|/h}$, and one checks immediately that the trace of G is g and that $||G||_1 = 2h\,||g||_1$, $\big|\big|\frac{\partial G}{\partial y}\big|\big|_1 = 2||g||_1$ and $\big|\big|\frac{\partial G}{\partial x}\big|\big|_1 = 2h\,||g'||_1 \le \frac{2}{\alpha}||g||_1$. Once one knows that the union of all V_h for $h > 0$ is dense one has a way to lift $f \in L^2(R)$ by writing it as a series $\sum_{n=1}^\infty g_n$, choosing $0 < \theta < 1$ and choosing g_1 such that $||f - g_1||_1 \le \theta\,||f||_1$, then g_2 such that $||(f - g_1) - g_2||_1 \le \theta\,||f - g_1||_1 \le \theta^2||f||_1$, and so on, so that $\sum_{n=1}^\infty ||g_n||_1| \le \frac{1}{1-\theta}||f||_1$.

The density is proved by approximating functions in $C_c(R)$, which is a dense subspace of $L^1(R)$. For $\varphi \in C_c(R)$ one constructs the interpolated[3] function $\Pi_h\varphi$ which is the function of V_h such that $\Pi_h\varphi(k\,h) = \varphi(k\,h)$ for all $k \in Z$; one checks easily that $|\Pi_h\varphi(x) - \varphi(x)| \le \omega(h)$ for all x, where ω is the modulus

[1] They were working on questions of scattering and they needed a space whose FOURIER transform has traces on spheres, with traces belonging to L^2; they introduced then the FOURIER transform of the space described here.

[2] When I was a student, I had tried to read Emilio GAGLIARDO's article, but not knowing much Italian at the time I had trouble understanding what he was doing with all these cubes which appeared in his proof. Many years after, I constructed my own proof, but although I have not tried to check if my understanding of Italian is better now and if I can understand his proof, I think that my argument must be identical to his original idea.

[3] Here we encounter the other meaning of the word interpolation, going back to LAGRANGE, a classical tool in Numerical Analysis.

of uniform continuity of φ, so that when $h_n \to 0$, the sequence $\Pi_{h_n}\varphi$ converges uniformly to φ and as $support(\Pi_h\varphi) \subset support(\varphi) + [-h, +h]$ one also has $||\Pi_{h_n}\varphi - \varphi||_1 \to 0$.

Jaak PEETRE has shown that there does not exist a linear continuous lifting from $L^1(R)$ into $W^{1,1}(R)$ (I have never looked for the proof).

21-724. Interpolation spaces

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Again the idea is shown on the case of R^2, and given a function $f \in L^2(R)$ one wants to construct $u \in \big(H^1(R^2), L^2(R^2)\big)_{1/2,1}$ whose trace on the x axis is f. For a (positive) mesh size h one considers the space E_h of functions in $L^2(R)$ which are affine on each interval $(k\,h, (k+1)h)$ for $k \in Z$, and continuous at the nodes $k\,h, k \in Z$; because there exists $0 < \alpha < \beta$ such that $\alpha(|g(0)|^2 + |g(1)|^2)^{1/2} \le \big(\int_0^1 |g(x)|^2\,dx\big)^{1/2} \le \beta(|g(0)|^2 + |g(1)|^2)^{1/2}$ for all affine functions on $(0,1)$, one deduces that for $g \in V_h$ one has $\sqrt{2}\alpha\sqrt{h}\big(\sum_{k\in Z}|g(k\,h)|^2\big)^{1/2} \le ||g||_2 = \big(\int_R |g(x)|^2\,dx\big)^{1/2} \le \sqrt{2}\beta\sqrt{h}\big(\sum_{k\in Z}|g(k\,h)|^2\big)^{1/2}$; the important observation is that $V_h \subset H^1(R)$ and for $g \in V_h$ one has $||g'||_2 = \frac{1}{\sqrt{h}}\big(\sum_{k\in Z}|g((k+1)h) - g(k\,h)|^2\big)^{1/2} \le \frac{2}{\sqrt{h}}\big(\sum_{k\in Z}|g(k\,h)|^2\big)^{1/2} \le \frac{C}{h}||g||_2$.

A function $g \in V_h$ is lifted to a function in $H^1(R^2)$ by the explicit formula $G(x,y) = g(x)e^{-|y|/h}$, and one checks immediately that the trace of G is g and that $||G||_2 = \sqrt{h}\,||g||_2$, $\big|\big|\frac{\partial G}{\partial y}\big|\big|_2 = \frac{1}{\sqrt{h}}||g||_2$ and $\big|\big|\frac{\partial G}{\partial x}\big|\big|_2 = \sqrt{h}\,||g'||_2 \le \frac{C}{\sqrt{h}}||g||_2$. One has then $||G||_{L^2(R^2)} \le \sqrt{h}\,||g||_2$ and $||G||_{H^1(R^2)} \le \frac{C}{\sqrt{h}}||g||_2$ for $0 < h < 1$, and therefore $||G||_{(H^1(R^2),L^2(R^2))_{1/2,1}} \le C||G||_{H^1(R^2)}^{1/2}||G||_{L^2(R^2)}^{1/2} \le C'||g||_2$. Then one uses the fact that the union of all V_h for $h > 0$ is dense, so that any $f \in L^2(R)$ can be written as a series $\sum_{n=1}^\infty g_n$, with $\sum_{n=1}^\infty ||g_n||_2| \le C\,||f||_2$.

Although functions in $H^{1/2}(R)$ are not continuous, as they are not even bounded, piecewise smooth functions which are discontinuous at a point do not belong to $H^{1/2}(R)$. For example, let $\varphi \in C_c^\infty(R)$ with $\varphi = 1$ near 0 and let $u = \varphi\,H$ where H is the HEAVISIDE function; then $u' = \delta_0 + \psi$ with $\psi \in C_c^\infty(R)$ and therefore $2i\,\pi\,\xi\mathcal{F}u(\xi) = 1 + \mathcal{F}\psi(\xi)$ so that $|\mathcal{F}u|$ behaves like $\frac{1}{2\pi\,|\xi|}$ near ∞, and therefore $(1 + |\xi|^{1/2})|\mathcal{F}u| \notin L^2(R)$, so $u \notin H^{1/2}(R)$.

As it seems that functions in $H^{1/2}(R)$ cannot have discontinuities at a point, one expects some kind of continuity, but of a different nature as the value at a point does not make sense. The following ideas have been introduced by Jacques-Louis LIONS and Enrico MAGENES, and some related work has been done by Pierre GRISVARD[1].

Lemma: If $u \in H^{1/2}(R)$ then one has $\frac{|u(x)-u(-x)|}{\sqrt{|x|}} \in L^2(R)$.

Proof: By HARDY's inequality one has $\big|\big|\frac{|u(x)-u(0)|}{|x|}\big|\big|_2 \le C||u'||_2$ for $u \in H^1(R)$, and similarly one has $\big|\big|\frac{|u(-x)-u(0)|}{|x|}\big|\big|_2 \le C||u'||_2$, and therefore $\big|\big|\frac{|u(x)-u(-x)|}{|x|}\big|\big|_2 \le 2C||u'||_2$ for all $u \in H^1(R)$. One considers the mapping $u \mapsto u - \check{u}$ which maps $H^1(R)$ into L^2 for the measure $\frac{dx}{|x|^2}$ and $L^2(R)$ into L^2 for the measure dx, and therefore it maps $H^{1/2}(R) = \big(H^1(R), L^2(R)\big)_{1/2,2}$ into the corresponding interpolation space, which is L^2 for the measure $\frac{dx}{|x|}$.■

Similarly, Jacques-Louis LIONS and Enrico MAGENES noticed that when considering the interpolation spaces $\big(H_0^1(\Omega), L^2(\Omega)\big)_{\theta,2}$ for a bounded open set with a LIPSCHITZ continuous boundary, it does give $H_0^{1-\theta}(\Omega)$ for $\theta \ne \frac{1}{2}$ (and one has $H_0^s(\Omega) = H^s(\Omega)$ for $0 \le s \le \frac{1}{2}$), but for $\theta = \frac{1}{2}$ it gives a new space, which they denoted $H_{00}^{1/2}(\Omega)$.

Lemma: If $u \in H_{00}^{1/2}(0,\infty) = \big(H_0^1(0,\infty), L^2(0,\infty)\big)_{1/2,2}$, then $\frac{u}{\sqrt{x}} \in L^2(0,\infty)$.

Proof: As $u \in H_0^1(0,\infty)$ implies $\frac{u}{x} \in L^2(0,\infty)$ by HARDY's inequality, one has $\big(H_0^1(0,\infty), L^2(0,\infty)\big)_{1/2,2} \subset \big(L^2\big(0,\infty;\frac{dx}{x^2}\big), L^2(0,\infty;dx)\big)_{1/2,2} = L^2\big(0,\infty;\frac{dx}{x}\big)$.■

A related result is that if $u \in H_0^1(\Omega)$ the extension $\widetilde{u}$ of u by 0 outside Ω belongs to $H^1(R^N)$ and similarly if $u \in H_0^s(\Omega)$ (closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$) then $\widetilde{u} \in H^s(R^N)$ for $0 \le s \le 1$ if $s \ne \frac{1}{2}$, but not for $s = \frac{1}{2}$; this is easy seen as $1 \in H_0^{1/2}(\Omega) = H^{1/2}(\Omega)$ and the extension by 0 is piecewise smooth and

[1] Pierre GRISVARD, French mathematician, -1994. He worked in Nice.

discontinuous and therefore is not in $H^{1/2}(R^N)$. Actually, $H^{1/2}_{00}(\Omega)$ can be characterized either as the space of functions in $H^{1/2}(\Omega)$ such that $\frac{u}{\sqrt{d(x)}} \in L^2(\Omega)$, where $d(x)$ is the distance of x to the boundary $\partial\Omega$, or as the space of functions $u \in H^{1/2}(\Omega)$ such that $\widetilde{u} \in H^{1/2}(R^N)$.

A related difficulty is that any partial derivative $\frac{\partial}{\partial x_j}$ maps $H^1(\Omega)$ into $L^2(\Omega)$ and $L^2(\Omega)$ into $H^{-1}(\Omega)$, and it does map $H^s(\Omega)$ into $H^{-s}(\Omega) = \big(H^s_0(\Omega)\big)'$ for $0 \le s \le 1$ and $s \ne \frac{1}{2}$ but not for $s = \frac{1}{2}$, because in this case it maps $H^{1/2}(\Omega)$ into the dual of $H^{1/2}_{00}(\Omega)$.

This technical difficulty appears when one solves boundary value problems like $-\Delta\, u = f$ in Ω with the boundary $\partial\Omega$ made of two disjoint pieces, Γ_D where a DIRICHLET condition is imposed and Γ_N where a NEUMANN condition is imposed; in the case where the parts Γ_D and Γ_N have a common boundary, the precise pairs of data allowed is a little technical to characterize. If all the boundary is Γ_N then the precise space is $H^{-1/2}(\partial\Omega)$, the dual of $H^{1/2}(\partial\Omega)$ (traces of functions of $H^1(\Omega)$), but the important point is that one cannot restrict an element of $H^{-1/2}(\partial\Omega)$ to a part Γ_N whose boundary is smooth enough, because restriction is the transposed of the operator of extension by 0, and this extension operation~does not act on $H^{1/2}(\Gamma_N)$ if the (N-1)-dimensional HAUSDORFF measure of Γ_D is positive.

These technical difficulties may seem quite academic, but some modelizations in Continuum Mechanics lead to using operations which are not defined in an obvious way, and it is important to understand if one should reject some laws as being unphysical or if one should try to overcome the mathematical difficulty that they represent. One such example is the static law of friction due to COULOMB[2], which involves a sign of a normal force at the boundary and an inequality on the strength of a tangential force at the boundary; if one uses Linearized Elasticity, the natural information coming from the finiteness of the stored elastic energy gives the various forces as normal traces of functions in $H(div;\Omega)$, i.e. elements in $H^{-1/2}(\partial\Omega)$; unfortunately, one cannot define the absolute value of an arbitrary element in $H^{-1/2}(\partial\Omega)$; however, one can define what a nonnegative element is by stating that it is a nonnegative measure, so the question is to find a way to express COULOMB's law which makes sense from a mathematical point of view, although there are indications that COULOMB's law is not exactly what real materials follow (as dynamics play a role).

[2] Charles Augustin DE COULOMB, French mathematician, 1736-1806.

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One has proved that for all integers $0 < k < m$ and all $1 \leq p \leq \infty$ one has $\big(W^{m,p}(R^N), L^p(R^N)\big)_{\theta,1} \subset W^{k,p}(R^N) \subset \big(W^{m,p}(R^N), L^p(R^N)\big)_{\theta,\infty}$ with $m(1-\theta) = k$. For $s > 0$ which is not an integer one defines the SOBOLEV space $W^{s,p}(R^N)$ as $\big(W^{m,p}(R^N), L^p(R^N)\big)_{\eta,p}$ with $m > s$ and $(1-\eta)m = s$, and one deduces from the reiteration theorem of Jacques-Louis LIONS and Jaak PEETRE that one also has $W^{s,p}(R^N) = \big(W^{m_1,p}(R^N), W^{m_2,p}(R^N)\big)_{\zeta,p}$ with equivalent norms if m_1, m_2 are nonnegative integers such that $s = (1-\zeta)m_1 + \zeta\, m_2$ for some $\zeta \in (0,1)$ (i.e. either $m_1 > s > m_2$ or $m_1 < s < m_2$), and this is still true even if m_1, m_2 are nonnegative reals numbers which are not necessarily integers, always under the condition $\zeta \in (0,1)$, of course.

Similarly, if for $s > 0$ and $1 \leq q \leq \infty$ one defines the BESOV space $B^{s,p}_q(R^N)$ as $\big(W^{m,p}(R^N), L^p(R^N)\big)_{\eta,q}$ with $m > s$ and $(1-\eta)m = s$, one deduces that for nonnegative reals s_1, s_2 such that $s = (1-\zeta)s_1 + \zeta\, s_2$ with $\zeta \in (0,1)$, one has $B^{s,p}_q(R^N) = \big(W^{s_1,p}(R^N), W^{s_2,p}(R^N)\big)_{\zeta,q}$ and $B^{s,p}_q(R^N) = \big(B^{s_1,p}_{q_1}(R^N), B^{s_2,p}_{q_2}(R^N)\big)_{\zeta,q}$ with equivalent norms, for all $1 \leq q_1, q_2 \leq \infty$.

If s is not an integer, one has $W^{s,p}(R^N) = B^{s,p}_p(R^N)$.

Although one has $H^1(R^N) = \big(H^2(R^N), L^2(R^N)\big)_{1/2,2}$, the case for $p = \infty$ is different and the space $W^{1,\infty}(R^N)$ which is the space of LIPSCHITZ continuous functions (often denoted $Lip(R^N)$), is a proper subspace of $\big(W^{2,\infty}(R^N), L^\infty(R^N)\big)_{1/2,\infty}$, which coincides with a space introduced by Antoni ZYGMUND and denoted $\Lambda_1(R^N)$, which is the space of functions $u \in L^\infty(R^N)$ for which there exists C such that $|u(x-h) + u(x+h) - 2u(x)| \leq C\,|h|$ for all $x, h \in R^N$, or equivalently $||\tau_h u + \tau_{-h} u - 2u||_\infty \leq C|h|$ for all $h \in R^N$.

One deduces also that if $s > k$ where k is a positive integer, then for every multi-index α with $|\alpha| = k$ the derivation D^α is linear continuous from $W^{s,p}(R^N)$ into $W^{s-|\alpha|,p}(R^N)$ and also from $B^{s,p}_q(R^N)$ into $B^{s-|\alpha|,p}_q(R^N)$, if $1 \leq p, q \leq \infty$. This follows immediately from the fact that for any integer $m \geq k$ the derivation D^α is linear continuous from $W^{m,p}(R^N)$ into $W^{m-|\alpha|,p}(R^N)$, and after choosing m_1, m_2 such that $k \leq m_1 < s < m_2$ and computed $\zeta \in (0,1)$ such that $s = (1-\zeta)m_1 + \zeta\, m_2$, one applies the interpolation property, and D^α maps continuously $\big(W^{m_1,p}(R^N), W^{m_2,p}(R^N)\big)_{\zeta,q} = B^{s,p}_q(R^N)$ into $\big(W^{m_1-|\alpha|,p}(R^N), W^{m_2-|\alpha|,p}(R^N)\big)_{\zeta,q} = B^{s-|\alpha|,p}_q(R^N)$.

For Ω an open subset of R^N, one may define $W^{s,p}(\Omega)$ for all positive real s which are not integers in at least two different ways; the first one as the space of restrictions of functions from $W^{s,p}(R^N)$, with the quotient norm $||u||_{W^{s,p}(\Omega)} = \inf_{U|_\Omega = u} ||U||_{W^{s,p}(R^N)}$, which will be denoted $X^{s,p}$ for the discussion; the second one as an interpolation space $\big(W^{m_1,p}(\Omega), W^{m_2,p}(\Omega)\big)_{\zeta,p}$ with m_1, m_2 nonnegative integers such that $s = (1-\zeta)m_1 + \zeta\, m_2$ with $0 < \zeta < 1$, which will be denoted $Y^{s,p}$ for the discussion (of course, one can also give two definitions for defining the spaces $B^{s,p}_q(\Omega)$).

Because the restriction to Ω is linear continuous from $W^{m_1,p}(R^N)$ into $W^{m_1,p}(\Omega)$ and also linear continuous from $W^{m_2,p}(R^N)$ into $W^{m_2,p}(\Omega)$, it is continuous from $W^{s,p}(R^N) = \big(W^{m_1,p}(R^N), W^{m_2,p}(R^N)\big)_{\zeta,p}$ into $Y^{s,p} = \big(W^{m_1,p}(\Omega), W^{m_2,p}(\Omega)\big)_{\zeta,p}$, and as every element of $X^{s,p}$ is the restriction of an element from $W^{s,p}(R^N)$, one deduces that $X^{s,p} \subset Y^{s,p}$.

If the boundary of Ω is smooth enough so that there exists a continuous extension P which maps $W^{m_1,p}(\Omega)$ into $W^{m_1,p}(R^N)$ and $W^{m_2,p}(\Omega)$ into $W^{m_2,p}(R^N)$, then it maps $Y^{s,p} = \big(W^{m_1,p}(\Omega), W^{m_2,p}(\Omega)\big)_{\zeta,p}$ into $W^{s,p}(R^N) = \big(W^{m_1,p}(R^N), W^{m_2,p}(R^N)\big)_{\zeta,p}$, and therefore every element of $Y^{s,p}$ is the restriction to Ω of an element of $W^{s,p}(R^N)$, i.e. one has $Y^{s,p} \subset X^{s,p}$.

The extension property has been shown for $W^{1,p}(\Omega)$ if Ω is bounded with a LIPSCHITZ continuous boundary, and an anologous situation has been described which works for $W^{k,p}(R^N_+)$ and $0 \leq k \leq m$, and it extends to the case of bounded open sets with smooth boundary. Actually STEIN has constructed an extension valid for all $W^{m,p}(\Omega)$ if Ω is bounded with a LIPSCHITZ continuous boundary.

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15. Wednesday April 19, 2000.

Before giving a characterization of $W^{s,p}(\Omega)$ in the case of a bounded open set with a LIPSCHITZ continuous boundary, it is useful to begin with the case of R^N. One starts with a preliminary result.

Lemma: If $0 < s < \infty$ and $1 \le p \le \infty$ one has $B^{s,p}_\infty = \big(W^{1,p}(R^N), L^p(R^N)\big)_{1-s,\infty} = \{u \in L^p(R^N)$, there exists C such that $||u - \tau_h u||_p \le C\,|h|^s$ for all $h \in R^N\}$.

Proof: If $u \in W^{1,p}(R^N)$ one has $||u - \tau_h u||_p \le |h|\,||grad(u)||_p$ and if $u \in L^p(R^N)$ one has $||u - \tau_h u||_p \le 2||u||_p$, and as the mapping $u \mapsto u - \tau_h u$ is linear, one finds that for $\theta \in (0,1)$ one has $||u - \tau_h u||_p \le C\,|h|^{1-\theta}||u||_{(W^{1,p}(R^N),L^p(R^N))_{\theta,\infty}}$ for all $u \in \big(W^{1,p}(R^N), L^p(R^N)\big)_{\theta,\infty}$.

Conversely, assume that $u \in L^p(R^N)$ and $||u - \tau_h u||_p \le C\,|h|^s$ for all $h \in R^N$. One decomposes $u = \rho_\varepsilon \star u + (u - \rho_\varepsilon \star u)$, where ρ_ε is a special smoothing sequence, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho_1\big(\frac{x}{\varepsilon}\big)$, with $\rho_1 \in C^\infty_c(R^N)$ and $\int_{R^N} \rho_1(x)\,dx = 1$, but one adds the hypothesis that ρ_1 is an even function. One has $u(x) - (\rho_\varepsilon \star u)(x) = \int_{R^N} \rho_\varepsilon(y)\big(u(x) - u(x-y)\big)\,dy$ and therefore $||u - \rho_\varepsilon \star u||_p \le \int_{R^N} |\rho_\varepsilon(y)|\,||u - \tau_y u||_p\,dy \le \int_{R^N} C\,|\rho_\varepsilon(y)|\,|y|^s\,dy = C'\,\varepsilon^s$. One has $||\rho_\varepsilon \star u||_p \le ||\rho_\varepsilon||_1 ||u||_p = C\,||u||_p$, and for any derivative $\partial_j = \frac{\partial}{\partial x_j}$ one has $\partial_j(\rho_\varepsilon \star u) = (\partial_j \rho_\varepsilon) \star u$, but one needs a better estimate than $||\partial_j(\rho_\varepsilon \star u)||_p \le ||(\partial_j \rho_\varepsilon)||_1 ||u||_p = \frac{C}{\varepsilon}||u||_p$, which has not used all the information on u; changing y into $-y$ in the integral, one can write $\partial_j(\rho_\varepsilon \star u)(x)$ as either $\int_{R^N} (\partial_j \rho_\varepsilon)(y) u(x-y)\,dy$ or $\int_{R^N} (\partial_j \rho_\varepsilon)(-y) u(x+y)\,dy$ or as the half sum of these two terms, and this is where having choosen for ρ_1 an even function is useful, as $\partial_j \rho_\varepsilon$ is an odd function and therefore $\partial_j(\rho_\varepsilon \star u)(x) = \frac{1}{2}\int_{R^N} (\partial_j \rho_\varepsilon)(y)\big(u(x-y) - u(x+y)\big)\,dy$, from which one deduces $||\partial_j(\rho_\varepsilon \star u)||_p \le \frac{1}{2}\int_{R^N} |(\partial_j \rho_\varepsilon)(y)|\,||\tau_y u - \tau_{-y} u||_p\,dy \le C\int_{R^N} |(\partial_j \rho_\varepsilon)(y)|\,|h|^s\,dy = C''\varepsilon^{s-1}$. For $E_0 = W^{1,p}(R^N)$ and $E_1 = L^p(R^N)$, this shows that $K(t,u) \le C_1(1+\varepsilon^{s-1}) + C_2 t\,\varepsilon^s$, and choosing $\varepsilon = \frac{1}{t}$ for $t \ge 1$, one obtains $K(t,u) \le C_3 t^{1-s}$ and for $0 < t < 1$ one has $K(t,u) \le t\,||u||_p$ (because in the case $E_0 \subset E_1$ one always has the decomposition $u = 0 + u$), and therefore $t^{-\theta}K(t,u) \in L^\infty(0,\infty)$ with $\theta = 1 - s$.■

Proposition: For $0 < s < 1$ and $1 \le p < \infty$, $W^{s,p}(R^N) = \{u \in L^p(R^N), \int_{R^N}\int_{R^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty\}$, or equivalently $u \in L^p(R^N)$ and $\int_{R^N} \big(\frac{||u-\tau_y u||_p}{|y|^s}\big)^p \frac{dy}{|y|^N} < \infty$.

Proof: Let $E_0 = W^{1,p}(R^N)$ and $E_1 = L^p(R^N)$. If $u \in W^{s,p}(R^N) = (E_0,E_1)_{\theta,p}$ with $\theta = 1 - s$, one has a decomposition $u = u_0 + u_1$ with $u_0 \in E_0, u_1 \in E_1$ and $||grad(u_0)||_p + t\,||u_1||_p \le 2K(t,u)$ and $t^{-\theta}K(t,u) \in L^p\big(0,\infty;\frac{dt}{t}\big)$. For $y \in R^N$ one has $||u - \tau_y u||_p \le ||u_0 - \tau_y u_0||_p + ||u_1 - \tau_y u_1||_p \le |y|\,||grad(u_0)||_p + 2||u_1||_p \le C\big(|y| + \frac{1}{t}\big)K(t,u)$; one chooses $t = \frac{1}{|y|}$ and one uses $||u - \tau_y u||_p \le C\,|y|\,K\big(\frac{1}{|y|},u\big)$. This gives $\int_{R^N} \big(\frac{||u-\tau_y u||_p}{|y|^s}\big)^p \frac{dy}{|y|^N} \le C\int_{R^N} \big(\frac{|y|\,K(\frac{1}{|y|},u)}{|y|^s}\big)^p \frac{dy}{|y|^N}$, and denoting by σ_{N-1} the $(N-1)$-HAUSDORFF measure of the unit sphere, it is $= C\,\sigma_{N-1}\int_0^\infty \big[|y|^{1-s}K\big(\frac{1}{|y|},u\big)\big]^p \frac{dy}{|y|} = C\,\sigma_{N-1}\int_0^\infty \big(t^{-\theta}K(t,u)\big)^p \frac{dt}{t} < \infty$.

Conversely, assume that $u \in L^2(R^N)$ and $\int_{R^N}\int_{R^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$, so that if one denotes $F(z) = ||u - \tau_z u||_p$ for $z \in R^N$, one has $\int_{R^N} \frac{F(z)}{|z|^{N+s\,p}}\,dz < \infty$. Let $\overline{F}(r)$ be the average of F on the sphere $|y| = r$, so that by HÖLDER inequality one has $\int_{|y|=r} |\overline{F}(y)|^p\,dH^{N-1} \le \int_{|y|=r} |F(y)|^p\,dH^{N-1}$, and therefore $\int_0^\infty \frac{|\overline{F}(r)|^p}{r^{s\,p}}\frac{dr}{r} = \frac{1}{\sigma_{N-1}}\int_{R^N} \frac{\overline{F}(z)}{|z|^{N+s\,p}}\,dz \le \int_{R^N} \frac{F(z)}{|z|^{N+s\,p}}\,dz < \infty$.

Then as for the preceding lemma one decomposes $u = \rho_\varepsilon \star u + (u - \rho_\varepsilon \star u)$ for an even special smoothing sequence with $support(\rho_1) \subset \overline{B(0,1)}$, and one obtains $||u - \rho_\varepsilon \star u||_p \le \int_{R^N} |\rho_\varepsilon(y)|\,||u - \tau_y u||_p\,dy = \int_{R^N} |\rho_\varepsilon(y)|\,F(y)\,dy \le \frac{C}{\varepsilon^N}\int_{|y|\le\varepsilon} F(y)\,dy = \frac{C}{\varepsilon^N}\int_{|y|\le\varepsilon} \overline{F}(y)\,dy = \frac{C\sigma_{N-1}}{\varepsilon}\int_0^\varepsilon \overline{F}(r)\,dr$; then using the fact that $\partial_j \rho_\varepsilon$ is odd, one obtains $||\partial_j(\rho_\varepsilon \star u)||_p \le \frac{1}{2}\int_{R^N} |\partial_j \rho_\varepsilon(y)|\,||\tau_y u - \tau_{-y} u||_p\,dy \le \frac{1}{2}\int_{R^N} |\partial_j\big(F(y) + F(-y)\big)\,dy \le \frac{C}{\varepsilon_{N-1}}\int_{|y|\le\varepsilon} \overline{F}(y)\,dy \le \frac{C\sigma_{N-1}}{\varepsilon^2}\int_0^\varepsilon \overline{F}(r)\,dr$. One deduces that $K(t,u) \le C\big(1 + \frac{1}{\varepsilon^2}\int_0^\varepsilon \overline{F}(r)\,dr\big) + \frac{C\,t}{\varepsilon}\int_0^\varepsilon \overline{F}(r)\,dr$, and therefore for $t > 1$ one takes $\varepsilon = \frac{1}{t}$ and one has $K(t,u) \le C + C\,t^2\int_0^{1/t} \overline{F}(r)\,dr$ (and $K(t,u) \le t\,||u||_p$ for $t < 1$). Then with $\theta = 1 - s$, one has $t^{-\theta}K(t,u) = C\,t^{-\theta} + C\,t^s t\int_0^{1/t} \overline{F}(r)\,dr$, and as the desired condition $t^{-\theta}K(t,u) \in L^p\big(0,\infty;\frac{dt}{t}\big)$ is invariant by the change of t into $\frac{1}{t}$, one has to show that the function

G defined on $(0,1)$ by $G(\varepsilon) = \frac{1}{\varepsilon} \int_0^\varepsilon \frac{\overline{F}(r)}{\varepsilon^s}\, dr$ belongs to $L^p\big(0,1; \frac{d\varepsilon}{\varepsilon}\big)$, as a consequence of the hypothesis that $\frac{\overline{F}(r)}{r^s} \in L^p\big(0,\infty; \frac{d\varepsilon}{\varepsilon}\big)$, and this is like deriving HARDY's inequality.■

Of course, the proof also applies to the case $p = \infty$, but is covered by the previous Lemma.

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16. Friday April 21, 2000.

One can now deduce a characterization of $W^{s,p}(\Omega)$ in the case of a bounded open set of R^N with a LIPSCHITZ continuous boundary, with $0 < s < 1$ and $1 \le p \le \infty$.

Proposition: If Ω is a bounded open set of R^N with a LIPSCHITZ continuous boundary, then for $0 < s < 1$ and $1 \le p < \infty$, $u \in W^{s,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and $\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$.

Proof: If $u \in W^{s,p}(\Omega)$, defined as the restriction to Ω of a function $U \in W^{s,p}(R^N)$, then one has $U \in L^p(R^N)$ and therefore by restriction $u \in L^p(\Omega)$, and $\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \int_\Omega \frac{|U(x)-U(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy \le \int_{R^N} \int_{R^N} \frac{|U(x)-U(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$.

Let $u \in L^p(\Omega)$ satisfy $\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$. For a partition of unity $\theta_i, i \in I$, each $\theta_i u$ satisfies a similar property, because $|(\theta_i u)(x) - (\theta_i u)(y)| \le |\theta_i(x)|\,|u(x)-u(y)| + |\theta_i(x)-\theta_i(y)|\,|u(y)| \le C\,|u(x)-u(y)| + C\,|x-y|\,|u(y)|$, using the assumption that each θ_i is a LIPSCHITZ continuous function. One deduces that $|(\theta_i u)(x) - (\theta_i u)(y)|^p \le C'\,|u(x)-u(y)|^p + C'\,|x-y|^p\,|u(y)|^p$, and one must show that $\int_\Omega \int_\Omega \frac{|x-y|^p |u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$; this is a consequence of $\int_\Omega \frac{|x-y|^p}{|x-y|^{N+s\,p}}\,dx \le M$ uniformly in $y \in \Omega$, which holds because if Ω has diameter d, the integral is bounded by $\int_{|z|\le d} \frac{|z|^p}{|z|^{N+s\,p}}\,dz = C \int_0^d \frac{dt}{t^\alpha} < \infty$, as $\alpha = 1 + s\,p - p < 1$. Using a local change of basis, one is led to consider the case of $\Omega_F = \{(x', x_N), x_N > F(x')\}$ for F LIPSCHITZ continuous, and one extends u by symmetry, defining $P\,u(x', x_N) = u(x', x_N)$ if $x_N > F(x')$ and $P\,u(x', x_N) = u(x', 2F(x') - x_N)$ if $x_N < F(x')$. If for $x = (x', x_N)$ one defines $\overline{x} = (x', 2F(x') - x_N)$, so that $\overline{(\overline{x})} = x$, then the integral $\int_{R^N} \int_{R^N} \frac{|P\,u(x) - P\,u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy$ can be cut into four parts, one is $I = \int_{\Omega_F} \int_{\Omega_F} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy$, which is finite by hypothesis; two parts have the form $\int_{\Omega_F} \int_{R^N \setminus \Omega_F} \frac{|u(x)-u(\overline{y})|^p}{|x-y|^{N+s\,p}}\,dx\,dy \le K^p I$ because $|x - \overline{y}| \le K\,|x-y|$ for all $x \in \Omega_F, y \in R^N \setminus \Omega_F$, and the fourth part is $\int_{R^N \setminus \Omega_F} \int_{R^N \setminus \Omega_F} \frac{|u(\overline{x})-u(\overline{y})|^p}{|x-y|^{N+s\,p}}\,dx\,dy \le K^p I$ because $|\overline{x} - \overline{y}| \le K\,|x-y|$ for $x, y \in R^N \setminus \Omega_F$; indeed the map $x = (x', x_N) \mapsto \big(x', x_N - F(x')\big)$ is LIPSCHITZ continuous from Ω_F onto R^N_+, and its inverse is LIPSCHITZ continuous as it is $z = (z', z_N) \mapsto \big(z', z_N + F(z')\big)$, and one is reduced to study the same inequalities for R^N_+, i.e. in the case $F = 0$, where one has $|x - \overline{y}| \le |x-y|$ if $x_N > 0 > y_N$, and $|\overline{x} - \overline{y}| = |x-y|$ for all x, y.■

Of course, the proof adapts to the case $p = \infty$, the only difference coming from the fact that the norm is not expressed by an integral. It is important to notice that without any regularity hypothesis on a set $A \subset R^N$, any LIPSCHITZ continuous function defined on A can be extended to R^N, and the same result is true for HÖLDER continuous functions. For showing this, one assumes that $C_1 \le u(x) \le C_2$ for all $x \in A$, and that there exists $\alpha \in (0,1]$ and $K \ge 0$ such that $|u(x) - u(y)| \le K\,|x-y|^\alpha$ for all $x, y \in A$, then one defines $v(x) = \sup_{y \in A} u(y) - K\,|x-y|^\alpha$, and this gives a HÖLDER continuous function which coincides with u on A, and then one truncates it by $w(x) = \min\{C_2, \max\{C_1, v(x)\}\}$.

One should notice that $W^{1,\infty}(\Omega)$ contains the space $Lip(\Omega)$ of LIPSCHITZ continuous functions, but may be different if Ω is not a bounded open set with a LIPSCHITZ continuous boundary, because $u \in W^{1,\infty}(\Omega)$ implies that there exists K such that $|u(x) - u(y)| \le K\,d_\Omega(x,y)$ where $d_\Omega(x,y)$ is the geodesic distance from x to y, the infimum of the lengths of paths from x to y which stay in Ω, and the geodesic distance from x to y could be much larger than the Euclidean distance from x to y.

As a way to ascertain the importance of the regularity of the boundary in proving some properties of SOBOLEV spaces, I describe a counter-example which I had constructed in order to answer (partially) a question that Sergei VODOP'YANOV[1] had asked a few years ago in a talk at CMU; I mentioned my result to my good friend Edward FRAENKEL[2], who had studied domains with irregular boundaries, and he later

[1] Sergei Konstantinovitch VODOP'YANOV, Russian mathematician; he works in the SOBOLEV Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk.

[2] Ludwig Edward FRAENKEL, German-born mathematician, 1927; his father, Eduard FRAENKEL, 1888-1970, was a classical scholar who emigrated to England in the 30s and was the first foreigner to obtain

mentioned it to O'FARRELL[3], who gave a more general construction. After learning about O'FARRELL's construction, I contacted Sergei VODOP'YANOV who mentioned that he had also done the case $p > 2$.

Proposition: For $N \geq 2$, there exists a (bounded) connected open set Ω such that $W^{1,\infty}(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$.
Proof: I only give the proof for $N = 2$ and for the case $2 < p < \infty$; for the case $1 \leq p \leq 2$, O'FARRELL has introduced a more technical construction. One defines $\Omega = A \cup B \bigcup_n C_n$, where A is the open set $\{(x,y), x > 0, y < 0, x^2 + y^2 < 1\}$, B is the open set $\{(x,y), x < 0, y < 0, x^2 + y^2 < 1\}$, and for $n \geq 1$ the passage C_n is defined in polar coordinates by $\{(x,y), 0 \leq \theta \leq \pi, 2^{-n} - \varepsilon_n < r < 2^{-n}\}$. The sequence ε_n satisfies $0 < \varepsilon_n < 2^{-n-1}$ so that the passages do not overlap but also $\varepsilon_n \to 0$ sufficiently rapidly so that $\sum_n \varepsilon_n 2^{n\,p} < \infty$ for every $p \in [1,\infty)$.

One checks immediately that the function u_* defined by $u_* = 0$ in A, $u_* = \pi$ in B and $u_* = \theta$ in all C_n belongs to $W^{1,p}(\Omega)$ for $1 \leq p < \infty$. This function u_* cannot be approached in $W^{1,p}(\Omega)$ by functions in $W^{1,\infty}(\Omega)$. Indeed, because A is a bounded open set with a LIPSCHITZ continuous boundary, functions in $W^{1,p}(A)$ have an extension in $W^{1,p}(R^2)$, which is a continuous function as $p > N = 2$, and therefore $u \in W^{1,p}(\Omega)$ implies $u \in C^0(\overline{A})$, and the linear form $L_A u = u(0)$ is continuous, where $u(0)$ is computed from the side of A. The same argument applies for B, and the linear form $L_B u = u(0)$ is continuous, where $u(0)$ is now computed from the side of B. One has $L_B(u_*) - L_A(u_*) = \pi$, and u_* cannot be approached by functions from $W^{1,\infty}(\Omega)$ because $L_B(u) = L_A(u)$ for all $u \in W^{1,\infty}(\Omega)$, and therefore any function v belonging to the closure of $W^{1,\infty}(\Omega)$ must satisfy $L_B(v) = L_A(v)$. Indeed, if $u \in W^{1,\infty}(\Omega)$ one has $|u(x) - u(y)| \leq ||grad(u)||_\infty d_\Omega(x,y)$ where $d_\Omega(x,y)$ is the shortest distance from x to y when one stays in Ω, and letting x tend to 0 from the side of A and y tend to 0 from the side of B, one has $d_\Omega(x,y) \to 0$ because there are arbitrary short paths by using the passages C_n for large n.∎

For the general case $1 \leq p < \infty$, O'FARRELL's starts with a CANTOR[4] set with positive measure in a segment imbedded in an open set of R^2, and he constructs passages through the complement of the CANTOR set in such a way that going from one side to the other by using the passages can be done with $d_\Omega(x,y) \leq C\, d(x,y)$ for all $x, y \in \Omega$; then he defines a function taking different values on both sides of the CANTOR set, which cannot be approached by functions in $W^{1,\infty}(\Omega)$ because they take the same value on both sides of the CANTOR set, and for functions in $W^{1,1}(\Omega)$ the restriction to any side of the CANTOR set is continuous.

The classical CANTOR set has measure zero, and is used in the construction of a nondecreasing function f_∞ on $[0,1]$, with $f(0) = 0, f(1) = 1$, but is constant on disjoints intervals $I_n, n \geq 1$ such that $\sum_n length(I_n) = 1$; this construction is often called the "devil's staircase", as it has infinitely many flat levels (steps) and it goes up from 0 to 1 without having any jump.

The particular function f_∞ constructed satisfies $f_\infty(1-x) = 1 - f_\infty(x)$ for all $x \in [0,1]$, and $f_\infty(3x) = 2f_\infty(x)$ for all $0 \leq x \leq \frac{1}{3}$, and it is HÖLDER continuous of order $\alpha = \frac{\log 3}{\log 2}$; it is the unique fixed point of the mapping T defined on functions φ which are continuous on $[0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$, and $T(\varphi) = \psi$ means $\psi(x) = \frac{1}{2}\varphi(3x)$ for $0 \leq x \leq \frac{1}{3}$, $\psi(y) = \frac{1}{2}$ for $x \in \left[0, \frac{1}{3}\right]$ and $\psi(z) = 1 - \psi(1-z)$ for $\frac{2}{3} \leq z \leq 1$; one usually starts from $f_0(x) = x$ for all x and one defines $f_n = T(f_{n-1})$ for $n \geq 1$ and f_n converges to f_∞ uniformly.

a prestigious chair at Oxford. Edward studied in Canada during World War II, and specialized in Fluid Dynamics; he later became professor in Cambridge, where his too mathematical approach irritated his colleagues who were working in the traditional British way of doing Applied Mathematics, where one writes expansions without wondering too much if they converge or if they really approach the solution of the real problem that one is interested in. Edward's approach is more akin to the French way of doing Mathematics with applications to real problems (which one should certainly criticize for being often too abstract), where one is supposed to prove all the assertions made. Edward was bound to have difficulties with his colleague for he used to construct counter-examples showing that the methods that were taught did not always work. He works at the University of Bath, England.

[3] Anthony G. O'FARRELL, Irish mathematician; he works at National University of Ireland, Maynooth.

[4] Georg Ferdinand Ludwig Philipp CANTOR, German mathematician, 1845-1918. He worked in Halle.

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The characterizations of $W^{s,p}(\Omega)$ and $W^{s,p}(R^N)$ provide a characterization of which functions $u \in W^{s,p}(\Omega)$ are such that the extension of u by 0 outside Ω, denoted $\widetilde{u}$, belong to $W^{s,p}(R^N)$.

Lemma: Let Ω be a bounded open set of R^N with a LIPSCHITZ continuous boundary. Then, for $0 < s < 1$ and $1 \le p \le \infty$ one has $\widetilde{u} \in W^{s,p}(R^N)$ if and only if $u \in W^{s,p}(\Omega)$ and $d^{-s}u \in L^p(\Omega)$, where $d(x)$ denotes the distance from x to the boundary $\partial\Omega$.

Proof: One shows the case $p < \infty$, the case $p = \infty$ being easier, using the fact that the functions used are HÖLDER continuous of order s. $\widetilde{u} \in L^p(R^N)$ is equivalent to $u \in L^p(\Omega)$ and $\int_{R^N}\int_{R^N} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$ is equivalent to $\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$ and $\int_\Omega\int_{R^N\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{N+s\,p}}\,dx\,dy < \infty$, i.e. $u \in W^{s,p}(\Omega)$ and $\varphi\,u \in L^p(\Omega)$ where $|\varphi(x)|^p = \int_{R^N\setminus\Omega} \frac{1}{|x-y|^{N+s\,p}}\,dy$ for $x \in \Omega$. Because $B\big(0,d(x)\big) \subset \Omega$ one has $|\varphi(x)|^p \le \int_{|z|\ge d(x)} \frac{1}{|z|^{N+s\,p}}\,dz = \frac{c_N}{d(x)^{s\,p}}$, so that $|\varphi(x)| \le C\,d(x)^{-s}$. This shows that if $u \in W^{s,p}(\Omega)$ and $d^{-s}u \in L^p(\Omega)$ then $\widetilde{u} \in W^{s,p}(R^N)$.

In order to prove the other implication, one uses a partition of unity, $\theta_i, i \in I$, and one notices that $v_i = \theta_i u \in W^{s,p}(\Omega)$ and if $\widetilde{u} \in W^{s,p}(R^N)$ one has $\widetilde{v_i} = \theta_i\widetilde{u} \in W^{s,p}(R^N)$, and as I is finite it is enough to show that for each i one has $d^{-s}v_i \in L^p(\Omega)$. This corresponds to proving that $|\varphi| \ge C\,d^{-s}$ in the case of Ω_F when F is LIPSCHITZ continuous; by using the mapping $(x', x_N) \mapsto \big(x', x_N - F(x')\big)$ whose inverse is $\big(x', x_N) \mapsto (x', x_N + F(x')\big)$ which are both LIPSCHITZ continuous, one has to consider the case $F = 0$, and in that case one has $\int_{y_N<0} \frac{1}{|x-y|^{N+s\,p}}\,dy = K\,|x_N|^{-s}$ if $x_N > 0$, by an argument of homogeneity, and K is the value of the integral for $x_N = 1$.■

The devil's staircase is an example of a function which is not absolutely continuous, a term equivalent to having the derivative in L^1, and the derivative in the sense of distribution is actually a nonnegative measure whose support is the CANTOR set, and it is useful to show Laurent SCHWARTZ's proof that nonnegative distributions are RADON measures.

Lemma: If a distributions $T \in \mathcal{D}'(\Omega)$ is nonnegative in the sense that $\langle T,\varphi\rangle \ge 0$ for all $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \ge 0$, then T is a nonnegative RADON measure.

Proof: Let K be a compact in Ω and let $\theta \in C_c^\infty(\Omega)$ satisfy $\theta \ge 0$ everywhere and $\theta = 1$ on K. Then for every $\varphi \in C_c^\infty(\Omega)$ with *support*$(\varphi) \subset K$, one has $-||\varphi||_\infty\theta \le \varphi \le ||\varphi||_\infty\theta$, and therefore $-||\varphi||_\infty\langle T,\theta\rangle \le \langle T,\varphi\rangle \le ||\varphi||_\infty\langle T,\theta\rangle$, i.e. $|\langle T,\varphi\rangle| \le C_K||\varphi||_\infty$, with $C_K = \langle T,\theta\rangle$. Then one extends this inequality to the case where $\varphi \in C_c(\Omega)$ with *support*$(\varphi) \subset K$, showing that T is a RADON measure, by applying the preceding inequality to $\rho_\varepsilon \star \varphi$ for a smoothing sequence ρ_ε (and using $C_{K'}$ for a larger compact set). Of course one also proves in this way that $\langle T,\varphi\rangle \ge 0$ for all $\varphi \in C_c(\Omega)$ such that $\varphi \ge 0$.■

In one dimension, one says that a function f has bounded variation if there exists a constant C such that for all N and all increasing sequences $x_1 < x_2 < \ldots < x_N$ one has $\sum_{i=1}^{N-1} |f(x_i) - f(x_{i+1})| \le C$. One proves then that such a function f has a limit on the left and a limit on the right at every point, with at most a countable number of points of discontinuity, and that $f = g - h$ where g and h are nondecreasing, and as one checks easily by regularization that the derivative of a nondecreasing function is a nonnegative RADON measure, one finds that if f has bounded variation then f' is a RADON measure with finite total mass (and the converse is true, because any RADON measure μ can be written as $\mu_+ - \mu_-$ for nonnegative RADON measures μ_+, μ_-, and every nonnegative RADON measure is the derivative in the sense of distributions of a nondecreasing function.

In order to define functions of bounded variation in more than one space dimension, one needs to define the space $\mathcal{M}_b$ of bounded RADON measures.

If K is a compact, then $C(K)$ the space of continuous functions on K equipped with the sup norm is a BANACH space, whose dual $\mathcal{M}(K)$ is the space of RADON measures on K, equipped with the dual norm $||\mu|| = \sup_{||\varphi||_\infty\le 1} |\langle\mu,\varphi\rangle|$. If M_n are distinct points in K then $\mu = \sum_n c_n\delta_{M_n}$ belongs to $\mathcal{M}(K)$ if and only if $\sum_n |c_n| < \infty$ and one has $||\mu|| = \sum_n |c_n|$.

If Ω is open, then $C_c(\Omega)$ and its dual $\mathcal{M}(\Omega)$ of all the RADON measures in Ω are not BANACH spaces (nor FRÉCHET spaces either), and if a sequence M_n of points tends to the boundary $\partial\Omega$ then $\mu = \sum_n c_n \delta_{M_n}$ belongs to $\mathcal{M}(\Omega)$ for all coefficients c_n (as each compact K only contains a finite number of these points).

The space $\mathcal{M}_b(\Omega)$, the space of bounded RADON measures, also called the space of measures with finite (total) mass or the space of measures with finite total variation, is the space of $\mu \in \mathcal{M}(\Omega)$ such that there exists C with $|\langle \mu, \varphi\rangle| \leq C||\varphi||_\infty$ for all $\varphi \in C_c(\Omega)$, and it can be shown that μ can be extended to all $\varphi \in C_b(\Omega)$, the space of bounded continuous functions in Ω, which is a BANACH space with the sup norm, so that $\mathcal{M}_b(\Omega)$ is the dual of $C_b(\Omega)$ and is a BANACH space. If M_n are distinct points in Ω then $\mu = \sum_n c_n \delta_{M_n}$ belongs to $\mathcal{M}_b(\Omega)$ if and only if $\sum_n |c_n| < \infty$ and one has $||\mu|| = \sum_n |c_n|$.

The extension to more than one space dimension has been studied by Ennio DE GIORGI, FEDERER and Wendell FLEMING, but the earlier names of TONELLI[1] and Lamberto CESARI[2] are often mentioned.

Definition: For an open set $\Omega \subset R^N$, a function u belongs to $BV(\Omega)$, the space of functions of bounded variation in Ω if $u \in L^1(\Omega)$ and $\frac{\partial u}{\partial x_j} \in \mathcal{M}_b(\Omega)$ for $j = 1, \ldots, N$.■

Proposition: $BV(R^N)$ is continuously imbedded in $L^{1^*,1}(R^N)$ for $N \geq 2$, and in $L^\infty(R)$ for $N = 1$. If ρ_ε is a smoothing sequence and $u \in BV(R^N)$ then $\rho_\varepsilon \star u$ is bounded in $W^{1,1}(R^N)$. $BV(R^N) = \{u \in L^1(R^N), ||\tau_h u - u||_1 \leq C\,|h| \text{ for all } h \in R^N\}$.

Proof: For $j = 1, \ldots, N$ and $\varphi \in C_c^\infty(R^N)$ one has $\langle \frac{\partial(\rho_\varepsilon \star u)}{\partial x_j}, \varphi\rangle = -\langle \rho_\varepsilon \star u, \frac{\partial \varphi}{\partial x_j}\rangle = -\langle u, \check{\rho}_\varepsilon \star \frac{\partial \varphi}{\partial x_j}\rangle = -\langle u, \frac{\partial(\check{\rho}_\varepsilon \star \varphi)}{\partial x_j}\rangle = \langle \frac{\partial u}{\partial x_j}, \check{\rho}_\varepsilon \star \varphi\rangle$, and therefore as $\check{\rho}_\varepsilon \star \varphi$ is continuous with compact support and has a sup norm $\leq C\,||\varphi||_\infty$ one deduces that $\big|\langle \frac{\partial(\rho_\varepsilon \star u)}{\partial x_j}, \varphi\rangle\big| \leq C\,||\varphi||_\infty$ and as $\frac{\partial(\rho_\varepsilon \star u)}{\partial x_j} \in C^\infty(\Omega)$ it means that $\big|\big|\frac{\partial(\rho_\varepsilon \star u)}{\partial x_j}\big|\big|_1 \leq C$. By Jaak PEETRE's improvement of SOBOLEV's imbedding theorem, $\rho_\varepsilon \star u$ stays in a bounded set of the LORENTZ space $L^{1^*,1}(R^N)$ if $N \geq 2$ (or a bounded set of $L^\infty(R)$ if $N = 1$); of course, $\rho_\varepsilon \star u$ converges to u in $L^1(R^N)$ strong, and because $L^{1^*,1}(R^N)$ is a dual (as will be proved later), one has $u \in L^{1^*,1}(R^N)$. For $u \in BV(R^N)$ one has $\tau_h u - u \in L^1(R^N)$, and $\rho_\varepsilon \star (\tau_h u - u) \to \tau_h u - u$ in $L^1(R^N)$ as $\varepsilon \to 0$, but as it is also $\tau_h(\rho_\varepsilon \star u) - (\rho_\varepsilon \star u)$ one has $||\tau_h(\rho_\varepsilon \star u) - (\rho_\varepsilon \star u)||_1 \leq |h|\,||grad(\rho_\varepsilon \star u)||_1 \leq C\,|h|$, which gives $||\tau_h u - u||_1 \leq C\,|h|$ for all $h \in R^N$. Conversely if $u \in L^1(R^N)$ and $||\tau_h u - u||_1 \leq C\,|h|$ for all $h \in R^N$, then for $h = t\,e_j$ one has $\frac{\tau_{t\,e_j} u - u}{t} \rightharpoonup \frac{\partial u}{\partial x_j}$ in the sense of distributions as $t \to 0$, but as $\frac{\tau_{t\,e_j} u - u}{t}$ is bounded in $L^1(R^N)$ and $L^1(R^N) \subset \big(C_b(R^N)\big)' = \mathcal{M}_b(R^N)$, the limit belongs to $\mathcal{M}_b(R^N)$.■

Therefore the fact that $u \in W^{1,p}(R^N)$ is equivalent to $u \in L^p(R^N)$ and $||\tau_h u - u||_p \leq C\,|h|$ for all $h \in R^N$ is true for $1 < p \leq \infty$ but not for $p = 1$. However the difference between $W^{1,1}(R^N)$ and $BV(R^N)$ is not seen for some interpolation spaces defined with these two spaces.

Lemma: For $0 < \theta < 1$ and $1 \leq q \leq \infty$ one has $\big(\mathcal{M}_b(R^N), L^\infty(R^N)\big)_{\theta,q} = \big(L^1(R^N), L^\infty(R^N)\big)_{\theta,q} = L^{p,q}(R^N)$ for $p = \frac{1}{1-\theta}$, and $\big(BV(R^N), \mathcal{M}_b(R^N)\big)_{\theta,q} = \big(BV(R^N), L^1(R^N)\big)_{\theta,q} = \big(W^{1,1}(R^N), \mathcal{M}_b(R^N)\big)_{\theta,q} = \big(W^{1,1}(R^N), L^1(R^N)\big)_{\theta,q} = B_q^{1-\theta,1}(R^N)$.

Proof: One uses the fact that $L^1(R^N) \subset \mathcal{M}_b(R^N)$ and for $u \in L^1(R^N)$ the norm of u in $L^1(R^N)$ and the norm of u in $\mathcal{M}_b(R^N)$ coincide. Let $E_0 = \mathcal{M}_b(R^N)$, $E_1 = L^\infty(R^N)$, and $F_0 = L^1(R^N)$, then $F_0 \subset E_0$ and therefore $(F_0, E_1)_{\theta,q} \subset (E_0, E_1)_{\theta,q}$; conversely $a \in (E_0, E_1)_{\theta,q}$ means $a = \int_0^\infty u(t)\,\frac{dt}{t}$ with $u(t) \in E_0 \cap E_1$ a.e. $t \in (0,\infty)$ and $t^{-\theta}\max\{||u(t)||_{E_0}, t\,||u(t)||_{E_1}\} \in L^q\big(0,\infty; \frac{dt}{t}\big)$, but then as $E_0 \cap E_1 = F_0 \cap E_1$ and $||u(t)||_{E_0} = ||u(t)||_{F_0}$, one has $u(t) \in F_0 \cap E_1$ a.e. $t \in (0,\infty)$ and $t^{-\theta}\max\{||u(t)||_{F_0}, t\,||u(t)||_{E_1}\} \in L^q\big(0,\infty; \frac{dt}{t}\big)$, and therefore $a \in (F_0, E_1)_{\theta,q}$. The same argument gives $\big(BV(R^N), \mathcal{M}_b(R^N)\big)_{\theta,q} = \big(BV(R^N), L^1(R^N)\big)_{\theta,q}$ and $\big(W^{1,1}(R^N), \mathcal{M}_b(R^N)\big)_{\theta,q} = \big(W^{1,1}(R^N), L^1(R^N)\big)_{\theta,q}$.

One observes that $\big(BV(R^N), L^1(R^N)\big)_{\theta,\infty} \subset \big(W^{1,1}(R^N), L^1(R^N)\big)_{\theta,\infty}$, and therefore these two spaces are equal, and one then uses the reiteration theorem of Jacques-Louis LIONS and Jaak PEETRE. Indeed, the linear map $u \mapsto \tau_h u - u \in L^1(R^N)$ has a norm $\leq C\,|h|$ in $BV(R^N)$ and a norm ≤ 2 on $L^1(R^N)$ and therefore

[1] Leonida TONELLI, Italian mathematician, 1885–1946. The Department of Mathematics of the University of Pisa is named "Leonida TONELLI".

[2] Lamberto CESARI, Italian-born mathematician, 1910–1990. He worked at University of Michigan, Ann Arbor.

a norm $\leq C\,|h|^{1-\theta}$ on $\left(BV(R^N), L^1(R^N)\right)_{\theta,\infty}$, i.e. one has $||\tau_h u - u||_1 \leq C\,|h|^{1-\theta}$ on this space, which is the characterization of elements of $\left(W^{1,1}(R^N), L^1(R^N)\right)_{\theta,\infty}$.■

21-724. Interpolation spaces

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Solving non linear partial differential equations sometimes requires a careful use of adapted functional spaces. The knowledge of the Theory of Interpolation helps for creating a large family of such spaces, and some of them may indeed be useful.

Many of the non linear partial differential equations which are studied have their origin is Continuum Mechanics or Physics, but very few mathematicians do spend the time to try to understand what the right equations and the right questions should be, and many work for years on distorted equations without knowing it; there are unfortunately many who know the defects of the models that they use but prefer to hide them in order to pretend that they work on some useful realistic problem. It is wiser to be aware of the defects of the models[1] and mention them, and explain what one tries to achieve by working on simplified or defectuous models; indeed if a technical difficulty has been identified on a realistic model, it is much better to try to overcome that particular difficulty by working first on a simplified model, which may have lost some of the realistic features of the problem that one would like to solve.

The space BV has been widely used in situations where solutions are discontinuous, but there are reasons to think that this functional space is not really adapted to most nonlinear partial differential equations where discontinuous solutions are found.

Of course, there are problems where the BV space is asked for, and some problems in Geometric Measure Theory are of this type, and there are applications to image processing for example, but one has imposed to look for a domain with finite perimeter (i.e. whose characteristic function belongs to BV), or to look for a set with finite $(N-1)$-dimensional HAUSDORFF measure.

The main class of partial differential equations where discontinuous solutions appear for intrinsic reasons is that of hyperbolic conservations laws; this class covers important situations in Continuum Mechanics, and too little is understood from a mathematical point of view. Because of their practical importance, numerical simulations of these problems are performed, for example in order to compute the flow of (compressible) air around an airplane. The (Franco-British) Concorde is the only commercial plane who flies over MACH[2] 1, but most commercial planes fly fast enough to require computations of transonic flows; indeed the speed of sound depends upon the temperature and the pressure, and the shape given to the wings of the plane makes more air go below the wing and creates a slight surpression below the wing but a high depression above the wing, and at the cruise velocity of large commercial jets the speed of sound in that depression (which sucks the plane upward) is then inferior to the velocity of the plane. Doing these numerical simulations has become much less expensive than using small scale planes in wind tunnels, and the shape of the plane can be improved (mostly for diminishing the fuel consumption of the plane, rarely for diminishing the noise it

[1] Some very good and very honest mathematicians may stay unaware of some practical limitations of the equations that they have studied, and an example was given to me by Jean LERAY, when he explained to me in 1984 the origin of the political difficulties that he had encountered. As an officer in the French army, he had been taken prisoner and he spent most of World War II in a German camp (while a famous member of the BOURBAKI group had dodged the draft), but he continued to do research, and even organized a university (of which he was chancellor) inside the camp. He told me that he had stopped working on NAVIER-STOKES equations, for fear that his results could be used by the German; fourty years after his decision he still seemed unaware that the works of mathematicians on this oversimplified model has hardly any practical use.

Nicolas BOURBAKI is the pseudonym of a group of mathematicians, mostly French; those who chose the name certainly knew about a French general named BOURBAKI.

Charles Denis Sauter BOURBAKI, French general, 1816-1897; of Greek ancestry, he declined an offer of the throne of Greece in 1862.

[2] Ernst MACH, Czech scientist, 1838-1916. He worked in Vienna. The MACH number is the ratio of the velocity of the plane to the speed of sound.

makes), but the mathematical knowledge of these questions, mostly due to the work of GARABEDIAN[3] and of Cathleen MORAWETZ[4], is still insufficient for corroborating the intuition of the engineers.

For $1 < p < \infty$, the BESOV space $B_\infty^{1/p,p}(R) = \big(W^{1,p}(R), L^p(R)\big)_{1/p',\infty} = \{u \in L^p(R), ||\tau_h u - u||_p \le C\,|h|^{1/p}$ for all $h \in R\}$ contains discontinuous (piecewise smooth) functions (while for $1 < q < \infty$, the space $B_q^{1/p,p}(R)$ does not contain piecewise smooth discontinuous functions). The excluded case $p = 1$ corresponds to $BV(R)$, and $p = \infty$ corresponds to $L^\infty(R)$, so why think that the case $1 < p < \infty$ is better?

Linear partial differential equations with constant coefficients can be solved by using elementary solutions, and for elliptic equations this leads to singular integrals which can be studied by CALDERÓN-ZYGMUND theorem, which requires $1 < p < \infty$. For example, solving $\Delta\, u = f$ for $f \in L^p(R^N)$ gives $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(R^N)$ for $i, j = 1, \ldots, N$, if $1 < p < \infty$, but the result is false for $p = 1$ or $p = \infty$, and $f \in L^1(R^N)$ does not imply that the derivatives $\frac{\partial u}{\partial x_i}$ belong to $BV(R^N)$.

There are however other spaces which can be used for replacement, using the HARDY space $\mathcal{H}^1(R^N)$ instead of $L^1(R^N)$, and the space $BMO(R^N)$ instead of $L^\infty(R^N)$, and indeed singular integrals acts from $\mathcal{H}^1(R^N)$ to itself and from $BMO(R^N)$ into itself; actually, the interpolation spaces between $\mathcal{H}^1(R^N)$ and $BMO(R^N)$ are the same than the already studied interpolation spaces between $L^1(R^N)$ and $L^\infty(R^N)$, but these results cannot be derived so easily. However there is another obstacle, which suggests that the choice $p = 2$ is the only right one.

The (scalar) wave equation in a general medium has the form $\rho(x)\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big) = 0$, where the coefficients $\rho\, a_{ij}, i, j = 1, \ldots, N$ belong to $L^\infty(R^N)$, satisfy the symmetry condition $a_{ji}(x) = a_{ij}(x)$ a.e. $x \in R^N$ for all $i, j = 1, \ldots, N$, satisfy the positivity property $\rho(x) \ge \rho_- > 0$ a.e. $x \in R^N$ and also satisfy the ellipticity property that for some $\alpha > 0$ one has $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \alpha\,|\xi|^2$ for all $\xi \in R^N$, a.e. $x \in R^N$. Under these conditions the CAUCHY problem is well posed if one imposes $u|_{t=0} = u_0 \in H^1(R^N)$ and $\frac{\partial u}{\partial t}\big|_{t=0} = u_1 \in L^2(R^N)$, and one has conservation of total energy, sum of the kinetic energy $\frac{1}{2}\int_{R^N} \rho(x)\big|\frac{\partial u}{\partial t}\big|^2\,dx$ and the potential energy $\frac{1}{2}\int_{R^N}\big(\sum_{i,j=1}^N a_{ij}(x)\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\big)\,dx$. It would seem natural to expect that with smooth coefficients, like for the simplified wave equation in a homogeneous isotropic material $\frac{\partial^2 u}{\partial t^2} - c^2\Delta\, u = 0$, one could take $u_0 \in W^{1,p}(R^N)$ and $u_1 = 0$ for example and find the solution $u(\cdot, t) \in W^{1,p}(R^N)$ for $t > 0$ (or for $t < 0$ as the wave equation is invariant through time reversal), but Walter LITTMAN[5] has shown that this only happens for $p = 2$.

Could the space $B_\infty^{1/2,2}(R^N)$ be used then for nonlinear hyperbolic equations where one expects some discontinuities to occur?

It is useful to compare $BV(R^N)$ with the BESOV space $B_\infty^{1/p,p}(R^N)$, which is the space of $u \in L^p(R^N)$ such that $||\tau_h u - u||_p \le C\,|h|^{1/p}$ for all $h \in R^N$.

Lemma: One has $BV(R^N) \cap L^\infty(R^N) \subset B_\infty^{1/p,p}(R^N)$, and more precisely $\big(BV(R^N), L^\infty(R^N)\big)_{1/p',p} \subset B_\infty^{1/p,p}(R^N)$.

If u is a characteristic function, then $u \in B_\infty^{1/p,p}(R^N)$ implies $u \in BV(R^N)$ (and conversely).

Proof: The linear mapping $u \mapsto \tau_h u - u$ is continuous from $BV(R^N)$ into $L^1(R^N)$ with norm $\le C\,|h|$, and from $L^\infty(R^N)$ into itself with norm ≤ 2, and therefore it is continuous from $\big(BV(R^N), L^\infty(R^N)\big)_{1/p',p}$ into $\big(L^1(R^N), L^\infty(R^N)\big)_{1/p',p} = L^p(R^N)$ with norm $\le C\,|h|^{1/p}$.

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[3] Paul R. GARABEDIAN, American mathematician. He works at the COURANT Institute of Mathematical Sciences, New York University.

[4] Cathleen SYNGE MORAWETZ, Canadian-born mathematician, 1923. She works at the COURANT Institute of Mathematical Sciences, New York University. Her father, John Lighton SYNGE, Irish mathematician, 1897-1995, was from 1946 to 1948 the Head of the Mathematics Department at the CARNEGIE Institute ot Technology in Pittsburgh, later to become CARNEGIE MELLON University.

Andrew CARNEGIE, Scottish-born businessman and philanthropist, 1835 - 1919.

Andrew William MELLON, American financier and philanthropist, 1855-1937.

[5] Walter LITTMAN, American mathematician. He works at University of Minnesota, Minneapolis.

If $u \in B_\infty^{1/p,p}(R^N)$ one has $\int_{R^N} |u(x-h)-u(x)|^p\,dx \le C\,|h|$ for all $h \in R^N$; if moreover u is a characteristic function, then $u(x-h)-u(x)$ can only take the values $-1, 0, 1$ and therefore $|u(x-h)-u(x)|^p = |u(x-h)-u(x)|$, from which one deduces that $||\tau_h u - u||_1 \le C\,|h|$ for every $h \in R^N$, and as one also has $||u||_1 = ||u||_p^p$, one deduces that $u \in BV(R^N)$ (the first part implying the converse).■

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19. Friday April 28, 2000.

Quasilinear hyperbolic equations have properties quite different from semilinear hyperbolic equations, mostly because discontinuities may appear even when initial data are very smooth. For second order equations modeled on the linear wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta\, u = 0$, semilinear equations are of the form $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta\, u = F\big(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\big)$ where the higher order part is linear with constant coefficients, while for quasilinear equations the higher order part is still linear but with coefficients which depend upon lower order derivatives; in the case $N = 1$ one has for example $\frac{\partial^2 u}{\partial t^2} - f\big(\frac{\partial u}{\partial x}\big)\frac{\partial^2 u}{\partial x^2} = 0$. This equation was first studied by POISSON in 1807, as a model for a compressible gas; the classical relation $p = c\,\rho$ leads to an incorrect value of the velocity of sound (which had been estimated by NEWTON), and LAPLACE seems to have proposed to use the relation $p = c\,\rho^\gamma$; POISSON had found some special solutions (which are called simple waves now), but he left them in an implicit form, so that it took fourty years before someone pointed out that there was a problem, which STOKES explained in 1848 by the formation of discontinuities. STOKES computed the correct jump conditions to impose for discontinuous solutions, by using the conservation of mass and the conservation of momentum, and these conditions were rediscovered by RIEMANN in 1860, but instead of being called the STOKES conditions or the STOKES-RIEMANN conditions, they are now known as the RANKINE[1]-HUGONIOT[2] conditions. The defects of these (isentropic) models were not obvious then, as Thermodynamics was barely in its infancy at the time, and even STOKES was wrongly convinced later by THOMSON (who later became Lord KELVIN) and by RAYLEIGH[3], who told him that his discontinuous solutions were not physical because they did not conserve energy[4].

The apparition of discontinuities is more easily seen on first order equations, for which a classical model[5] is the BURGERS equation $\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$. Using the method of characteristic curves, one finds easily the critical time of existence of a smooth solution.

Lemma: Let u_0 be a smooth bounded function on R. If u_0 is nondecreasing, there exists a unique smooth solution of $\frac{\partial u}{\partial t} + c\,u\,\frac{\partial u}{\partial x} = 0$ for $t > 0$ satisfying $u(x, 0) = u_0(x)$ for $x \in R$. If $\inf_{x\in R} \frac{du_0}{dx}(x) = -\alpha < 0$, there is a unique smooth solution for $0 < t < T_c = \frac{1}{\alpha}$, and there is no smooth solution over an interval $(0, T)$ with $T > T_c$.

Proof: Assume that there exists a smooth solution for $0 < t < T$. For $y \in R$ one defines the characteristic curve with initial point y by $\frac{dz(t)}{dt} = u(z(t), t)$ and $z(0) = y$, and one deduces that $\frac{d[u(z(t),t)]}{dt} = \frac{\partial u}{\partial x}(z(t), t)\,\frac{dz(t)}{dt} + \frac{\partial u}{\partial t}(z(t), t) = 0$, and therefore $u(z(t), t) = u(z(0), 0) = u_0(y)$ for $0 < t < T$; this gives $\frac{dz(t)}{dt} = u_0(y)$, i.e. $z(t) = y + t\,u_0(y)$. This shows that on the line $x = y + t\,u_0(y)$ for $0 < t < T$ the smooth solution is given by $u(x, t) = u_0(y)$. If u_0 is nondecreasing, the mapping $y \mapsto y + t\,u_0(y)$ is a global diffeomorphism from R onto R for any $t > 0$ (it is increasing), and therefore there exists a unique global smooth

[1] William John Macquorn RANKINE, Scottish engineer, 1820-1872. He worked in Glasgow.

[2] Pierre Henri HUGONIOT, French engineer, 1851-1887.

[3] John William STRUTT, Third Baron RAYLEIGH, English physicist, 1842-1919. He received the NOBEL prize in Physics in 1904. He held the CAVENDISH Professorship at Cambridge, 1879-1884, after MAXWELL.

[4] It shows that around 1880 (when STOKES was editing his work of 1848 and did not reproduce his derivation of the jump condition in his complete works, and apologized for his “mistake”), RAYLEIGH, STOKES and THOMSON (who became Lord KELVIN in 1892) did not understand that mechanical energy could be transformed into heat. If one has learned Thermodynamics, one should not disparage these great scientists of the 19$^{\text{th}}$ Century for their curious mistake, and one should recognize that there are things which take time to understand (and one learns now also that Thermodynamics is not about dynamics.). Actually, Thermodynamics is still a subject which is not so well understood, and mathematicians should pay more attention to it; ignoring Thermodynamics, and publishing too much on isentropic equations, for example, tends to make engineers and physicists believe that mathematicians do not know what they are talking about.

[5] The function u has the dimension $L\,T^{-1}$ of a velocity; some physicists prefer to write $\frac{\partial u}{\partial t} + c\,u\,\frac{\partial u}{\partial x} = 0$, where c is a characteristic velocity and u has no dimension.

solution for all $t > 0$. If there exist $y_1 < y_2$ with $u_0(y_1) > u_0(y_2)$, then the characteristic line with initial point y_1 catches upon the characteristic line with initial point y_2 and a smooth solution cannot exist up to the time of encounter of the two characteristic lines as it would have to take two different values at their intersection; one can easily check that for any $T > T_c$ one can find two characteristic lines which intersect before T. One can check directly that the solution is well defined for $0 < t < T$ because the mapping $y \mapsto y + t\,u_0(y)$ is a global diffeomorphism from R onto R for $0 < t < T_c$, and in order to show that there is no solution on $(0,T)$ with $T > T_c$, one defines $v = \frac{\partial u}{\partial x}$ and one checks that $\frac{d[v(z(t),t)]}{dt} = [v(z(t),t)]^2$, so that $v(z(t),t) = \frac{u_0'(y)}{1+t\,u_0'(y)}$, and therefore the smooth solution satisfies $\lim_{t\to T_c} \sup_{x\in R} \frac{\partial u}{\partial x} = +\infty$.■

The analog of the implicit formula used by POISSON would be to say that the solution must satisfy $u(x,t) = u_0\big(x - t\,u(x,t)\big)$, and seeing the limitation in time on this formula is less obvious.

After the critical time T_c one cannot have smooth solutions, and the correctly defined solution will be discontinuous at some points, and therefore the product of u by $\frac{\partial u}{\partial x}$ is not defined, and it is important to write the equation in conservation form, $\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0$, and to consider the weak solutions, i.e. the solutions in the sense of distributions. The jump conditions which STOKES had computed correspond to saying that if $\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0$ and u, v are continuous on each side of a curve with equation $x = \varphi(t)$ and $s = \varphi'(t)$ denotes the velocity of the (possible) discontinuity at time t, then one has $jump(v) = s\,jump(u)$, where $jump(w) = w\big(\varphi(t)_+, t\big) - w\big(\varphi(t)_-, t\big) = w_+ - w_-$ (noticing that changing x into $-x$ changes v into $-v$, changes the sign of $jump(u)$ and of $jump(v)$, but changes also the sign of s).

Unfortunately, there are too many weak solutions; for example for $u_0 = 0$ there is a global smooth solution which is $u = 0$, but there are infinitely many weak solutions by taking x_0 arbitrary, $a > 0$ arbitrary, and defining $u(x,t) = 0$ for $x < x_0 - t\,a$, $u(x,t) = -2a$ for $x - t\,a < x_0$, $u(x,t) = 2a$ for $0 < x < t\,a$, and $u(x,t) = 0$ for $t\,a < x$.

In order to reject all unphysical weak solutions, one decides then to keep only the discontinuities for which $u_- > u_+$; the method of characteristic curves gives a local LIPSCHITZ continuous solution when one starts from a nondecreasing function and by continuity it gives such a LIPSCHITZ continuous solution in the other case $u_- < u_+$, i.e. when one starts from the discontinuous function jumping from u_- up to u_+ (called a rarefaction wave). This selection criterium, called an "entropy" condition[6], extends to the more general equation $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$ only if the nonlinearity f is a convex function; the case of a general (smooth) f has been solved by Olga OLEINIK[7], and OLEINIK's criterium is that one accepts or rejects a discontinuity according to the position of the chord joining $\big(u_-, f(u_-)\big)$ and $\big(u_+, f(u_+)\big)$ with respect to the graph of the function f, accepting the discontinuity $u_- > u_+$ if and only the chord is above the graph, and accepting the discontinuity $u_- < u_+$ if and only the chord is below the graph.

Under OLEINIK's criterium there is a unique piecewise smooth solution, but in order to study weak solutions without having to assume that they are piecewise smooth, Eberhard HOPF[8] derived the equivalent formulation $\frac{\partial \varphi(u)}{\partial t} + \frac{\partial \psi(u)}{\partial x} \le 0$ in the sense of distributions for all convex ("entropy") functions[9] φ, where ψ is a corresponding "entropy flux", defined by $\psi' = \varphi' f'$. Peter LAX extended the idea to systems of conservation laws, $\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$, with $U(x,t) \in R^p$, but not all functions φ on R^p are entropies, because if one has $\nabla\varphi\,\nabla F = \nabla\psi$ then there are compatibilty conditions that φ must satisfy, i.e. $curl(\nabla\varphi\,\nabla F) = 0$, and the trivial entropies $\varphi(U) = \pm U_i$ for $i = 1, \ldots, p$ just correspond to the given system of equations[10].

One way to construct such admissible weak solutions is to consider a regularization by artificial viscosity $\frac{\partial U_\varepsilon}{\partial t} + \frac{\partial F(U_\varepsilon)}{\partial x} - \varepsilon\,\frac{\partial^2 U_\varepsilon}{\partial x^2} = 0$ for $\varepsilon > 0$. The scalar case with $f(u) = \frac{u^2}{2}$ (also called the BURGERS-HOPF equation) was first solved by Eberhard HOPF using a nonlinear change of function which transforms the equation into

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[6] By analogy with conditions imposed by Thermodynamics for the system of compressible gas dynamics.

[7] Olga Arsen'evna OLEINIK, Russian mathematician, born in 1925. She works at Moscow State University.

[8] Eberhard HOPF, German-born mathematician, 1902-1983. He emigrated to United States in 1949, and worked at Indiana University, Bloomington, where I met him in 1980.

[9] Again, this is only by analogy with Thermodynamics, and these "entropies" are rarely related with the thermodynamic entropy.

[10] It is not clear if the right notion of solution has been found, but all the physical examples seem to be endowed with a strictly convex "entropy", which sometimes is the total energy!

the linear heat equation, and this transformation was also found by Julian COLE[11], and is now known as the HOPF-COLE transform. The scalar case with a general f was solved by Olga OLEINIK, and the scalar case with more than one space variable[12], $\frac{\partial u}{\partial t}+\sum_{j=1}^N \frac{\partial f_j(u)}{\partial x_j}=0$ was obtained by KRUZHKOV[13].

The important difference between the scalar case and the vectorial case (regularized by artificial viscosity), is that there are simple BV estimates for the scalar case, which are unknown for the vector case; the BV estimates are used to prove convergence by a compactness argument, but any uniform estimate in a BESOV space $(B_q^{s,p})_{loc}(R^N)$ with $s>0$ would be sufficient. Unfortunately, the estimates for the scalar case are based on the maximum principle, and the same argument cannot be extended to systems.

There is another method due to James GLIMM[14], which proves existence for some systems if the total variation is small enough.

I have introduced another approach, based on the Compensated Compactness Method which I had partly developed with François MURAT[15], which does not need estimates in BESOV spaces, but which requires a special understanding of how to use entropies to generate a kind of compactness; Ron DIPERNA[16] was the first to find a way to apply my method to systems.

Of course, the preceding list of methods is not exhaustive, and there has been other partially succesful approaches.

The BV estimate in the scalar case can be linked to a $L^1(R)$ contraction property, which was noticed by Barbara KEYFITZ[17] in one dimension and by KRUZHKOV in dimension N. This property is strongly related to the maximum principle, and I have noticed with Michael CRANDALL[18] that if a map S from $L^1(\Omega)$ into itself satisfies $\int_\Omega S(f)\,dx=\int_\Omega f\,dx$ for all f, then S is a L^1 contraction if and only if S is order preserving; as order preserving properties do not occur for realistic systems, one cannot expect L^1 contraction properties for systems; Michael CRANDALL and Andrew MAJDA[19] have later applied the same idea for discrete approximations.

The simplest discrete approximation of $\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$ is the LAX-FRIEDRICHS scheme, $\frac{1}{\Delta t}\big(U_i^{n+1}-\frac{1}{2}(U_{i-1}^n+U_{i+1}^n)\big)+\frac{1}{2\Delta x}\big(f(U_{i+1}^n)-f(U_{i-1}^n)\big)=0$, where U_i^n is expected to approximate $U(i\,\Delta x,n\,\Delta t)$; starting with a bounded initial data u_0, satisfying $\alpha\le u_0(x)\le\beta$ a.e. $x\in R$, one chooses for example $U_i^0=\frac{1}{\Delta x}\int_{i\,\Delta x}^{(i+1)\Delta x}u_0(y)\,dy$ for all i, and the explicit scheme generates the numbers U_i^n for $n>0$, but one must impose the COURANT-FRIEDRICHS-LEWY[20] condition (known as the CFL condition) $\frac{\Delta t}{\Delta x}\sup_{v\in[\alpha,\beta]}|f'(v)|\le 1$, which imposes that the numerical velocity of propagation $\frac{\Delta x}{\Delta t}$ is at least equal to the real velocity of propagation; under this condition one has $\alpha\le U_i^n\le\beta$ for all i and all $n>0$, and it is exactly the condition which imposes that U_i^{n+1} is a nondecreasing function of U_{i-1}^n and of U_{i+1}^n, and the $l^1(Z)$ contraction property follows. Of course this approach creates solutions such that $||\tau_h u(\cdot,t)-u(\cdot,t)||_1\le||\tau_h u_0-u_0||_1$, which gives a BV estimate if $u_0\in BV(R)$. This scheme is only of order 1 and tends to smooth out the discontinuities too much, but higher order schemes are not order preserving; there is however a class of higher order schemes, called TVD schemes (total variation diminishing), for which the total variation is not increasing.

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[11] Julian D. COLE, American mathematician. He works at Rensselaer Polytechnic Institute, Troy, NY.

[12] A scalar equation in N variables is not a good physical model for $N>1$, because it implies a very strong anisotropy of the space (due to a particular direction of propagation).

[13] Stanislav Nikolaevich KRUZHKOV, Russian mathematician, 1936-1997. He worked at Moscow State University.

[14] James G. GLIMM, American mathematician, born in 1934. He works at State University of New York, Stony Brook.

[15] François MURAT, French mathematician, born in 1947. He works at University of Paris VI (Pierre et Marie CURIE).

[16] Ronald J. DIPERNA, American mathematician, -1989. He worked at University of California in Berkeley.

[17] Barbara Lee QUINN KEYFITZ, American mathematician. She works at University of Houston.

[18] Michael G. CRANDALL, American mathematician, born in 1940. He works at University of California, Santa Barbara.

[19] Andrew J. MAJDA, American mathematician. He works at the COURANT Institute for Mathematical Sciences, New York University.

[20] Hans LEWY, German-born mathematician, 1904-1988. He received the WOLF prize in 1984. He emigrated to United States in the 30s and worked at University of California, Berkeley.

Obtaining BV estimates for general systems of conservation laws, or more generally obtaining some estimates on fractional derivatives using Interpolation spaces is certainly a difficult open question, and some new ideas or some new functional spaces may be needed for that important question.

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20. Monday May 1, 2000.

The possibility of defining the spaces $(E_0,E_1)_{\theta,p}$ for $0<\theta<1$ and $1\le p\le\infty$ as spaces of traces has been mentioned, and it is time to explain the proof. One notices that if $t^{-\theta}K(t,a)$ is bounded then one can decompose $a=a_0(t)+a_1(t)$ with $||a_0(t)||_0+t\,||a_1(t)||_1\le 2K(t,a)\le C\,t^\theta$ and therefore $a_1(t)\to 0$ in E_1 as $t\to\infty$, and because the traces are taken as $t\to 0$, there will be a change of t into $\frac{1}{t}$.

Proposition: Let $0<\theta<1$ and $1\le p\le\infty$. If $v(t)\in E_0$ and $v'(t)\in E_1$ with $t^\theta||v(t)||_0\in L^p\big(0,\infty;\frac{dt}{t}\big)$ and $t^\theta||v'(t)||_1\in L^p\big(0,\infty;\frac{dt}{t}\big)$, then $v(0)\in(E_0,E_1)_{\theta,p}$. Conversely, every element of $(E_0,E_1)_{\theta,p}$ can be written as $v(0)$ with v satisfying the preceding properties.

Proof: Let $a\in(E_0,E_1)_{\theta,p}$, i.e. $e^{-n\,\theta}K(e^n,a)\in l^p(Z)$. For $n\in Z$ one chooses a decomposition $a=a_{0,n}+a_{1,n}$ with $a_{0,n}\in E_0, a_{1,n}\in E_1$ such that $||a_{0,n}||_0+e^n||a_{1,n}||_1\le 2K(e^n,a)$, and one notices that $||a_{0,n+1}-a_{0,n}||_1=||a_{1,n+1}-a_{1,n}||_1\le 2e^{-(n+1)}K(e^{n+1},a)+2e^{-n}K(e^n,a)\le 4e^{-n}K(e^n,a)$. One defines the function u with values in E_0 by $u(e^n)=a_{0,n}$, and one extends u to be affine in each interval (e^n,e^{n+1}); this gives $||u(t)||_0\le\max\{||a_{0,n}||_0,||a_{0,n+1}||_0\}\le 2K(e^{n+1},a)\le 2e\,K(e^n,a)$ on the interval (e^n,e^{n+1}); in the case $p=\infty$, one deduces that $t^{-\theta}||u(t)||_0\le e^{-n\,\theta}2e\,K(e^n,a)$ on the interval (e^n,e^{n+1}) and therefore $t^{-\theta}||u(t)||_0\in L^\infty\big(0,\infty;\frac{dt}{t}\big)$, while if $1\le p<\infty$ one has $\int_{e^n}^{e^{n+1}}t^{-\theta\,p}||u(t)||_0^p\,\frac{dt}{t}\le e^{-n\,\theta\,p}(2e)^pK(e^n,a)^p$ and therefore $t^{-\theta}||u(t)||_0\in L^p\big(0,\infty;\frac{dt}{t}\big)$. On the interval (e^n,e^{n+1}) one has $u'=\frac{a_{0,n+1}-a_{0,n}}{e^{n+1}-e^n}$, and therefore $||u'(t)||_1\le\frac{4e^{-n}K(e^n,a)}{e^{n+1}-e^n}=\frac{4}{e-1}e^{-2n}K(e^n,a)$; in the case $p=\infty$, one deduces that $t^{2-\theta}||u'(t)||_1\le e^{(2-\theta)(n+1)}\frac{4}{e-1}e^{-2n}K(e^n,a)=\frac{4e^{2-\theta}}{e-1}e^{-\theta\,n}K(e^n,a)$ on the interval (e^n,e^{n+1}) and therefore $t^{2-\theta}||u'(t)||_1\in L^\infty\big(0,\infty;\frac{dt}{t}\big)$, while if $1\le p<\infty$ one has $\int_{e^n}^{e^{n+1}}t^{(2-\theta)p}||u'(t)||_1^p\,\frac{dt}{t}\le e^{(2-\theta)(n+1)p}\frac{4^p}{(e-1)^p}e^{-2n\,p}K(e^n,a)^p=C\,e^{-\theta\,n\,p}K(e^n,a)^p$ and therefore $t^{2-\theta}||u'(t)||_1\in L^p\big(0,\infty;\frac{dt}{t}\big)$. Defining $v(s)=u\big(\frac{1}{s}\big)$ gives $v(s)\to a$ as $s\to 0$ and moreover $t^\theta||v(t)||_0\in L^p\big(0,\infty;\frac{dt}{t}\big)$ and $t^\theta||v'(t)||_1\in L^p\big(0,\infty;\frac{dt}{t}\big)$.

Conversely, assume that $t^\theta||v(t)||_0$ and $t^\theta||v'(t)||_1$ belong to $L^p\big(0,\infty;\frac{dt}{t}\big)$. Then for $t>\varepsilon>0$ one has $||v(t)-v(\varepsilon)||_1\le\int_\varepsilon^t||v'(s)||_1\,ds\le C\,t^{1-\theta}$, so that $v(t)$ tends to a limit in E_0+E_1 as s tends to 0. Using then the decomposition $v(0)=v(t)+\big(v(0)-v(t)\big)$, one has $t^\theta||v(t)||_0\in L^p\big(0,\infty;\frac{dt}{t}\big)$, and from $\frac{||v(0)-v(t)||_1}{t}\le\frac{1}{t}\int_0^t||v'(s)||_1\,ds$ one deduces by HARDY's inequality that $t^{\theta-1}||v(0)-v(t)||_1\in L^p\big(0,\infty;\frac{dt}{t}\big)$; then one changes t into $\frac{1}{t}$, or one notices that it says that $v(0)\in(E_1,E_0)_{1-\theta,p}$, which is $(E_0,E_1)_{\theta,p}$.■

The initial definition of trace spaces by Jacques-Louis LIONS and Jaak PEETRE used four parameters and considered functions satisfying $t^{\alpha_0}u\in L^{p_0}(0,\infty;E_0)$ with $t^{\alpha_1}u'\in L^{p_1}(0,\infty;E_1)$, for suitable parameters $\alpha_0,\alpha_1,p_0,p_1$; they had noticed that the family depended on at most three parameters by changing t into t^λ, but they had also introduced the important parameter θ; it was Jaak PEETRE who later[1] found that the family depended only upon two parameters, and developed the simpler K and J methods that have been followed in this course.

With the same arguments used in the preceding proposition, the characterization of trace spaces is similar to studying the following variant, where one defines $L_{p_0,p_1}(t,a)=\inf_{a=a_0+a_1}||a_0||_0^{p_0}+t\,||a_1||_1^{p_1}$, and one defines $(E_0,E_1)_{\theta,p;L}$ as the space of $a\in E_0+E_1$ such that $t^{-\theta}L_{p_0,p_1}(t,a)\in L^p\big(0,\infty;\frac{dt}{t}\big)$. The lack of homogeneity may look strange, and if trace spaces had not been defined before it would not even be obvious that $(E_0,E_1)_{\theta,p;L}$ is a vector subspace, and it is actually a space already defined.

[1] Jacques-Louis LIONS's interests had switched to other questions concerning the use of Functional Analysis in linear and then nonlinear partial differential equations, in optimization and control problems and in their numerical approximations. After writing his books with Enrico MAGENES, his interest in interpolation spaces became marginal, but he used the ideas when necessary; after finding a nonlinear framework for interpolating regularity for variational inequalities, he thought of a generalization and he probably found that it was a good problem for a student instead of investigating the question himself. Developping this idea made the first part of my thesis, and the second part answered another (slightly academic) question that he had thought of, and I characterized the traces of functions satisfying $u^3\in L^2\big(0,T;H^1(\Omega)\big)$ and $\frac{\partial u}{\partial t}\in L^2\big(0,T;L^2(\Omega)\big)$.

Lemma: For $0 < \theta < 1$ and $1 \leq p_0, p_1, p \leq \infty$, one has $(E_0, E_1)_{\theta,p;L} = (E_0, E_1)_{\overline{\theta},\overline{p}}$, with $\overline{\theta}$ defined by $\frac{1-\overline{\theta}}{\overline{\theta}} = \frac{1-\theta}{\theta}\frac{p_0}{p_1}$, and $\overline{p} = \big((1-\theta)p_0 + \theta\, p_1\big)p$.

Proof: If one defines $K_q(t,a) = \big(\inf_{a=a_0+a_1} ||a_0||_0^q + t^q||a_1||_1^q\big)^{1/p}$ and one lets q tend to ∞, one obtains $K_\infty(t,a) = \inf_{a=a_0+a_1} \max\{||a_0||_0, t\,||a_1||\}$, and one has $K_\infty(t,a) \leq K(t,a) \leq 2K_\infty(t,a)$ for all $a \in E_0 + E_1$. Geometrically, for the GAGLIARDO set associated to a, i.e. $\{(x_0,x_1)$, there exists a decomposition $a = a_0 + a_1$ with $||a_0||_0 \leq x_0, ||a_1||_1 \leq x_1\}$, the line $x_0 = t\,x_1$ intersects it at $x_0 = K_\infty(t,a), x_1 = \frac{K_\infty(t,a)}{t}$. Similarly one defines $L_\infty(s,a) = \inf_{a=a_0+a_1} \max\{||a_0||_0^{p_0}, s\,||a_1||_1^{p_1}\}$, and one has $L_\infty(s,a) \leq L_{p_0,p_1}(s,a) \leq 2L_\infty(s,a)$ for all $a \in E_0+E_1$, and in order to find the same point of the GAGLIARDO set one chooses $s = t^{p_1}K_\infty(t,a)^{p_0-p_1}$, so that $K_\infty(t,a)^{p_0} = s\,\frac{K_\infty(t,a)^{p_1}}{t^{p_1}}$, and one deduces $L_\infty(s,a) = K_\infty(t,a)^{p_0}$; then one notices that $t \mapsto s = t^{p_1}K_\infty(t,a)^{p_0-p_1}$ is a good change of variable, because $t \mapsto K_\infty(t,a)$ is nondecreasing and $t \mapsto \frac{K_\infty(t,a)}{t}$ is nonincreasing, and one deduces that $\frac{ds}{s} = C(t)\frac{dt}{t}$ with $\min\{p_0,p_1\} \leq C(t) \leq \max\{p_0,p_1\}$ for all $t > 0$, and therefore $\int_0^\infty s^{-\theta\,p}L_\infty(s,a)^p\,\frac{ds}{s} < \infty$ is equivalent to $\int_0^\infty t^{-\theta\,p\,p_1}K_\infty(s,a)^{p\,p_0-\theta\,p(p_0-p_1)}\,\frac{dt}{t} < \infty$. This gives the condition $\overline{p} = p\,p_0 - \theta\,p(p_0-p_1) = (1-\theta)p_0\,p + \theta\,p_1\,p$, and $\overline{\theta}\overline{p} = \theta\,p_1\,p = \overline{p} - (1-\theta)p_0 p$, so that $(1-\overline{\theta})\overline{p} = (1-\theta)p_0 p$ and eliminating $\overline{p}$ between these last two formulas gives the desired formula for $\overline{\theta}$.■

Corollary: If $u \in W^{1,p}(R^N)$, then its trace on $x_N = 0$ satisfies $\gamma_0 u \in W^{1/p',p}(R^{N-1}) = B_p^{1/p',p}(R^{N-1}) = \big(W^{1,p}(R^{N-1}), L^p(R^{N-1})\big)_{1/p,p}$.

Proof: Using x_N as the variable t (and using only the fact that $u \in W^{1,p}(R_+^N)$), one finds that $u \in W^{1,p}(R^N)$ implies $u \in L^p\big(0,\infty; W^{1,p}(R^{N-1})\big)$ and $u' \in L^p\big(0,\infty; L^p(R^{N-1})\big)$, and this corresponds to $\theta = \frac{1}{p}$ for $E_0 = W^{1,p}(R^{N-1})$ and $E_1 = L^p(R^{N-1})$.■

In defining interpolation spaces, one has not really used the fact that E_0 and E_1 are normed vector spaces, and one can extend the theory to the case of commutative (Abelian) groups, and moreover the norm can be replaced by a quasi-norm, satisfying $[a] \geq 0, [-a] = [a]$ for all a, $[a] = 0$ if and only if $a = 0$, and the c-triangle inequality $[a+b] \leq c([a]+[b])$ for all a, b (one calls then $[\cdot]$ a c-norm). One notices that if one defines ρ by $(2c)^\rho = 2$ then there is a 1-norm $||\cdot||$ such that $||a|| \leq [a]^\rho \leq 2||a||$ for all a, and such a norm is defined by $||a|| = \inf_{a=\sum_{i=1}^n a_i}[a_i]^\rho$, where the infimum is taken over all n and all decompositions of a.

One can define the space $(E_0,E_1)_{\theta,p}$ if E_0, E_1 are quasi-normed Abelian groups, and with $0 < \theta < 1$ as usual, but for a larger family of p, as one may take $0 < p \leq \infty$, obtaining a quasi-normed space in the case $0 < p < 1$, even if E_0 and E_1 are normed spaces.

If $||\cdot||$ is a 1-norm and for $\alpha > 0$ one defines $[u] = ||u||^\alpha$, then $[\cdot]$ is a c-norm if one has $(a+b)^\alpha \leq c(a^\alpha + b^\alpha)$ for all $a, b \geq 0$, and one checks easily that one may take $c = 1$ if $\alpha \leq 1$ and $c = 2^{\alpha-1}$ if $\alpha \geq 1$. One may then consider the variant $L_{p_0,p_1}(t,a)$ for $0 < p_0, p_1 \leq \infty$, as a particular case of using quasi-norms.

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For characterizing the dual of $(E_0, E_1)_{\theta,p}$ for $0 < \theta < 1$ and $1 \le p < \infty$, one needs a few technical results.

First of all there is a new hypothesis, that $E_0 \cap E_1$ is dense in E_0 and dense in E_1; this implies that E_0' and E_1' are subspaces of $(E_0 \cap E_1)'$, because if j_0 is the injection of $E_0 \cap E_1$ into E_0, then j_0^T is a continuous mapping from E_0' into $(E_0 \cap E_1)'$ whose range is dense, while the hypothesis that the range of j_0 is dense implies that j_0^T is injective. Therefore $E_0' \cap E_1'$ and $E_0' + E_1'$ are well defined. One denotes $||\cdot||_k$ the norm on E_k and by $||\cdot||_{*k}$ the norm on E_k' for $k = 1, 2$.

Lemma: Assume that $E_0 \cap E_1$ is dense in E_0 and dense in E_1. For $t > 0$, the dual space of $E_0 \cap E_1$ equipped with the norm $J(t, a; E_0, E_1) = \max\{||a||_0, t\,||a||_1\}$ is $E_0' + E_1'$ equipped with the norm $K\big(\frac{1}{t}, b; E_0', E_1'\big) = \inf_{b_0+b_1=b}\big(||b_0||_{*0} + \frac{1}{t}\,||b_1||_{*1}\big)$.

Proof: If $a \in E_0 \cap E_1$ and $b \in E_0' + E_1'$ with a decomposition $b = b_0 + b_1, b_0 \in E_0', b_1 \in E_1'$, then one has $|\langle b, a\rangle| \le |\langle b_0, a\rangle| + |\langle b_1, a\rangle| \le ||b_0||_{*0}||a||_0 + \frac{1}{t}||b_1||_{*1}t\,||a||_1 \le \big(||b_0||_{*0} + \frac{1}{t}||b_1||_{*1}\big)\max\{||a||_0, t\,||a||_1\}$, and therefore by minimizing among all the decompositions of b one deduces that $|\langle b, a\rangle| \le K\big(\frac{1}{t}, b; E_0', E_1'\big)J(t, a; E_0, E_1)$ for all $a \in E_0 \cap E_1, b \in E_0' + E_1', t > 0$.

Conversely, let L be a linear continuous form on $E_0 \cap E_1$ equipped with the norm $J(t, a; E_0, E_1)$, and let $M = ||L||$, so that $|L(a)| \le M\,\max\{||a||_0, t\,||a||_1\}$ for all $a \in E_0 \cap E_1$; one uses HAHN-BANACH theorem in order to find a linear continuous form L_* on $E_0 \times E_1$, equipped with the norm $||(a_0, a_1)|| = \max\{||a_0||_0, t\,||a_1||_1\}$, which extends the linear form $\widetilde{L}$ defined on the diagonal of $(E_0 \cap E_1) \times (E_0 \cap E_1)$ by $\widetilde{L}(a, a) = L(a)$, for which one has $|\widetilde{L}(a, a)| = |L(a)| \le M\,\max\{||a||_0, t\,||a||_1\} = M\,||(a, a)||$, so that $||\widetilde{L}|| \le M$ and therefore there exists an extension L_* on $E_0 \times E_1$ satisfying $||L_*|| \le M$. Any linear continuous form on $E_0 \times E_1$ can be written as $(a_0, a_1) \mapsto \langle b_0, a_0\rangle + \langle b_1, a_1\rangle$ with $b_0 \in E_0'$ and $b_1 \in E_1'$, and the norm of that linear continuous form is obviously $\le ||b_0||_{*0} + \frac{1}{t}||b_1||_{*1}$, but it is actually equal to that quantity because one can choose $a_0 \in E_0$ with $||a_0||_0 = 1$ and $\langle b_0, a_0\rangle = ||b_0||_{*0}$ and $a_1 \in E_1$ with $||a_1||_1 = \frac{1}{t}$ and $\langle b_1, a_1\rangle = \frac{1}{t}\,||b_1||_{*1}$ (again by an application of HAHN-BANACH theorem); one deduces then that $L(a) = \widetilde{L}(a, a) = \langle b_0, a\rangle + \langle b_1, a\rangle = \langle b_0 + b_1, a\rangle$ for all $a \in E_0 \cap E_1$, and $||b_0||_{*0} + \frac{1}{t}||b_1||_{*1} \le M$, so that L is given by the element $b = b_0 + b_1 \in E_0' + E_1'$, which satisfies $K\big(\frac{1}{t}, b; E_0', E_1'\big) \le M$.

It remains to show that b is defined in a unique way, i.e. that $\langle b, a\rangle = 0$ for all $a \in E_0 \cap E_1$ implies $b = 0$; indeed, because $|\langle b_0, a\rangle| = |\langle b_1, a\rangle| \le C\,||a||_1$ for all $a \in E_0 \cap E_1$ and $E_0 \cap E_1$ is dense in E_1, b_0 extends in a unique way to an element of E_1', which then coincides with $-b_1$ on the dense subspace $E_0 \cap E_1$ of E_1, so that one has $b_0, b_1 \in E_0' \cap E_1'$ and $b_0 + b_1 = 0$.

Lemma: Assume that $E_0 \cap E_1$ is dense in E_0 and dense in E_1. For $s > 0$, the dual space of $E_0 + E_1$ equipped with the norm $K(s, a; E_0, E_1) = \inf_{a_0+a_1=a}(||a_0||_0 + s\,||a_1||_1)$ is $E_0' \cap E_1'$ equipped with the norm $J\big(\frac{1}{s}, b; E_0', E_1'\big) = \max\big\{||b||_{*0}, \frac{1}{s}\,||b||_{*1}\big\}$.

Proof: If $a \in E_0 + E_1$ and $b \in E_0' \cap E_1'$ with a decomposition $a = a_0 + a_1$, then one has $|\langle b, a\rangle| \le |\langle b, a_0\rangle| + |\langle b, a_1\rangle| \le ||b||_{*0}||a_0||_0 + \frac{1}{s}\,||b||_{*1}s\,||a_1||_1 \le \max\big\{||b||_{*0}, \frac{1}{s}\,||b||_{*1}\big\}(||a||_0 + s\,||a||_1)$, and therefore by minimizing among all the decompositions of a one deduces that $|\langle b, a\rangle| \le J\big(\frac{1}{s}, b; E_0', E_1'\big)K(s, a; E_0, E_1)$ for all $a \in E_0 + E_1, b \in E_0' \cap E_1', s > 0$.

Conversely, if L is a linear continuous form on $E_0 + E_1$ equipped with the norm $K(s, a; E_0, E_1)$, then it is a linear continuous form on E_0 and also a linear continuous form on E_1 and therefore L is given by an element $b \in E_0' \cap E_1'$.

For computing the norm of L, for $0 < \varepsilon < 1$, $x, y > 0$ one chooses $a_0 \in E_0$ with $||a_0||_0 = x$ and $\langle b, a_0\rangle \ge (1-\varepsilon)x\,||b||_{*0}$ and $a_1 \in E_1$ with $||a_1||_1 = y$ and $\langle b, a_1\rangle \ge (1-\varepsilon)y\,||b||_{*1}$, so that $a = a_0 + a_1$ satisfies $K(s, a; E_0, E_1) \le x + s\,y$ and $|\langle b, a\rangle| \ge (1-\varepsilon)(x\,||b||_{*0} + y\,||b||_{*1})$, and therefore $||L|| \ge (1-\varepsilon)\frac{x\,||b||_{*0}+y\,||b||_{*1}}{x+s\,y}$, and letting ε tend to 0 and either x or y tend to 0, one finds $||L|| \ge \max\big\{||b||_{*0}, \frac{1}{s}\,||b||_{*1}\big\}$.■

The main result about duality for interpolation spaces, which has already been mentioned for LORENTZ spaces, is the following.

Proposition: Assume that $E_0\cap E_1$ is dense in E_0 and dense in E_1. Then, for $0<\theta<1$ and $1\le q<\infty$, one has $\big((E_0,E_1)_{\theta,q}\big)'=(E_0',E_1')_{\theta,q'}$ with equivalent norms, where $\frac{1}{q}+\frac{1}{q'}=1$.

Proof: We shall prove that $\big((E_0,E_1)_{\theta,q;J}\big)'\subset(E_0',E_1')_{\theta,q';K}$ and $(E_0',E_1')_{\theta,q';J}\subset\big((E_0,E_1)_{\theta,q;K}\big)'$, and the result will follow from the identity (with equivalent norms) between the interpolation spaces defined by the J-method and the K-method.

In order to prove the first inclusion, one takes $a'\in\big((E_0,E_1)_{\theta,q;J}\big)'$, and as a' defines a linear continuous form on $E_0\cap E_1$, a preceding lemma asserts that for every $\varepsilon>0$ there exists a sequence $b_n\in E_0\cap E_1$ such that $J(2^n,b_n;E_0,E_1)=1$ and $K(2^{-n},a';E_0',E_1')-\varepsilon\min\{1,2^{-n}\}\le\langle a',b_n\rangle$. For a sequence α_n such that $2^{-\theta\,n}\alpha_n\in l^q(Z)$, one defines $a(\alpha)=\sum_{n\in Z}\alpha_n\,b_n$, and one has $a(\alpha)\in(E_0,E_1)_{\theta,q;J}$ with $||a(\alpha)||_{\theta,q;J}\le||2^{-\theta\,n}\alpha_n||_q$ because of the normalization choosen for the sequence b_n; the particular choice for the sequence b_n implies $\sum_{n\in Z}\alpha_n\big(K(2^{-n},a';E_0',E_1')-\varepsilon\min\{1,2^{-n}\}\big)\le\langle a',a(\alpha)\rangle\le||a'||\,||a(\alpha)||_{\theta,q;J}\le||a'||\,||2^{-\theta\,n}\alpha_n||_q$ (notice that $\alpha_n\min\{1,2^{-n}\}\in l^1$); by letting ε tend to 0 one deduces that $\sum_{n\in Z}\alpha_nK(2^{-n},a';E_0',E_1')\le||a'||\,||2^{-\theta\,n}\alpha_n||_q$ for all such sequences α_n. Because $\sum_{n\in Z}\alpha_n\,\beta_n\le M\,||2^{-\theta\,n}\alpha_n||_q$ for all sequences α_n is equivalent to $||2^{\theta\,n}\beta_n||_{q'}\le M$, one deduces that $||2^{\theta\,n}K(2^{-n},a';E_0',E_1')||_{q'}\le||a'||$, i.e. $a'\in(E_0',E_1')_{\theta,q';K}$.

In order to prove the second inclusion, one takes $a'\in(E_0',E_1')_{\theta,q';J}$, and one writes $a'=\sum_n a_n'$ with $a_n'\in E_0'\cap E_1'$ and $2^{-\theta\,n}J(2^n,a_n';E_0',E_1')\in l^{q'}(Z)$; then for $a\in(E_0,E_1)_{\theta,q;K}$ one has $|\langle a',a\rangle|\le\sum_n|\langle a_n',a\rangle|\le\sum_n J(2^n,a_n';E_0',E_1')K(2^{-n},a;E_0,E_1)\le M\,||2^{\theta\,n}K(2^{-n},a;E_0,E_1)||_q\le M\,||a||_{\theta,q;K}$.■

When I was a student I had noticed that for $0<\theta<1$ the space $(E_0',E_1')_{\theta,1}$ is actually a dual, although not the dual of $(E_0,E_1)_{\theta,\infty}$; I had mentioned that to my advisor, Jacques-Louis LIONS, and he had told me that Jaak PEETRE had already made that observation[1]. The idea is to observe that l^1 is the dual of c_0, and that one can define a new[2] interpolation space modeled on c_0, considering that the usual interpolation space indexed by θ,p is actually modeled on $l^p(Z)$. For $0<\theta<1$ and two BANACH spaces E_0,E_1 continuously imbedded into a common topological vector space, one defines the space $(E_0,E_1)_{\theta;c_0}$ as the space of $a\in E_0+E_1$ such that $2^{-\theta\,n}K(2^n,a)\in c_0(Z)$, equipped with same norm as $(E_0,E_1)_{\theta,\infty}$ (this space is the closure of $E_0\cap E_1$ in $(E_0,E_1)_{\theta,\infty}$). The proof of the previous Proposition easily extends to show that the dual of $(E_0,E_1)_{\theta;c_0}$ is $(E_0',E_1')_{\theta,1}$ with an equivalent norm.

Another useful result concerning interpolation spaces is the question of compactness.

Proposition: If A is a linear mapping from a normed space F into $E_0\cap E_1$, such that A is linear continuous from F into E_0 and compact from F into E_1, then for $0<\theta<1$ the mapping A is compact from F into $(E_0,E_1)_{\theta,1}$ (and therefore compact from F into $(E_0,E_1)_{\theta,p}$ for $1\le p\le\infty$).

If B is a linear mapping from E_0+E_1 into a normed space G, such that B is linear continuous from E_0 into G and compact from E_1 into G, then for $0<\theta<1$ the mapping B is compact from $(E_0,E_1)_{\theta,\infty}$ into G (and therefore compact from $(E_0,E_1)_{\theta,p}$ into G for $1\le p\le\infty$).

Proof: If $||f_n||_F\le 1$, then $A\,f_n$ is bounded in E_0 and belongs to a compact subset of E_1, so that a subsequence f_m converges in E_1 and therefore is a CAUCHY sequence; one has $||x||_{\theta,1}\le C\,||x||_0^{1-\theta}||x||_1^\theta$ for all $x\in(E_0,E_1)_{\theta,1}$, and applying this inequality to $x=f_m-f_{m'}$ one deduces that $A\,f_m$ is a CAUCHY sequence in $(E_0,E_1)_{\theta,1}$, and therefore A is compact from F into $(E_0,E_1)_{\theta,1}$.

Let $||e_n||_{\theta,\infty}\le 1$ so that for each $\varepsilon>0$ there exists a decomposition $e_n=e_n^0+e_n^1$ with $||e_n^0||_0+\varepsilon\,||e_n^1||_1\le 2K(\varepsilon,e_n)\le 2\varepsilon^\theta$; then $||B\,e_n^0||_G\le 2||B||_{\mathcal{L}(E_0,G)}\varepsilon^\theta$, and $B\,e_n^1$ belongs to a compact subset of G so that a subsequence $B\,e_m^1$ converges in G, and therefore for this subsequence one has $\limsup_{m,m'\to\infty}||B\,e_m^1-B\,e_{m'}^1||_G\le 4||B||_{\mathcal{L}(E_0,G)}\varepsilon^\theta$; using CANTOR's diagonal subsequence argument one finds that $B\,e_n$ contains a converging subsequence in G, and therefore B is compact from $(E_0,E_1)_{\theta,\infty}$ into G.■

[1] It is quite natural that in the process of doing research one finds results which have already been found before, and Ennio DE GIORGI had once said “chi cerca trova, chi ricerca ritrova” (the first part reminds of the “seek and you will find” from the gospels, but the play on the prefix does not work well in English, although one could replace seek and find by search and discover in order to use research and rediscover). Sometimes, one may find the result by a different method and it may be worth publishing if the new proof is simpler than the previous one, or if it contains ideas which could be useful for other problems; of course, one should mention the author of the first proof, even if he/she has not published it.

[2] Obviously, one can describe more general classes of interpolation spaces, and Jaak PEETRE has actually developed a quite general framework for doing that.

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22. Friday May 5, 2000.

In 1974/75, I spent the year at University of Wisconsin, Madison, and I often discussed with Michael CRANDALL; among our joint results that we did not publish there was a question related to interpolation. The motivation for looking at the problem was some kind of generalization which had been published, for which it was not clear if there was any example showing that it was indeed a genuine[1] generalization, and as our result did not cover the same situations than the published theorem, it might well have been more general than the previous ones in some cases. Although we started by proving some observations for linear mappings, and then extended the method to a nonlinear setting, I present the results in reverse order.

Proposition: Let Ω be a bounded (LEBESGUE) measurable subset of R^N and let F be a nonlinear mapping from $L^\infty(\Omega)$ into itself satisfying the following properties:

i) F is LIPSCHITZ continuous from $L^\infty(\Omega)$ into itself,

ii) F is monotone (in the L^2 sense), i.e. $\int_\Omega \big(F(u_2)-F(u_1)\big)(u_2-u_1)\,dx \geq 0$ for all $u_1,u_2 \in L^\infty(\Omega)$,

then for every $p>2$, F is LIPSCHITZ continuous from $L^p(\Omega)$ into itself (i.e. F is LIPSCHITZ continuous with respect to the L^p distance, and hence it extends in a unique way as a LIPSCHITZ continuous mapping from $L^p(\Omega)$ into itself).

Proof: Let M_∞ be the LIPSCHITZ constant for F with respect to the L^∞ distance. For $0<\varepsilon<\frac{1}{M_\infty}$ and $v\in L^\infty(\Omega)$ there is a unique $u\in L^\infty(\Omega)$ solution of the equation $u+\varepsilon\,F(u)=v$, as it is the unique fixed point of the mapping $u\mapsto v-\varepsilon\,F(u)$, which is a strict contraction; moreover the mapping $v\mapsto u$ is LIPSCHITZ continuous with a constant $\leq\frac{1}{1-\varepsilon\,M_\infty}$, and if one defines the mapping G_ε by $G_\varepsilon(v)=v-u=\varepsilon\,F(u)$ then G_ε is LIPSCHITZ continuous with a constant $\leq\frac{\varepsilon\,M_\infty}{1-\varepsilon\,M_\infty}$.

For $v_1,v_2\in L^\infty(\Omega)$, one subtracts $u_1+\varepsilon\,F(u_1)=v_1$ from $u_2+\varepsilon\,F(u_2)=v_2$ and one multiplies by $F(u_2)-F(u_1)$, giving $\varepsilon\int_\Omega|F(u_2)-F(u_1)|^2\,dx\leq\varepsilon\int_\Omega|F(u_2)-F(u_1)|^2\,dx+\int_\Omega\big(F(u_2)-F(u_1)\big)(u_2-u_1)\,dx=\int_\Omega\big(F(u_2)-F(u_1)\big)(v_2-v_1)\,dx$, and therefore $||G_\varepsilon(v_2)-G_\varepsilon(v_1)||_2\leq||v_2-v_1||_2$. In particular G_ε has a unique extension to $L^2(\Omega)$ (which is a contraction).

Having shown that G_ε is LIPSCHITZ continuous on $L^\infty(\Omega)$ with a constant $\leq\frac{\varepsilon\,M_\infty}{1-\varepsilon\,M_\infty}$ and LIPSCHITZ continuous on $L^2(\Omega)$ with a constant ≤ 1, one deduces by nonlinear interpolation that G_ε is LIPSCHITZ continuous on $L^p(\Omega)$ with a constant $\leq\big(\frac{\varepsilon\,M_\infty}{1-\varepsilon\,M_\infty}\big)^\theta$ if θ is defined by $\frac{1}{p}=\frac{\theta}{\infty}+\frac{1-\theta}{2}$, and as one has $\theta=\frac{p-2}{p}>0$ one deduces that the constant may be made small by taking ε small. Assuming then that ε has been choosen small enough so that the LIPSCHITZ constant of G_ε in $L^p(\Omega)$ is $\leq K<1$, then for every $u_1,u_2\in L^\infty(\Omega)$, one defines $v_j=u_j+\varepsilon\,F(u_j)$ for $j=1,2$, and one deduces that $\varepsilon\,||F(u_2)-F(u_1)||_p=||G(v_2)-G(v_1)||_p\leq K\,||v_2-v_1||_p\leq K\,(||u_2-u_1||_p+\varepsilon\,||F(u_2)-F(u_1)||_p)$, and therefore $||F(u_2)-F(u_1)||_p\leq M_p||u_2-u_1||_p$ with $M_p=\frac{K}{\varepsilon(1-K)}$.∎

In the case of linear mappings, the idea is that one may use unbounded (densely defined) closed operators by considering their resolvents, i.e. the bounded operators $(A-\lambda\,I)^{-1}$ for some $\lambda\in C$. In particular, if for a real HILBERT space H a closed unbounded operator A has a dense domain and satisfies $\langle A\,u,u\rangle\geq 0$ for all $u\in D(A)$, then for $\lambda<0$ the resolvent exists and is a contraction. Then for $\varepsilon>0$ the bounded operator $I-(I+\varepsilon\,A)^{-1}$ is also a contraction, while if A is bounded its norm is $O(\varepsilon)$, and by interpolation it will have a small norm in an interpolation space and it will show that the operator is bounded in that space. The

[1] My advisor had once mentioned that when reading a too abstract article one should first look at the examples. On a preceding occasion, I had applied his advice and shown that a generalization of the LAX-MILGRAM lemma was not a genuine one, and that all the examples of the proposed new theory could be dealt with in a classical way, once a particular observation had been made; a friend insisted that I publish the observation, and it became my shortest article. Of course, in such situations, it is better to avoid mentioning names, and one should remember that even the best mathematicians have made mistakes (I was told that a great mathematician had his ego a little bruised after publishing a perfectly good proof, when he realized that no object satisfied all the hypotheses of his theorem, which therefore was a quite useless one).

preceding Proposition has followed the same scenario in a nonlinear setting, but one can deduce more in a linear setting by using the spectral radius of an operator[2].

Lemma: Let A be a linear mapping from E_0+E_1 into itself satisfying $A\in\mathcal{L}(E_0,E_0)$ with spectral radius ρ_0 and $A\in\mathcal{L}(E_1,E_1)$ with spectral radius ρ_1 (where E_0 and E_1 are BANACH spaces). Then for $0<\theta<1$ and $1\le p\le\infty$, the spectral radius $\rho(\theta,p)$ of A on $(E_0,E_1)_{\theta,p}$ satisfies the inequality $\rho(\theta,p)\le\rho_0^{1-\theta}\rho_1^{\theta}$.
Proof: One uses the fact that $\rho(A)=\lim_{n\to\infty}||A^n||^{1/n}$, and the fact that $||A^n||_{\mathcal{L}((E_0,E_1)_{\theta,p},(E_0,E_1)_{\theta,p})}\le C\,||A^n||_{\mathcal{L}(E_0,E_0)}^{1-\theta}||A^n||_{\mathcal{L}(E_1,E_1)}^{\theta}$ with C depending upon which equivalent norm is used for $(E_0,E_1)_{\theta,p}$ (but not on n); taking the power $\frac{1}{n}$ and letting n tend to ∞ gives the result.■

If f is holomorphic, then the spectrum of $f(A)$ is the image by f of the spectrum of A, and using the preceding lemma one deduces that if K_0 is the spectrum of A in E_0, K_1 is the spectrum of A in E_1, and $K_{\theta,p}$ is the spectrum of A in $(E_0,E_1)_{\theta,p}$, then for every holomorphic function f one has $\max_{z\in K_{\theta,p}}|f(z)|\le(\max_{z\in K_0}|f(z)|)^{1-\theta}(\max_{z\in K_1}|f(z)|)^{\theta}$. I once asked Ciprian FOIAS[3] if he knew some situation where the spectrum strongly depends upon the space used, but I did not understand his answer.

I do not know any good example of applications of these results obtained with Michael CRANDALL, which is one reason why I never wrote that proof before, but I find interesting the fact that with respect to interpolation a monotonicity property is almost like a LIPSCHITZ condition. Actually, the last inequality giving a localization of the spectrum suggests that one could develop notions of interpolation of sets.

Finally, I want to mention a result which I found a few years ago, while I was teaching a graduate course on mathematical methods in control, because I wanted to explain the following result of Yves MEYER[4], which he had used in connection with a control problem[5].

Lemma: Let $d\mu$ be a RADON measure on R and $T>0$. A necessary and sufficient condition that there exists a constant $C(T)$ such that $\int_R|\mathcal{F}f(\xi)|^2\,d\mu\le C(T)\int_0^T|f(x)|^2\,dx$ for all function $f\in L^2(R)$ which vanish outside $(0,T)$ is that $\sup_{k\in Z}\mu([k,k+1])\le C'<\infty$.■

After looking at his proof, I found that with very little change one could prove the following variant.

Lemma. Let $d\mu$ be a RADON measure on R. The condition $\sup_{k\in Z}\mu([k,k+1])\le C'<\infty$ is equivalent to the existence of a constant C such that $\int_R|\mathcal{F}f(\xi)|^2\,d\mu\le C\left(\int_R|f(x)|^2\,dx\right)^{1/2}\left(\int_R(1+x^2)|f(x)|^2\,dx\right)^{1/2}$ for all functions f such that $\int_R(1+x^2)|f(x)|^2\,dx<\infty$.■

Using the caracterisation of the space $(E_0,E_1)_{1/2,1}$ of Jacques-Louis LIONS and Jaak PEETRE, it means that one can replace the right side of the inequality by the norm of f in a corresponding interpolation space; here E_1 is $L^2(R)$ for the LEBESGUE measure dx while E_0 is L^2 for the measure $(1+x^2)\,dx$, and it is not difficult to characterize $(E_0,E_1)_{1/2,1}$ as the space of functions $f\in L^2(R)$ such that $\sum_{k\ge1}\left(\int_{2^k\le|x|\le2^{k+1}}2^k|f(x)|^2\right)^{1/2}dx<\infty$. For proving that the condition is necessary, Yves MEYER considers a function $\varphi\in L^2(R)$ whose FOURIER transform does not vanish on $(0,1)$ (if φ_n converges to δ_0 then $\mathcal{F}\varphi_n$ converges to 1), and applies the inequality to f defined by $f(x)=e^{2i\pi\,k\,x}\varphi(x)$ so that for $\xi\in[k,k+1]$

[2] If $A\in\mathcal{L}(E,E)$ for a BANACH space E, the spectrum of A is the nonempty closed set of $\lambda\in C$ for which $A-\lambda I$ is not invertible, and the spectral radius $\rho(A)$ is the maximum of $|\lambda|$ for λ in the spectrum of A.

[3] Ciprian FOIAS, Rumanian-born mathematician. He was my colleague in Orsay in 1978/79. He works at Indiana University, Bloomington.

[4] Yves F. MEYER, French mathematician, born in 1939. He was my colleague in Orsay from 1975 to 1979. He works at Ecole Normale Supérieure, Cachan, France.

[5] The title of his article mentioned the control of deformable structures in space, but only contained a result of control for the scalar wave equation, although a little idealistic, as the control was applied at a point inside the domain. I guess that Jacques-Louis LIONS had understood that the control of flexible structures in space is an important question, but because Elasticity with large displacement is a too difficult subject, and even the linearized version of Elasticity is a complicated hyperbolic system, he had started by considering questions related to a scalar wave equation, but he had probably forgotten to point out how far these questions really are from controlling large deformable structures.

one has $|\mathcal{F}f(\xi)| \ge \gamma = \min_{\eta\in(0,1)} |\mathcal{F}\varphi(\eta)|$. That the condition is sufficient is a consequence of the following lemma.

Lemma: There exists a constant C'' such that $\sum_{k\in Z} \sup_{\xi\in[k,k+1]} |\mathcal{F}f(\xi)|^2 \le C''\big(\int_R |f(x)|^2\,dx\big)^{1/2}\big(\int_R (1+x^2)|f(x)|^2\,dx\big)^{1/2}$ for all functions $f \in L^2(R)$ such that $\int_R (1+x^2)|f(x)|^2\,dx < \infty$.

Proof: Let $a_k = \big(\int_{[k,k+1]} |\mathcal{F}f(\xi)|^2\,d\xi\big)^{1/2}$ and $b_k = \big(\int_{[k,k+1]} \big|\frac{d\mathcal{F}f(\xi)}{d\xi}\big|^2\,d\xi\big)^{1/2}$; the usual proof of continuity of functions of the SOBOLEV space $H^1\big((0,1)\big)$ gives $\sup_{\xi\in[k,k+1]} |\mathcal{F}f(\xi)|^2 \le K^2 a_k (a_k^2+b_k^2)^{1/2}$, and summing in k and using CAUCHY-SCHWARZ gives $\sum_{k\in Z} \sup_{\xi\in[k,k+1]} |\mathcal{F}f(\xi)|^2 \le K^2\big(\int_R |\mathcal{F}f(\xi)|^2\,d\xi\big)^{1/2}\big(\int_R (|\mathcal{F}f(\xi)|^2 + \big|\frac{d\mathcal{F}f(\xi)}{d\xi}\big|^2\,d\xi\big)^{1/2}$, which is essentially the desired result.■

With the remark concerning interpolation, and exchanging the roles of f and $\mathcal{F}f$, this lemma says a little more that the fact that the functions from $\big(H^1(R), L^2(R)\big)_{1/2,1}$ are continuous and tend to 0 at ∞, as it gives some precise way how the functions tend to 0 at ∞, as it implies that $\sum_{n\in Z} \sup_{x\in[n,n+1]} |u(x)|^2 < \infty$.

In the case of functions with support in $(0,T)$, my variant gives the same growth in $1+T$ found by Yves MEYER, who notices that the growth is optimal for large values of T, because if f is the characteristic function of $(0,T)$, then $\int_R |\mathcal{F}f(\xi)|^2\,d\xi = c\,T$. In the case of R^N one can easily generalize the proof and consider RADON measures $d\mu \ge 0$ for which there exists a constant C such that $\mu(Q) \le C$ for every cube of size 1, and consider functions f such that $\sum_{k\ge 1} \big(\int_{2^k\le|x|\le 2^{k+1}} 2^{kN}|f(x)|^2\big)^{1/2}\,dx < \infty$; for functions with support in a bounded set K, one obtains a constant growing like $\big(1+diameter(K)\big)^N$, but it is not clear to me if the diameter of K is the correct geometric quantity to use in such an inequality.