Introduction to structured equations in biology

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Chapter 1

Examples of stage structured population equations

Population biology is certainly the oldest area of biology where mathematics has been used. Speaking usually of large populations, partial differential equations (PDE) play a natural role. The recent book of H. Thieme [16] gives a general view of this subject.

This Chapter presents several examples of stage structured equations, mostly based on the Lecture Note by the author [14].

1.1 The renewal equation: demography and cell division cycle

1.1.1 Setting the model

The simplest model to understand why other variables than space can enter naturally in PDE models of biology is certainly the renewal equation for demography. Consider a 'closed' population with no immigration, neither emigration. Neglect also for the time being death and consider only aging and birth.

The population density $n(t,x)$ of individuals of age $x>0$ at time t then satisfies

$$n(t+s,x+s)=n(t,x), \qquad \forall s\geq 0.$$

As a consequence, differentiating in s, and taking $s=0$, we find

$$\frac{\partial}{\partial t}n(t,x)+\frac{\partial}{\partial x}n(t,x)=0 \quad t\geq 0,\, x\geq 0,. \tag{1.1}$$

This equation has to be complemented by the 'boundary condition' at $x=0$, i.e. the number of newborns at time t; this is given by the quantity

$$n(t,x=0)=\int_0^\infty B(y)n(t,y)dy, \tag{1.2}$$

where B denotes the birth rate of the population (that vanishes certainly for $x \approx 0$ and x large).

These two last equations form the so-called renewal equation. They were introduced by McKendrick for epidemiology, then x denotes the age in the disease, an important factor for the epidemic spread.

The same model was re-discovered some years later by von Foerster [18] for the cell division cycle. In this context it is natural stipulate that cells, after division, take some time x^*, with a well reported variability, before dividing again. We arrive with the variant of the previous renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + k(x)\, n(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\ \\ n(t, x=0) = 2 \int k(y) n(t,y) dy, \\ \\ n(t=0, x) = n^0(x). \end{cases} \tag{1.3}$$

Here $k(x)$ denotes the division rate. A possible choice is

$$k(x) = A \mathbb{1}_{\{x \geq x^*\}}.$$

1.1.2 Age structured equations and Volterra equations

Consider the system (1.3) and set

$$K(x) = \int_0^x k(y) dy.$$

Therefore

$$\frac{\partial}{\partial t}[e^{K(x)} n(t,x)] + \frac{\partial}{\partial x}[e^{K(x)} n(t,x)] = 0, \quad t \geq 0,\, x \geq 0,$$

and $e^{K(x)} n(t,x)$ is constant along the characteristics $e^{K(x-s)} n(t-s, x-s) = e^{K(x)} n(t,x)$. We choose successively $s = x$ and $s = t$

$$e^{K(x)} n(t,x) = \begin{cases} n(t-x, 0) = b(t-x), & x < t, \\ \\ e^{K(x-t)} n^0(x-t), & x > t. \end{cases}$$

Inserting this information and defining $b(t)$ as the birth term

$$b(t) = n(t, x=0), \tag{1.4}$$

we find the so-called *Volterra integral equation*

$$\begin{cases} b(t) = 2 \int k(y) e^{-K(y)} b(t-y) dy + b^0(t), \\ \\ b^0(t) = 2 \int k(y) e^{K(y-t) - K(y)} n^0(y-t). \end{cases} \tag{1.5}$$

This equation can be solved by the standard Cauchy-Lipschitz theory.

1.1.3 The limit of deterministic birth age

In order to simplify the analysis, it is often assumed the division occurs exactly at a certain age x^*. This means we are now interested in the limit $A \to \infty$ for the following division rate

$$k(x) = k_A(y) = A\mathbb{I}_{\{x \geq x^*\}}.$$

Then, the weight $k_A(y)e^{-K_A(y)} \geq 0$ has the following properties

$$\int_0^\infty k_A(y)e^{-K_A(y)}dy = 1, \qquad k_A(y)e^{-K_A(y)} = 0 \text{ for } y < x^*,$$

$$k_A(y)e^{-K_A(y)} = Ae^{-A(y-x^*)} \xrightarrow[A\to\infty]{} 0, \qquad \text{for } y > x^*.$$

Therefore, we have : $k_A(y)e^{-K_A(y)} \xrightarrow[A\to\infty]{} \delta(a - x^*)$.

Inserting this in (1.5) gives, in the limit $A \to \infty$ (after passing to the limit also in $b^0_A(t)$),

$$b(t) = 2b(t - x^*) + b^0(t),$$

1.2 Finite resources and nonlinearities

We give two examples of nonlinearities which we take from two different areas of biosciences: ecology and epidemiology.

1.2.1 A nonlinear model from ecology

Nonlinear models are typically possible when including a limited resource $S(t)$ shared by all the population. The level of nutrient controls growth, death and birth. This yields models as

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}\left[(x_m \frac{S(t)}{1+S(t)} - x)n(t,x)\right] + d(x, S(t))\, n(t,x) = 0, \\ \\ n(t, x = 0) = \int B(y, S(t))n(t,y)dy, \\ \\ n(t = 0, x) = n^0(x), \end{cases} \tag{1.6}$$

here we take naturally $0 \leq x \leq x_m$. If x represents the size of the micro-organism under consideration, this equation expresses that x increases as long as the nutrient level is high enough and diminishes when it is too low for maintening the organism which needs are proportional to it size x.

This PDE is coupled with a differential equation for the resources which expresses their outflow (degration), inflow (with rate S^0), their consumption For instance one can write a balance law

$$\frac{d}{dt}S(t) + S(t) = S^0 - \frac{S(t)}{1+S(t)} \int_0^{x_m} x^2 n(t,x)dx. \tag{1.7}$$

It is also usual to assume that the resources are available with a faster time scale (adiabatic assumption). Then, we replace the differential equation on $S(t)$ by its steady state, the mere algebraic equation

$$S(t) + \frac{S(t)}{1+S(t)} \int_0^{x_m} x^2 n(t,x)dx = S^0.$$

1.2.2 The Kermack-McKendrick model for epidemic spread

Ordinary differential equations have been used for a long time to describe the spread of an epidemy in a population. The simplest are called SIR (Susceptible, Infected, Resistant), an extension is the SEIR (Exposed) model, and reads

$$\begin{cases} \frac{dS}{dt} &= \beta_S(S+I+R) - \mu_S S - \gamma_S S\, I, \\ \frac{dI}{dt} &= \gamma_S S\, I - \mu_I I - \beta_R I, \\ \frac{dR}{dt} &= \beta_R I - \mu_R R. \end{cases} \tag{1.8}$$

In order to improve the validity of the model, Kermack and McKendrick [10, 8] proposed to take into account the variable infectivity level, and removal rate, depending on the age in the disease. They arrive at

$$\frac{d}{dt}S(t) = B - \mu_S S(t) - \lambda_S(t)S(t),$$

$$\lambda_S(t) := \int_0^\infty \kappa(x)n(t,x)dx,$$

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + (\mu_I + \beta_n(x))n(t,x) = 0, \\ n(t,x=0) = \lambda_S(t)S(t), \end{cases}$$

$$\frac{d}{dt}R(t) = \int_0^\infty \beta_n(x)n(t,x)dx.$$

The interpretation is that $n(t,x)$ is the density of population at age x of infection and this replaces the compartment $I(t)$ in the SIR system (1.8). Individuals are infected (at age $x=0$ in the infection stage) from susceptible that are getting the virus by encounter with infected at a rate $\lambda_S(t)$ depending of the time x elapsed since infection through a rate $\kappa(x)$. Notice again the quadratic term for transition from susceptible to infected. The advantage of the Kermack-McKendrick model is that one can take into account

The book [3] is a very good and recent account of the subject.

1.3 Age structure with quiescence

For physiological applications, it is often considered that cells can be in two states, proliferative (entered in the cell cycle) or quiescent (they perform their duty of cell but do not proliferate). This issue is often related to the notion of stem cells which are also the subject of particular modelling, a subject we do not touch here.

1.3.1 Model with proliferative and quiescent cells

Consider now an age structured population that can be either in prolifarative (and $p(t,x)$ denotes the population density) or quiescent state (and $q(t,x)$ denotes the population density). Neglecting the death term in quiescent state, the dynamics is described by the system

$$\begin{cases} \frac{\partial}{\partial t}q(t,x) + \frac{\partial}{\partial x}q(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\ \\ q(t,x=0) = \int B(y)p(t,y)dy, \\ \\ \frac{\partial}{\partial t}p(t,x) + d_p(t)p(t,x) = r(t,x)q(t,x), \\ \\ p(t=0,x) = p^0(x), \qquad q(t=0,x) = q^0(x). \end{cases} \tag{1.9}$$

1.3.2 Age structured equations and delay equations

When d_p only depends on time, we can reduce this system to a delay equation. We set

$$b(t) = \int_0^\infty B(y)p(t,y)dy,$$

we find, multiplying the equation on p by $B(x)$ and integrating dx,

$$\frac{d}{dt}b(t) + d_p(t)b(t) = \int_0^\infty r(t,x)B(x)q(t,x)dx.$$

But using again the caracteristics (see Section 1.1.2), we know that

$$q(t,x) = \begin{cases} q(t-x,0) = b(t-x), & x < t, \\ \\ q^0(x-t), & x > t. \end{cases}$$

Therefore we arrive at the *delay equation*

$$\frac{d}{dt}b(t) + d_p(t)b(t) = \int_0^t r(t,x)B(x)b(t-x)dx + \int_t^\infty r(t,x)B(x)q^0(x-t)dx.$$

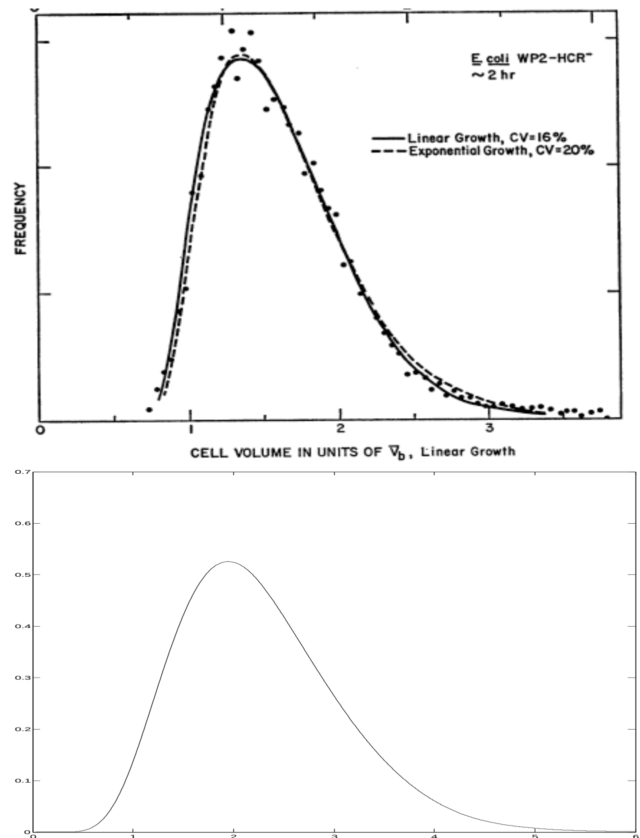

Figure 1.1: TOP:EXPERIMENTAL DATA FOR SIZE DISTRIBUTION IN *E. coli*. BOTTOM: NUMERICAL SIMULATION OF EQUATION (1.10).

1.4 Size structured models (equal mitosis)

Size is usually a better physiological trait to structure cell populations as bacterium or yeast. Then x is the mass (or length, or volume) of the cell. Assuming equal mitosis, i.e., that cells divide exactly in two equal new cells, the equation reads as follows (for $t \geq 0$, $x \geq 0$),

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x}\big[g(x)\, n(t,x)\big] + B(x)\, n(t,x) = 4\, B(2x)\, n(t,2x), \\ \\ n(t, x=0) = 0, \\ \\ n(t=0,x) = n^0(x). \end{cases} \tag{1.10}$$

The term $\big[g(x)\, n(t,x)\big]$ describes the growth of cells using the (unlimited) nutrients. The term $4\, B(2x)\, n(t,2x)$ describes the division of cells of size $2x$ in two cells of size x, the term $B(x)\, n(t,x)$ takes into account the loose of cells of size x that divide in cells of size $x/2$.

The dynamics of this model is also characterized by a growth, both of total biomass and number of cells. The first natural question is to find the growth exponent. The second question is to find the typical repartition of cells (we will show that this concept makes sense) which results from the two opposite effects described by the model; growth by the differential term and x decay by the algebraic term.

In order to show that this model exhibits growth, one can notice two idendities. The first one is to consider the total number of cells and compute (this is formal at this level, assuming that one can integrate on the half line)

$$\frac{d}{dt}\int_0^\infty n(t,x)dx+\int_0^\infty B(x)n(t,x)dx = \int_0^\infty 4\,B(2x)\,n(t,2x)dx = 2\int_0^\infty B(x)n(t,x)dx,$$

therefore

$$\frac{d}{dt}\int_0^\infty n(t,x)dx = \int_0^\infty B(x)n(t,x)dx,$$

in words, this means that the total number of cells only increases thanks to the division rate B.

One can also compute the biomass. Multiplying by x and integrating by parts we calculate

$$\begin{aligned}\tfrac{d}{dt}\int_0^\infty xn(t,x)dx - \int_0^\infty g(x)n(t,x)dx + \int_0^\infty xB(x)n(t,x)dx &= \int_0^\infty 4\,x\,B(2x)\,n(t,2x)dx \\ &= \int_0^\infty xB(x)n(t,x)dx,\end{aligned}$$

therefore

$$\frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx,$$

in words, biomass only increases by use of nutrients.

1.5 Size structured models (asymmetric division)

The division is not always symmetric and a daughter cell can be much smaller that the mother cell. The above model can be generalized to tak ethis into account. We arrive at an equation also calles 'aggregation-fragmentation' because it arises in physics to describe such phenomena e.g. for polymers.

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}\big[g(x)\,n(t,x)\big] + B(x)\,n(t,x) = 2\int B(y)\kappa(x,y)\,n(t,y)dy, \\ n(t,x=0) = 0, \\ n(t=0,x) = n^0(x). \end{cases} \tag{1.11}$$

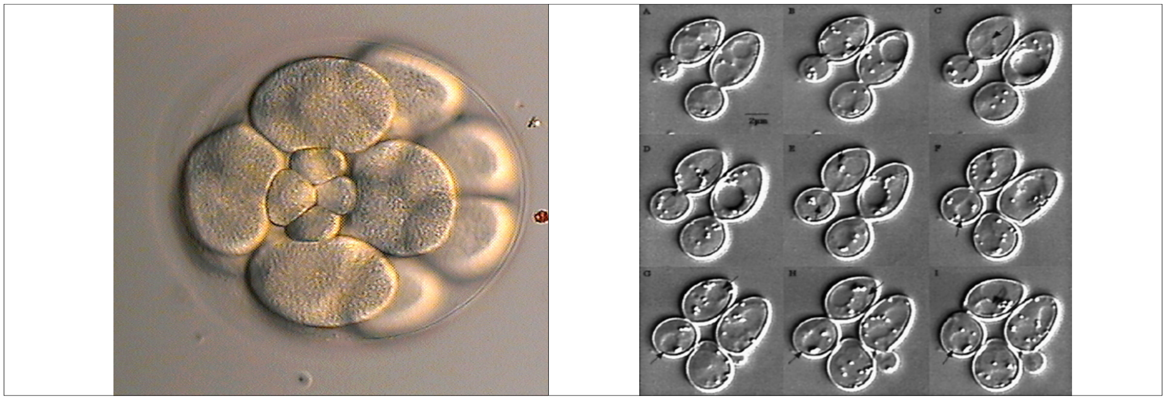

Figure 1.2: Asymmetric cell division in yeast.

Here $B(y)$ is the division rate of cells of sizes y and $\kappa(x,y)$ is the probality that such a cell gives a daughter cell of size $x \leq y$. It is natural to assume that (i) daughter cells are smaller than the mother cell, (ii) the division event gives two cells exactly, (iii) the toal mass is conserved in the division. These are expresses by the idendities

$$\kappa(x,y) = 0 \qquad \text{for } x > y, \tag{1.12}$$

$$\int_0^\infty \kappa(x,y)dx = 1, \tag{1.13}$$

$$\int_0^\infty x\ \kappa(x,y)dx = y/2. \tag{1.14}$$

Notice that this last equality is a consequence the first two and of the natural symmetry assumption

$$\kappa(x,y) = \kappa(y-x,x).$$

As an exercise, one can notice that the same relations as in Section 1.4 hold for growth of the number of cells and total biomass (still formally, assuming that one can integrate on the half line)

$$\frac{d}{dt}\int_0^\infty n(t,x)dx = \int_0^\infty B(x)n(t,x)dx,$$

$$\frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx.$$

For various choices of κ we can recover models we have already encountered. Let us give examples.

(i) The renewal (age structured) equation (1.1)–(1.2) can be recovered using,

$$\kappa(x,y) = \frac{1}{2}\big(\delta(x=0) + \delta(x=y)\big). \tag{1.15}$$

(ii) Equal mitosis, as in section 1.4, is the special case

$$\kappa(x,y) = \delta(x = y/2). \tag{1.16}$$

(iii) Uniform division is the case

$$\kappa(x,y) = \frac{1}{y}\, \mathbb{1}_{\{0\le x\le y\}}. \tag{1.17}$$

1.6 Nonlinear aggregation-fragmentation for prion

As a nonlinear example for size structure model we mention the model of prion proliferation following [6, 1].

The most widely accepted explanation of prion self-replication is that the 'normal' prion protein (with density denoted by $V(t)$ below) is generated normally (in the brain). Infected individual are characterized by the presence of aggregates (polymers) of the same protein (that changes conformation by contact with the prion polymers to adopt this capacity of aggregation). The density of aggregates of size $x > 0$ is denoted by $u(x,t)$. The corresponding continuous model reads, with possibly nonconstant coefficients,

$$\begin{cases} \dfrac{dV(t)}{dt} &= \lambda - V(t)\left[\gamma + \displaystyle\int_0^\infty \tau(x)u(x,t)\, dx\right], \\ \dfrac{\partial}{\partial t}u(x,t) &= -V(t)\dfrac{\partial}{\partial x}\big(\tau(x)u(x,t)\big) - [\mu(x)+\beta(x)]u(x,t) \\ & \qquad\qquad\qquad +2\int_x^\infty \beta(y)\kappa(x,y)\, u(y,t)\, dy, \\ u(x_0,t) &= 0, \end{cases} \tag{1.18}$$

together with appropriate initial conditions.

Chapter 2

Weak solutions to the renewal equation

In this Chapter we aim at building a solution in the weak (distribution) sense to the renewal equation

$$
\begin{cases}
\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x}[g(t,x)n(t,x)] + d(t,x)\, n(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\
\\
g(t,0)\ n(t,x=0) = \int B(t,y)n(t,y)dy, \\
\\
n(t=0,x) = n^0(x),
\end{cases}
\tag{2.1}
$$

We assume that

$$
0 \leq d \in L^\infty(\mathbb{R}^+), \quad 0 \leq B \in L^\infty(\mathbb{R}^+), \tag{2.2}
$$

$$
g(t,x) \in C_{\mathrm{b}}^1(\mathbb{R}^+ \times \mathbb{R}^+), \quad g(t,0) \geq g_m > 0, \tag{2.3}
$$

$$
n^0 \in L^1 \cap L^\infty(\mathbb{R}^+). \tag{2.4}
$$

Also we define

$$
\bar{B} = \sup_{t\geq 0,\ x \geq 0} B(t,x). \tag{2.5}
$$

We define the weak solutions (or distributional solutions), as follows:

Definition 2.1 *A function $n \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{R}^+\big)$ satisfies the renewal equation (2.1) in the weak (distribution) sense, if $\int_0^\infty B(t,x)|n(t,x)|dx \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and for all $T > 0$ and all* test function $\Psi \in C^1_{\mathrm{comp}}\big([0,T] \times [0,\infty[\big)$ *such that $\psi(T,x) \equiv 0$, we have*

$$
\begin{aligned}
-\int_0^T \int_0^\infty \quad & n(t,x) \left\{ \frac{\partial \Psi(t,x)}{\partial t} + g(t,x) \frac{\partial \Psi(t,x)}{\partial x} - d(x)\Psi(t,x) \right\} dx\ dt \\
& = \int_0^\infty n^0(x)\Psi^0(x)dx + \int_0^T \Psi(t,0) \int_0^\infty B(t,x)n(t,x)dx\ dt.
\end{aligned}
\tag{2.6}
$$

A motivation for such a definition is that

Theorem 2.1 *Whenever $n \in C^1\big([0,\infty[\times[0,\infty[\big)$ is a classical solution to the renewal equation (2.1), it is also a weak solution.*

Proof. Multiply (2.1) by the test function Ψ and integrate by parts on $[0,T]\times\mathbb{R}^+$. $\square$

It turns out that C^1 bounds are usually too strong for practical purposes. For instance $n(t,x)$ is obviously discontinous if

$$g(0,0)\ n^0(0) \neq \int_0^\infty B(x) n^0(x)dx.$$

Weak solutions, as often in PDEs, are the good concept because they exist and are unique in a wide class for the coefficients and the solution itself. For instance they are well adapted to study how discrete versions of the renewal equation converge to a 'continuous' solution. We prove here the following result in this direction,

Theorem 2.2 *We assume (2.2)– (2.4), then there is a unique weak solution $n \in L^\infty\big(0,T;L^1(\mathbb{R}^+)\big)$, for all $T>0$, to the renewal equation (2.1) and it satisfies*

$$n \in C\big(\mathbb{R}^+;L^1(\mathbb{R}^+)\big)$$

$$\int_0^\infty |n(t,x)|dx \leq \int_0^\infty |n^0(x)|dx\ e^{\|(B-d)_+\|_\infty t},$$

$$|n(t,x)| \leq \max\Big(\max_y |n^0(y)|, \frac{\bar{B}}{g_m} \sup_{0\leq s\leq t} \int_0^\infty |n(s,y)|dy\Big),$$

For two different initial data, we have L^1 uniform contraction

$$\int_0^\infty |n(t,x)-\tilde{n}(t,x)|dx \leq \int_0^\infty |n^0(x)-\tilde{n}^0(x)|dx e^{\|(B-d)_+\|_\infty t}.$$

The end of this Chapter is devoted to the proof of this Theorem. We begin with uniqueness and turn to existence through a discrete version.

2.1 The adjoint problem

As a preliminary to the uniqueness proof, we consider the adjoint equation to (2.1) with a source term $S(t,x)$ on a given time interval $[0,T]$,

$$\begin{cases} -\frac{\partial}{\partial t}\psi(t,x) - g(t,x)\frac{\partial}{\partial x}\psi(t,x) + d(t,x)\,\psi(t,x) = \psi(t,0)B(t,x) + S(t,x),\ x\geq 0, \\ \psi(t=T,x)=0. \end{cases} \tag{2.7}$$

Lemma 2.1 *Assume (2.2)– (2.3) and* $d, B \in C^1(\mathbb{R}^+)$, $S \in C^1_{\text{comp}}\big([0,T[\times\mathbb{R}^+\big)$, B *with compact support, then there is a unique* C^1 *solution to the adjoint equation (2.7). Moreover* $\psi(t,x)$ *vanishes for* $x \geq R > 0$ *for some* R *depending on the data. Moreover we have,*

$$\sup_{0\leq t\leq T, x\in\mathbb{R}^+} |\psi(t,x)| \leq C(T,\bar{B},\bar{d})\|S\|_\infty.$$

Proof. We use the method of characteristics based on the solution to the differential system parametrized by the Cauchy data (t,x)

$$\begin{cases} \frac{d}{ds}X(s) = g(s,X(s)), \quad t\leq s\leq T, \\ \\ X(t) = x \geq 0. \end{cases}$$

The solutions exist thanks to the Cauchy-Lipschitz theorem and $X(s)\geq 0$ thanks to assumption (2.3).

Then, setting

$$\widetilde{\psi}(s) = \psi(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma}, \quad \widetilde{B}(s) = B(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma},$$

$$\widetilde{S}(s) = S(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma},$$

we rewrite equation (2.7) as

$$\begin{aligned} \frac{d}{ds}\widetilde{\psi}(s) &= [\tfrac{\partial}{\partial t}\psi + g\tfrac{\partial}{\partial x}\psi - d\psi]e^{\int_s^t d(\sigma,X(\sigma))d\sigma}\Big|_{(s,X(s))} \\ \\ &= -\psi(s,0)\widetilde{B}(s) - \widetilde{S}(s), \end{aligned}$$

Next, we integrate between $s=t$ and $s=T$ and using the Cauchy data at $t=T$ and the idendity $\widetilde{\psi}(t) = \psi(t,x)$,

$$\psi(t,x) = \int_t^T [\psi(s,0)\widetilde{B}(s;t,x) + \widetilde{S}(s;t,x)]ds. \tag{2.8}$$

In order to make it more clear, we have recorded that the $\widetilde{\cdot}$ quantities depend also on (t,x), i.e., $\widetilde{B}(s) = \widetilde{B}(s;t,x)$, $\widetilde{S}(s) = \widetilde{S}(s;t,x)$.

This integral equation can be solved first for $x=0$. Then, equation (2.8) is reduced to the Volterra equation

$$\psi(t,0) = \int_t^T [\psi(s,0)\widetilde{B}(s;t,0) + \widetilde{S}(s;t,0)]ds, \qquad 0\leq t\leq T$$

which has a unique solution thanks to the (backward) Cauchy-Lipschitz theorem that vanishes for $t = T$. By the C^1 regularity of the data, we also have $\psi(t, 0) \in C^1$.

Since $\psi(t, 0)$ is now known, formula (2.8) gives us the explicit form of the solution for all (t, x). Notice that, in the compact support statement, $\widetilde{\psi}(t, x)$ vanishes for $x \geq R$ where R denotes the size of the support of B and S in x, plus $T\|g\|_\infty$.

The uniform bound on ψ also follows from formula (2.8), and the C^1 regularity of the data shows that $\psi(\cdot, \cdot) \in C^1$. $\square$

2.2 Uniqueness

With the help of the adjoint problem that we have studied in the previous section, we can prove the uniqueness of weak solutions. We use the classical Hilbert uniqueness method. The idea is simple: when the coefficients d, B satisfy the assumptions of Lemma 2.1, we can use the solution ψ to (2.7) as a test function in the weak formulation (2.6). For the difference between two possible solutions with the same initial data, we arrive at

$$-\int_0^T \int_0^\infty n(t,x) \quad \left\{ \tfrac{\partial \Psi(t,x)}{\partial t} + g(t,x) \tfrac{\partial \Psi(t,x)}{\partial x} - d(x)\Psi(t,x) + \Psi(t,0)B(t,x) \right\} dx\ dt$$

$$= \int_0^\infty n^0(x)\Psi^0(x)dx = 0.$$

Taking into account (2.7) , we arrive at

$$\int_0^T \int_0^\infty n(t,x)S(t,x)dx\ dt = 0,$$

for all functions $S \in C^1_{\text{comp}}$, and this implies $n \equiv 0$.

When the coefficient are merely bounded and B does not have compact support, we consider a regularized family $d_p \to d$, $B_p \to B$ where the convergence holds a.e. with uniform bounds and d_p, B_p satisfying the assumptions of Lemma 2.1. Then, for a given function $S \in C^1_{\text{comp}}$, we solve (2.7) and call ψ_p its solution. Inserting it in the solution of weak solutions we obtain

$$\int_0^T \int_0^\infty n(t,x)S(t,x)dx\ dt = R_p,$$

$$R_p = \int_0^T \int_0^\infty n(t,x) \left\{ [d_p - d(x)]\Psi_p(t,x) + \Psi_p(t,0)[B(t,x) - B_p(t,x)] \right\} dx\ dt,$$

and using that ψ_p is uniformly bounded, we deduce that

$$|R_p| \leq C \int_0^T \int_0^\infty |n(t,x)| \left\{ |d_p - d(x)| + |B(t,x) - B_p(t,x)| \right\} dx\, dt.$$

Because $n \in L^1([0,T] \times \mathbb{R}^+)$ and because of the uniform bounds on d_p, B_p and the convergence a.e., we deduce immediately that

$$R_p \underset{p \to \infty}{\longrightarrow} 0.$$

Therefore, we have recovered the idendity $\displaystyle\int_0^T \int_0^\infty n(t,x) S(t,x) dx\ dt = 0$, for all functions $S \in C^1_{\text{comp}}$, and this implies again $n \equiv 0$.

This concludes the uniqueness result stated in the Theorem 2.2. $\square$

2.3 A semi-discrete approximation

For proving the existence part of the Theorem 2.2, we use a semi-discrete approximation and pass to the limit. To do so, we need some notations. We fix $h > 0$ and we set

$$x_i = ih, \qquad x_{i+1/2} = (i + 1/2)h, \qquad i \in \mathbb{N}, \tag{2.9}$$

$$d_{i+1/2}(t) = \frac{1}{h} \int_{x_i}^{x_{i+1}} d(t,x)\, dx, \qquad B_{i+1/2}i = \frac{1}{h} \int_{x_i}^{x_{i+1}} B(t,x)\, dx, \tag{2.10}$$

$$g_i(t) = g(t, x_i), \qquad i \in \mathbb{N} \tag{2.11}$$

and we truncate the indices i by some finite number I, with $x_I = hI \underset{h \to 0}{\longrightarrow} \infty$.

The semi-discrete model is to find a vector function $n \in C^1(\mathbb{R}^+; \mathbb{R}^I)$ solving the differential system, for $0 \leq i \leq I - 1$

$$\begin{cases} h \frac{d}{dt} n_{i+1/2}(t) + g_{i+1}(t) n_{i+1}(t) - g_i(t) n_i(t) + h d_{i+1/2}(t) n_{i+1/2}(t) = 0, \\[2ex] g_0(t)\ n_0(t) = h \displaystyle\sum_{0 \leq i \leq I-1} B_{i+1/2}(t)\ n_{i+1/2}(t), \\[2ex] n_{i+1/2}(0) = n^0_{i+1/2} = \frac{1}{h} \displaystyle\int_{x_i}^{x_{i+1}} n^0(x)\, dx. \end{cases} \tag{2.12}$$

We use here a standard upwind scheme for values $n_i(t)$ and $i \geq 1$,

$$n_i(t) = \begin{cases} n_{i-1/2}(t) \ \text{ for } \ g_i(t) > 0, \\[2ex] n_{i+1/2}(t) \ \text{ for } \ g_i(t) < 0. \end{cases} \tag{2.13}$$

The boundary points are special:

• for $i = 0$, $g_0(t) > 0$ by assumption (2.3), and we need a value $n_0(t)$ which has been defined by the second equation of (2.12),

• for $i = I$ and $g_I(t) < 0$ we define $n_I(t) = 0$.

Theorem 2.3 *Assume (2.2)–(2.4), and set* $\bar{B} = \sup_{t\geq 0,\, x\geq 0} B(t,x)$. *Then we have the following estimates. For* $t \geq 0$,

$$\sum_{0\leq i\leq I-1} h|n_{i+1/2}(t)| \leq \sum_{0\leq i\leq I-1} h|n^0_{i+1/2}|e^{\|(B-d)_+\|_\infty t} \leq M(t), \tag{2.14}$$

with $M(t) = \|n^0\|_1 e^{\|(B-d)_+\|_\infty t}$. *Also, for* $h\|g'\|_\infty < g_m$ *and* $0 \leq t \leq T$,

$$\sum_{0\leq i\leq I-1} |n_{i+1/2}(t)| \leq \max\big(\max_{0\leq i\leq I-1} |n^0_{i+1/2}|, \frac{\bar{B}}{g_m} M(T)\big) e^{2\|g'\|_\infty t}. \tag{2.15}$$

For an initial data such that $\int x\, n^0(x)dx < \infty$, *then*

$$\frac{d}{dt}\max_{0\leq i\leq I-1} x_{i+1/2}|n_{i+1/2}(t)| \leq 2M(T)\|g\|_\infty. \tag{2.16}$$

And, for two different initial data n^0 *and* $\tilde{n}^0$, *we have (contraction property),*

$$\sum_{0\leq i\leq I-1} |n_{i+1/2}(t) - \tilde{n}_{i+1/2}(t)| \leq \sum_{0\leq i\leq I-1} |n^0_{i+1/2} - \tilde{n}^0_{i+1/2}|e^{\|(B-d)_+\|_\infty t}. \tag{2.17}$$

Proof. For the *integrability property* (2.14), we multiply the first equation of (2.12) by $\mathrm{sgn}(n_{i+1/2}(t))$ and obtain

$$h\frac{d}{dt}|n_{i+1/2}(t)| + g_{i+1}(t)|n_{i+1}(t)| - g_i(t)|n_i(t)| + hd_{i+1/2}(t)|n_{i+1/2}(t)| \leq 0.$$

Indeed, either $g_{i+1}(t) > 0$ and $n_{i+1}(t) = n_{i+1/2}(t)$ therefore $|n_{i+1}(t)| = n_{i+1}(t)\mathrm{sgn}(n_{i+1/2}(t))$, or $g_{i+1}(t) < 0$ and the inequality is obvious. The same argument holds for the term with $g_i(t)$.

We sum up on i this inequalities and use $n_0(t)$ from (2.12), we find

$$h\frac{d}{dt}\sum_{i=0}^{I-1}|n_{i+1/2}(t)| + g_I(t)|n_I(t)| - g_0(t)|n_0(t)| + h\sum_{i=0}^{I-1} d_{i+1/2}(t)|n_{i+1/2}(t)| \leq 0.$$

Because $n_I(t) = 0$ for $g_I(t) < 0$, we deduce

$$\begin{aligned} h\tfrac{d}{dt}\sum_{i=0}^{I-1}|n_{i+1/2}(t)| + h\sum_{i=0}^{I-1} d_{i+1/2}(t)|n_{i+1/2}(t)| &\leq g_0(t)|n_0(t)| \\ &= h|\sum_{i=0}^{I-1} B_{i+1/2}(t)n_{i+1/2}(t)| \\ &\leq h\sum_{i=0}^{I-1} B_{i+1/2}(t)|n_{i+1/2}(t)|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} h\tfrac{d}{dt}\sum_{i=0}^{I-1}|n_{i+1/2}(t)| &\leq h\sum_{i=0}^{I-1}[B_{i+1/2}(t) - d_{i+1/2}(t)]|n_{i+1/2}(t)| \\ &\leq h\|(B-d)_+\|_\infty \sum_{i=0}^{I-1}|n_{i+1/2}(t)| \end{aligned}$$

and (2.14) follows.

For the L^∞ *property* (2.15), we use the same computation as before for $\frac{d}{dt}|n_{i+1/2}(t)|$ Then, we begin with estimating $|n_{1/2}(t)|$ writing

$$\frac{d}{dt}|n_{1/2}(t)| + \frac{g_1(t)}{h}|n_{1/2}(t)| \leq \frac{g_0(t)}{h}|n_0(t)|,$$

and thus, setting $G_i(t) = \int_0^t g_i(s)ds$ (notice that $g_1(t) > 0$ by the smallness assumption in h), we have

$$\begin{aligned} |n_{1/2}(t)| &\leq e^{-G_1(t)/h}|n^0_{1/2}| + e^{-G_1(t)/h}\int_0^t \tfrac{g_0(s)}{h}|n_0(s)|e^{G_1(s)/h}ds \\ &\leq e^{-G_1(t)/h}|n^0_{1/2}| + e^{-G_1(t)/h}\sup_{0\leq s\leq t}[|n_0(s)|\frac{g_0(t)}{g_1(t)}]\int_0^t \frac{g_1(s)}{h}e^{G_1(s)/h}ds \\ &\leq e^{-G_1(t)/h}|n^0_{1/2}| + \sup_{0\leq s\leq t}[|n_0(s)|\frac{g_0(t)}{g_1(t)}][1 - e^{-G_1(t)/h}] \\ &\leq \max\left(|n^0_{1/2}|, \sup_{0\leq s\leq t}[|n_0(s)|\frac{g_0(t)}{g_1(t)}]\right) \\ &\leq \max\left(|n^0_{1/2}|, \tfrac{\bar{B}}{g_m}M(T)(1 + h\|g'\|_\infty)\right). \end{aligned}$$

Next we set $S(t) = \max_{0\le i\le I-1} |n_{1/2}(t)|e^{-Gt}$, with $G = 2\|g'\|_\infty$. At a given time t, this maximum is attained for some index j and when $j \ge 1$, we can write

$$\begin{aligned} \tfrac{d}{dt}\left[|n_{j+1/2}(t)|e^{-Gt}\right] &\le \tfrac{e^{-Gt}}{h}[g_j(t)|n_j(t)| - g_{j+1}(t)|n_{j+1}(t)|] - |n_{j+1/2}(t)|Ge^{-Gt} \\ &\le 0. \end{aligned} \tag{2.18}$$

To see this last statement, one argues depending on the signs of $g_j(t)$ and $g_{j+1}(t)$. For instance when $g_{j+1}(t) \ge 0$, we have

$$\begin{aligned} \tfrac{e^{-Gt}}{h}[g_j(t)|n_j(t)| - g_{j+1}(t)|n_{j+1}(t)|] &\le \tfrac{e^{-Gt}}{h}[g_j - g_{j+1}(t)]|n_j(t)| \\ &\le e^{-Gt}\|g'\|_\infty |n_{j+1/2}(t)|, \end{aligned}$$

and the above inequality follows. We leave to the reader the case $g_j(t) \le 0$ that is similar. The remaining case is $g_j(t) \ge 0$, $g_{j+1}(t) \le 0$, then we write (because $g(t,x)$ vanishes in between)

$$\begin{aligned} \tfrac{e^{-Gt}}{h}[g_j(t)|n_j(t)| - g_{j+1}(t)|n_{j+1}(t)|] &\le 2\tfrac{e^{-Gt}}{h}\|g'\|_\infty |n_{j+1/2}(t)| \\ &\le e^{-Gt}\|g'\|_\infty |n_{j+1/2}(t)|, \end{aligned}$$

and the inequality again follows.

From the inequality (2.18), we conclude (this is standard) that $S(t)$ also decreases as long as the maximum is not attained for the index $i = 0$, in which case $S(t) > \max\left(|n^0_{1/2}|, \frac{\bar B}{g_m}M(T)(1 + h\|g'\|_\infty)\right)$. In other words

$$S(t) \le \max\left(S(0), |n^0_{1/2}|, \frac{\bar B}{g_m}M(T)(1 + h\|g'\|_\infty)\right),$$

or simply

$$S(t) \le \max\left(\max_{0\le i\le I-1} |n^0_{i+1/2}|, \frac{\bar B}{g_m}M(T)(1 + h\|g'\|_\infty)\right),$$

which is exactly the statement (2.15).

For the *x-moment property* (2.16), we still use the same computation as before for $\frac{d}{dt}|n_{i+1/2}(t)|$ and obtain

$$\begin{aligned}
\tfrac{d}{dt}\sum_{0\le i\le I-1} hx_{j+1/2}(t)|n_{j+1/2}(t)| &\le \sum_{0\le i\le I-1} x_{j+1/2}(t)[g_i(t)|n_i(t)| - g_{i+1}(t)|n_{i+1}(t)|] \\
&\le \sum_{0\le i\le I-1} hg_i(t)|n_i(t)| \\
&\le 2\|g\|_\infty \sum_{0\le i\le I-1} h|n_{i+1/2}(t)| \\
&\le 2\|g\|_\infty M(t),
\end{aligned}$$

and the result is proved.

The *contraction statement* (2.17) follows by substracting the equation (2.12) for two different initial data and applying the L^1 estimate (2.14). □

2.4 Limit as $h \to 0$

We first need to introduce notations that allow a better understanding, in continuous terms, of the previous inequalities. With the previous notations, we set

$$n_h^0(x) = \sum_{i=0}^{I-1} n^0_{i+1/2}\mathbb{1}_{\{x_i<x<x_{i+1}\}},$$

$$n_h(t,x) = \sum_{i=0}^{I-1} n_{i+1/2}(t)\mathbb{1}_{\{x_{i-1}<x<x_i\}},$$

$$b_h(t) = \sum_{i=0}^{I-1} B_i n_{i+1/2}(t)h = \int_0^{x_I} B_h(x)n_h(t,x)dx, \quad \text{for } \; t^k \le t < t^{k+1},$$

with

$$B_h(t,x) = \sum_{i=0}^{I-1} B_{i+1/2}(t)\mathbb{1}_{\{x_{i-1}<x<x_i\}}, \qquad d_h(t,x) = \sum_{i=0}^{I-1} d_{i+1/2}(t)\mathbb{1}_{\{x_{i-1}<x<x_i\}}.$$

We extend these functions by 0 to the half line $x \ge 0$.

We deduce from (2.10) (and we recall that $hI \to \infty$ as $h \to 0$) the strong convergence results

$$d_h(t,x) \underset{h\to 0}{\longrightarrow} d(x) \quad \text{a.e. and} \quad 0 \le d_h \le \|d\|_\infty, \tag{2.19}$$

$$B_h(t,x) \underset{h\to 0}{\longrightarrow} B(x) \quad \text{a.e. and} \quad 0 \le B \le \bar{B}, \tag{2.20}$$

$$n_h^0(x) \xrightarrow[h\to 0]{} n^0(x) \quad \text{a.e. and in } L^1(\mathbb{R}^+), \quad \|n_h^0\|_\infty \le \|n^0\|_\infty. \tag{2.21}$$

Also, from the estimates in Theorem 2.3, we deduce that

Corollary 2.1 *Assume (2.2)–(2.4) and $xn^0 \in L^1(\mathbb{R}^+)$. Then, for a.e. $t \ge 0$, the function $n_h(t,x) \ge 0$ satisfies*

$$\int_0^\infty |n_h(t,x)|dx \le M(t),$$

for $h\|g'\|_\infty < g_m$ and $0 \le t \le T$,

$$\|n_h(t)\|_\infty \le \max\big(\|n^0\|_\infty, \frac{\bar{B}}{g_m}M(T)\big)e^{2\|g'\|_\infty t},$$

$$\frac{d}{dt}\int_0^\infty x|n_h(t,x)|dx \le 2\|g\|_\infty M(t),$$

and, for two different initial data n^0 and $\tilde{n}^0$,

$$\int_0^\infty |n_h(t,x) - \tilde{n}_h(t,x)|dx \le \int_0^\infty |n_h^0(x) - \tilde{n}_h^0(x)|dx \; e^{\|(B-d)_+\|_\infty t}.$$

As a consequence, there are functions

$$n \in L^\infty\big(0,T; L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\big), \qquad b \in L^\infty\big(0,T\big),$$

and a sequence $h(k) \longrightarrow 0$ as $k \to \infty$, so that the functions $n_k(t,x) := n_{h(k)}(t,x)$ and $b_k(t) := b_{h(k)}(t)$ satisfies, for all $T > 0$,

$$n_k \to n \quad L^\infty\big(0,T; L^\infty(\mathbb{R}^+)\big)\text{weak-*},$$

$$b_k \to b \quad L^\infty\big(0,T; L^\infty(\mathbb{R}^+)\big)\text{weak-*},$$

We are now ready to stae our main result

Theorem 2.4 *Assume (2.2)–(2.4) and $xn^0 \in L^1(\mathbb{R}^+)$. Then, for a.e. $t \ge 0$, we have*

$$b(t) = \int_0^\infty B(t,x)n(t,x)dx.$$

and n is weak solution to (2.1) in the sense of Definition (2.1). Moreover, the full family n_h converges weakly (as above) to n.

Proof. *We first prove the equality.* We decompose the integrals as

$$\int_0^T \int_0^\infty \psi(t)\beta(t,x)n_k(t,x)dt\ dx \to \int_0^\infty \int_0^\infty \psi(t)\beta(x)n_k(t,x)dt\ dx,$$

by the definition of weak-* convergence, for all $T > 0$ and all $\psi \in L^\infty(0,T)$, $\beta \in L^1((0,T)\times \mathbb{R}^+)$. Therefore we can write, for $\beta_A(t,x) = B(t,x)\mathbb{1}_{\{x\le A\}}$

$$\int_0^T \psi(t)b_k(t)dt = \int_0^T \int_0^\infty \psi(t)\beta_A(t,x)n_k(t,x)dxdt + O(\frac{1}{A}),$$

because we can use (2.16) to estimate

$$\int_0^T \int_A^\infty |\psi(t)|B(t,x)|n_k(t,x)|dxdt \le \frac{1}{A}\bar{B}\int_0^T \psi(t)\int_0^\infty xn_k(t,x)dxdt = O(\frac{1}{A}).$$

Therefore in the limit $k \to \infty$

$$\int_0^T \psi(t)b(t)dt = \int_0^T \int_0^\infty \psi(t)\beta_A(t,x)n(t,x)dxdt + O(\frac{1}{A}),$$

and in the limit $A \to \infty$ (recall $n \in L^\infty\big(0,T;L^1(\mathbb{R}^+)\big)$,

$$\int_0^T \psi(t)b(t)dt = \int_0^T \int_0^\infty \psi(t)B(t,x)n(t,x)dxdt.$$

Then, we prove that n *is a weak solution.* We use test functions $\psi \in C^1\big([0,T]\times[0,\infty[\big)$, such that for some $T > 0$ and $R > 0$, $\psi(T,x) = 0$ and $\psi(t,x) = 0$ for $x > R$. We define,

$$\psi_{i+1/2}(t) = \frac{1}{h}\int_{x_i}^{x_{i+1}} \psi(t,x)dx\ dt. \tag{2.22}$$

Then, we successively rewrite (2.12) after testing against $\psi_i(t)$ as:

$$\begin{aligned}
\int_0^T \sum_{i=0}^{I-1} \big[h\frac{dn_{i+1/2}(t)}{dt}\ \psi_{i+1/2}(t) \quad &+[g_{i+1}(t)n_{i+1}(t) - g_i(t)n_i(t)]\psi_{i+1/2}(t)\\
&+d_{i+1/2}(t)n_{i+1/2}(t)\psi_{i+1/2}(t)\big]dt = 0,\\
-\int_0^T \sum_{i=0}^{I-1} \big[h\frac{d\psi_{i+1/2}(t)}{dt}\ n_{i+1/2}(t) \quad &+g_{i+1}(t)n_{i+1}(t)[\psi_{i+1/2}(t) - \psi_{i-1/2}(t)]\\
&-d_{i+1/2}(t)n_{i+1/2}(t)\psi_{i+1/2}(t)\big]dt\\
&= \sum_{i=0}^{I-1} n^0_{i+1/2}\psi_{i+1/2}(t=0) + g_0(t)n_0(t),
\end{aligned}$$

$$-\int_0^T \int_0^\infty \quad n_h(t,x)\tfrac{d}{dt}\psi(t,x) - d_h(t,x)n_h(t,x)\psi(t,x)dxdt$$

$$-\int_0^T \sum_{i=0}^{I-1} g_{i+1}(t)n_{i+1}(t)[\psi_{i+1/2}(t) - \psi_{i-1/2}(t)]$$

$$= \int_0^\infty n_h^0(x)\psi(t=0,x)dx + \int_0^\infty b_h(t) \int_0^h \psi(t,y)\frac{dy}{h}\; dt.$$

We can now pass to the weak-strong limit in all terms of this equality for the subsequence n_k, with the convergences we have derived before. The only term which is not obvious is the flux (second line). To treat it, we fix $\varepsilon > 0$ (the strategy is to let $h(k)$ vanish before ε) and consider the sets

$$A_\varepsilon = \{(t,x) \text{ s.t. } |g(t,x)| \leq \varepsilon\},$$

$$B_\varepsilon = \{(t,x) \text{ s.t. } \; g(t,x) > \; \varepsilon \; \},$$

$$C_\varepsilon = \{(t,x) \text{ s.t. } g(t,x) < -\varepsilon\}.$$

Because ψ is Lipschitz continuous and has compact support, on the indices corresponding to the set A_ε the flux term is of order $0(\varepsilon)$. For those corresponding to B_ε, we obtain with an obvious abuse of notations,

$$\int_0^T \sum_{i\in B_\varepsilon} \quad g_{i+1}(t)n_{i+1}(t)[\psi_{i+1/2}(t) - \psi_{i-1/2}(t)]$$

$$= \int_0^T \int_{B_\varepsilon} g(t,x)n_h(t,x)\psi'(t,x)dxdt + O(h)$$

$$\to \int_0^T \int_{B_\varepsilon} g(t,x)n(t,x)\psi'(t,x)dxdt.$$

Arguing in the same way on C_ε, and letting ε go to zero we obtain the weak formulation.

Finally, we prove that the full family converges . Because of the uniqueness result in Section 2.2, any subsequence extracted from n_h converges to the same limit. Therefore the full family converges. □

Chapter 3

Generalized relative entropy

3.1 Generalized relative entropy: finite dimension

We begin with describing the General Entropy Inequality in the case of matrices and we deal with two theories where it applies to give an entropy based understanding of time relaxation. In the framework of Perron-Frobenius eigenvalue theorem it explains why the associated dynamic converges to the first (positive) eigenvector (once correctly normalized). In the framework of Floquet's eigenvalue theorem it explains why the associated dynamic converges to the (positive) periodic solution (once correctly normalized).

3.1.1 The Perron-Frobenius theorem

Let $a_{ij} > 0$, $1 \leq i, j \leq d$, be the coefficients of a matrix $A \in M_{d\times d}(\mathbb{R})$ (there are interesting issues with the case $a_{ij} \geq 0$ but we try to keep simplicity here). The Perron-Frobenius theorem (see [15] for instance) tells us that A has a first eigenvalue $\lambda > 0$ associated with a positive right eigenvector $N \in \mathbb{R}^d$, and a positive left eigenvector $\phi \in \mathbb{R}^d$

$$\begin{cases} A.N = \lambda N, & N_i > 0 \quad \text{for} \quad i = 1, \dots, d, \\[2ex] \phi.A = \lambda \phi, & \phi_i > 0 \quad \text{for} \quad i = 1, \dots, d. \end{cases}$$

For later purposes, it is convenient to normalize these vectors, so that they are now uniquely defined. We choose

$$\sum_{i=1}^{d} N_i = 1, \qquad \sum_{i=1}^{d} N_i \; \phi_i = 1.$$

We set $\widetilde{A} = A - \lambda Id$ and consider the evolution equation

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^0. \tag{3.1}$$

The solutions to this system converge as $t \to \infty$ with an exponential rate. Indeed, the following result is classical

Proposition 3.1 *For positive matrices A and solutions to the differential system (3.1), we have,*

$$\rho := \sum_{i=1}^{d} \phi_i n_i(t) = \sum_{i=1}^{d} \phi_i n_i^0, \tag{3.2}$$

$$\sum_{i=1}^{d} \phi_i |n_i(t)| \le \sum_{i=1}^{d} \phi_i |n_i^0|, \tag{3.3}$$

$$\underline{C} N_i \le n_i(t) \le \overline{C} N_i \quad \textit{with constants given by} \quad \underline{C} N_i \le n_i^0 \le \overline{C} N_i, \tag{3.4}$$

and there is a constant $\alpha > 0$ such that, with ρ given in (3.2), we have

$$\sum_{i=1}^{d} \phi_i N_i \big(\frac{n_i(t) - \rho N_i}{N_i}\big)^2 \le \sum_{i=1}^{d} \phi_i N_i \big(\frac{n_i^0 - \rho N_i}{N_i}\big)^2 e^{-\alpha t}. \tag{3.5}$$

Here, we wish to justify it with an entropy inequality.

Proposition 3.2 *Let $H(\cdot)$ be a convex function on $\mathbb{R}$, then the solution to (3.1) satisfies*

$$\frac{d}{dt} \quad \sum_{i=1}^{d} \phi_i N_i H\big(\frac{n_i(t)}{N_i}\big)$$

$$= \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\big(\frac{n_i(t)}{N_i}\big)[\frac{n_j(t)}{N_j} - \frac{n_i(t)}{N_i}] - H\big(\frac{n_j(t)}{N_j}\big) + H\big(\frac{n_i(t)}{N_i}\big) \right]$$

$$\le 0.$$

Definition 3.1 *We call General Relative Entropy, the quantity* $\displaystyle\sum_{i=1}^{d} \phi_i N_i H\big(\frac{n_i(t)}{N_i}\big)$.

Proof of Proposition 3.2. We denote by $\widetilde{a_{ij}}$ the coefficients of the matrix $\widetilde{A}$ and compute

$$\begin{aligned} \tfrac{d}{dt} \textstyle\sum_i \phi_i N_i H\big(\tfrac{n_i(t)}{N_i}\big) &= \textstyle\sum_{i,j} \phi_i H'\big(\tfrac{n_i(t)}{N_i}\big) \widetilde{a_{ij}} n_j(t) \\ &= \textstyle\sum_{i,j} \phi_i \widetilde{a_{ij}} N_j H'\big(\tfrac{n_i(t)}{N_i}\big)\big[\tfrac{n_j(t)}{N_j} - \tfrac{n_i(t)}{N_i}\big], \end{aligned}$$

because the additional $\frac{n_i(t)}{N_i}$ term vanishes since $\widetilde{A}.N = 0$. But we also have, again thanks to the equation on N and ϕ, that

$$\sum_{i,j} \phi_i \widetilde{a_{ij}} N_j \big[H\big(\frac{n_j(t)}{N_j}\big) - H\big(\frac{n_i(t)}{N_i}\big)\big] = 0.$$

Combining these two identities, we arrive to the equality in Proposition 3.2. The inequality follows because only the coefficients out of the diagonal, that satisfy $\widetilde{a_{ij}} = a_{ij} \geq 0$, enters here. □

Proof of Proposition 3.1. Notice that, as a special case of H in Proposition 3.2, we can choose $H(u) = u$, which being convex together with $-H$ gives the equality

$$\frac{d}{dt} \sum_{i=1}^{d} \phi_i n_i(t) = 0.$$

And (3.2) follows. In particular this identifies the value ρ mentioned in (3.2).

The second statement (3.3) follows immediately by choosing the (convex) entropy function $H(u) = |u|$.

As for the third statement (3.4), let us consider for instance the upper bound. It follows choosing the (convex) entropy function $H(u) = (u - \overline{C})_+^2$ because for this nonnegative function we have

$$\sum_{i=1}^{d} \phi_i N_i H\big(\frac{n_i^0}{N_i}\big) = 0.$$

Therefore, because the General Relative Entropy decays, it remains zero for all times,

$$\sum_{i=1}^{d} \phi_i N_i H\big(\frac{n_i(t)}{N_i}\big) = 0,$$

which proves the result.

It remains to prove the exponential time decay statement (3.5). To do that, we work on

$$h(t,x) = n(t,x) - \rho N,$$

which verifies $\int \phi[n(t,x) - \rho N]dx = 0$ and satisfies the same equation as n. Then, we use the quadratic entropy function $H(u) = u^2$ and the General Entropy Inequality gives

$$\frac{d}{dt} \sum_{i=1}^{d} \phi_i N_i \left(\frac{h_i(t)}{N_i}\right)^2 = - \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{h_j(t)}{N_j} - \frac{h_i(t)}{N_i}\right)^2 \leq 0.$$

Then, we need a discrete Poincaré inequality

Lemma 3.1 *Being given $\phi_i > 0$, $N_i > 0$, $a_{ij} > 0$ for $i = 1,\dots,d$, $j = 1,\dots,d$, $i \neq j$, there is a constant $\alpha > 0$ such that for all vector m of components m_i, $1 \le i \le d$ satisfying*

$$\sum_{i=1}^{d} \phi_i m_i = 0,$$

we have (Poincaré inequality)

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2 \ge \alpha \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i}{N_i}\right)^2.$$

With this lemma, we conclude

$$\frac{d}{dt}\sum_{i=1}^{d} \phi_i N_i \left(\frac{h_i(t)}{N_i}\right)^2 \le -\alpha \sum_{i=1}^{d} N_i \left(\frac{h_i(t)}{N_i}\right)^2,$$

and then, (3.5) follows by a simple use of the Gronwall lemma. □

Proof of Lemma 3.1. After renormalizing the vector m (when it does not vanish, otherwise the result is obvious), we may suppose that

$$\sum_{i=1}^{d} \phi_i m_i = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i}{N_i}\right)^2 = 1.$$

Then we argue by contradiction. If such a α does not exist, this means that we can find a sequence of vectors $(m^k)_{(k\ge1)}$ such that

$$\sum_{i=1}^{d} \phi_i m_i^k = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i^k}{N_i}\right)^2 = 1, \qquad \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j^k}{N_j} - \frac{m_i^k}{N_i}\right)^2 \le 1/k.$$

After extraction of a subsequence, we may pass to the limit $m^k \to \bar{m}$ and this vector satisfies

$$\sum_{i=1}^{d} \phi_i \bar{m}_i = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{\bar{m}_i}{N_i}\right)^2 = 1,$$

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{\bar{m}_j}{N_j} - \frac{\bar{m}_i}{N_i}\right)^2 = 0.$$

Therefore, from this last relation, for all i and $j = 1,\dots,d$, we have

$$\frac{\bar{m}_i}{N_i} = \frac{\bar{m}_j}{N_j} := \nu.$$

By the zero sum condition, we have $\nu = 0$ because

$$\nu \sum_{i=1}^{d} \phi_i = 0.$$

In other words, $\bar{m} = 0$ which contradicts the normalization and thus such a α should exist. $\square$

Remark 3.1 *1. The matrix with (positive) coefficients $b_{ij} = \phi_i \; a_{ij} \; N_j$ is doubly stochastic, i.e., the sum of the lines and columns is 1 (see for instance[15]).*
2. Notice that $a_{ii} - \lambda < 0$ because $\sum_j \widetilde{a_{ij}} N_j = 0$. Therefore the matrix C with coefficients $c_{ij} = \frac{1}{N_i}\widetilde{a_{ij}} \; N_j$ is that of a Markov process. In other words, we set $y_i = x_i/N_i$, then it satisfies

$$\frac{d}{dt} y_i(t) = c_{ij} y_j(t),$$

and the vector $(1, 1, \ldots, 1)$ is the (positive) eigenvector associated to the eigenvalue 0 of the matrix C, i.e., $c_{ii} = \sum_{j \neq i} c_{ij}$ and $c_{ij} \geq 0$. Then, $(N_i \phi_i)_{(i=1,\ldots,d)}$ is the invariant measure of the Markov process. In particular this explains the entropy property which is classical for Markov processes ([17]).

3.1.2 The Floquet theory

We now consider T-periodic coefficients $a_{ij}(t) > 0$, $1 \leq i, j \leq d$, i.e., $a_{ij}(t+T) = a_{ij}(t)$. And we denote by $A(t) \in M_{d \times d}$ the corresponding matrix. Again our motivation comes from several questions in biology where such structures arise as seasonal rhythm, circadian rhythm, see [5] for instance.

The Floquet theorem tells us that there is a first 'Floquet eigenvalue' $\lambda_{per} > 0$ and two positive T-periodic functions $N(t) \in \mathbb{R}^d$, $\phi(t) \in \mathbb{R}^d$ that are periodic solutions (uniquely defined up to multiplication by a constant) to the differential systems

$$\frac{d}{dt} N(t) = [A(t) - \lambda_{per} Id].N(t), \tag{3.6}$$

$$\frac{d}{dt} \phi(t) = \phi(t).[A(t) - \lambda_{per} Id]. \tag{3.7}$$

Up to a normalization, these elements $(\lambda_{per} > 0, N(t) > 0, \phi(t))$ are unique and we normalize again as

$$\int_0^T \sum_{i=1}^{d} N_i(t) dt = 1, \qquad \int_0^T \sum_{i=1}^{d} \phi_i(t) N_i(t) dt = 1.$$

We recall that this case of Floquet theory (which applies to more general situations than positive matrices) is an application of Perron-Frobenius theorem to the resolvent matrix

$$S(t) = e^{\int_0^t A(s)ds},$$

which has positive coefficients also.

Again, we set $\widetilde{A(t)} = A(t) - \lambda_{per} Id$ and consider the differential system

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^0.$$

In the present context we obtain the following version of Proposition 3.1,

Proposition 3.3 *For positive matrices A we have,*

$$\rho := \sum_{i=1}^{d} \phi_i(t) n_i(t) = \sum_{i=1}^{d} \phi_i(t=0) n_i^0, \tag{3.8}$$

$$\sum_{i=1}^{d} \phi_i(t)|n_i(t)| \leq \sum_{i=1}^{d} \phi_i(t=0)|n_i^0|, \tag{3.9}$$

if for some constants, we have $\underline{C} N_i(t=0) \leq n_i^0 \leq \overline{C} N_i(t=0)$, *then*

$$\underline{C} N_i(t) \leq n_i(t) \leq \overline{C} N_i(t), \tag{3.10}$$

and there is a constant $\alpha > 0$ such that

$$\sum_{i=1}^{d} \phi_i(t) N_i(t) \big(\frac{n_i(t) - \rho N_i(t)}{N_i(t)}\big)^2 \leq \sum_{i=1}^{d} \phi_i^0 N_i^0 \big(\frac{n_i^0 - \rho N_i^0}{N_i^0}\big)^2 e^{-\alpha t}. \tag{3.11}$$

Again, this can be justified thanks to entropy inequalities.

Proposition 3.4 *Let $H(\cdot)$ be a convex function on $\mathbb{R}$, then we have*

$$\begin{aligned}
\tfrac{d}{dt} \quad & \sum_{i=1}^{d} \phi_i(t) N_i(t) H\big(\frac{n_i(t)}{N_i(t)}\big) \\
&= \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\big(\frac{n_i}{N_i}\big)[\frac{n_j}{N_j} - \frac{n_i}{N_i}] - H\big(\frac{n_j}{N_j}\big) + H\big(\frac{n_i}{N_i}\big) \right] \\
&\leq 0.
\end{aligned}$$

These two propositions are variants of the corresponding ones in Perron-Frobenius theorem and we leave the proofs to the reader. Adapting the Lemma 3.1 requires an additional compactness argument based on the Ascoli-Arzela Theorem.

3.2 Generalized relative entropy: parabolic and integral PDE's

We now explain the notion of General Relative Entropy on continuous models. We begin with the most classical equation, namely the parabolic equation for the unknown $n(t,x)$,

$$\frac{\partial n}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial n}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i n\big) + dn = 0, \qquad x \in \mathbb{R}^d, \tag{3.12}$$

where the coefficients depend on t and x, $d \equiv d(t,x)$ (no sign assumed), $b_i \equiv b_i(t,x)$, and the symmetric matrix $A(t,x) = \big(a_{ij}(t,x)\big)_{1\leq i,j \leq d}$ satisfies $A(t,x) \geq 0$. We could possibly set the equation on a domain and assume Dirichlet, zero-flux, mixed or periodic boundary conditions and then include them in the above calculation.

Here, it is not obvious to derive a priori bounds on the solution $n(t,x)$, by opposition to the case $A \geq \nu Id > 0$, $b_i \equiv 0$, $d(x) \geq 0$ where we have, multiplying the equation by $n|n|^{p-2}$ with $p > 1$,

$$\frac{d}{dt}\int \frac{|n(t,x)|^p}{p}dx + \frac{4\nu(p-1)}{p^2}\int |\nabla n^{p/2}|^2 dx \leq 0.$$

Indeed the only remarkable property of (3.12) is the mass conservation and L^1 contraction principle

$$\frac{d}{dt}\int n(t,x)dx + \int d(t,x)n(t,x)dx = 0,$$

$$\frac{d}{dt}\int \big(n(t,x)\big)_+ dx + \int d(t,x)\big(n(t,x)\big)_+ dx \leq 0.$$

On the other hand the conservative Fokker-Planck equation is very standard when $b = -\nabla V$ for some convex potential with enough growth at infinity

$$\frac{\partial n}{\partial t} - \Delta n - \operatorname{div}(\nabla V\, n) = 0.$$

Then, one has $N = e^{-V}$ and the relative entropy $\int n \ln\left(\frac{n}{N}\right) dx$ is a standard object. It decays with time. Of course, here we still have the family $\int NH\left(\frac{n}{N}\right) dx$ of relative entropies. All of these entropies, for all convex functions $H(\cdot)$, decays in time, and not only $H(u) = u\ln(u)$.

3.2.1 Coefficients independent of time

In the case of coefficients independent of time, and depending on the values of $a_{ij}(x)$, $b(x)$ and $d(x)$, the solution can exhibit exponential growth or decay as $t \to \infty$.

Therefore, we will assume that 0 is the first eigenvalue and, following the Krein-Rutman theorem (see [2]), we also assume that we can find two functions $N(x) > 0$, $\phi(x) > 0$, such that

$$\begin{cases} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial N}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(x)N\big) + d(x)N = 0, \\ \\ \qquad\qquad N(x) > 0, \qquad \int N(x)dx = 1, \end{cases} \tag{3.13}$$

$$\begin{cases} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\big) - \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}\phi + d(x)\phi = 0, \\ \\ \qquad\qquad \phi(x) > 0, \qquad \int N(x)\phi(x)dx = 1. \end{cases} \tag{3.14}$$

These are the first eigenvectors; N for the direct problem and ϕ for the dual operator. Notice that such eigenelements do not always exist but there are standard examples, namely when $d \equiv 0$, $A = Id$ and there is a potential V such that $b = -\nabla V$. Then, one can readily check that solutions to (3.13)–(3.14) are

$$N = e^{-V} \qquad \phi \equiv 1,$$

when $V(x) \to \infty$ as $|x| \to \infty$ fast enough in order to fulfill the integrability conditions.

The general relative entropy property of the parabolic equation (3.12) can be expressed as

Lemma 3.2 *For coefficients independent of t, assume that there exist eigenelements N, ϕ satisfying (3.13)–(3.14). Then for all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to (3.12) with sufficient decay in x to zero at infinity ($|n^0| \leq CN$), we have*

$$\frac{d}{dt} \int \phi(x)\; N(x)\; H\big(\tfrac{n(t,x)}{N(x)}\big)dx$$

$$= -\int \phi\; N\; H''\big(\tfrac{n(t,x)}{N(x)}\big)\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\;\frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)\; dx \leq 0.$$

For conservative equations, i.e., $d \equiv 0$, it is usual to take $\phi \equiv 1$, and then the corresponding principle is classical (especially related to stochastic differential equations and Markov processes, [17]).

Proof of Lemma 3.2. We just calculate (leaving the intermediary steps to the reader)

$$\frac{\partial}{\partial t}\big(\frac{n}{N}\big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[a_{ij}\frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)\big] + 2N\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\;\frac{\partial}{\partial x_j}\big(\frac{1}{N}\big) + b\cdot\nabla\big(\frac{n}{N}\big) = 0.$$

Therefore, for any smooth function H, we arrive at

$$\frac{\partial}{\partial t}H\big(\tfrac{n}{N}\big) \quad -\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\big[a_{ij}\frac{\partial}{\partial x_j}H\big(\frac{n}{N}\big)\big]+2N\sum_{i,j=1}^{d}a_{ij}\frac{\partial}{\partial x_i}H\big(\frac{n}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{1}{N}\big)$$

$$+b\cdot\nabla H\big(\tfrac{n}{N}\big)+H''\big(\tfrac{n}{N}\big)\sum_{i,j=1}^{d}a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)=0.$$

At this stage we can 'undo' the calculation that lead from an equation on n to an equation on n/N and we arrive at

$$\frac{\partial}{\partial t}NH\big(\tfrac{n}{N}\big) \quad -\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\big[a_{ij}\frac{\partial}{\partial x_j}NH\big(\frac{n}{N}\big)\big]+NH''\big(\frac{n}{N}\big)\sum_{i,j=1}^{d}a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)$$

$$+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\ \big[b_i NH\big(\frac{n}{N}\big)\big]+dNH\big(\frac{n}{N}\big)=0.$$

Finally, combining it with the equation on ϕ, we deduce that

$$\frac{\partial}{\partial t}\phi NH\big(\tfrac{n}{N}\big) \quad -\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\big[\phi a_{ij}\frac{\partial}{\partial x_j}NH\big(\frac{n}{N}\big)\big]+\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\big[a_{ij}NH\big(\frac{n}{N}\big)\frac{\partial}{\partial x_j}\phi\big]$$

$$+\phi NH''\big(\tfrac{n}{N}\big)\sum_{i,j=1}^{d}a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)=0.$$

After integration in x (because we have assumed sufficient decay in x to zero at infinity), we arrive at the result stated in Lemma 3.2. $\square$

This lemma can be used in the directions indicated in section §3.1 (a priori estimates, long time convergence to a steady state) and we refer to [13, 11, 12] for specific examples.

As far as long time convergence is concerned, we notice that, as in Lemma 3.1, a control of entropy by entropy dissipation is useful for exponential convergence in as $t\to\infty$ as in (3.5). For the quadratic entropy, this follows from the Poincaré inequality

$$\nu\int\phi N\left(\frac{m}{N}\right)^2\leq 2\int\phi N\sum_{i,j=1}^{d}a_{ij}\frac{\partial}{\partial x_i}\big(\frac{m}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{m}{N}\big),\quad\text{when}\ \int\phi m=0.$$

Such inequalities, as well as log-Sobolev inequalities, are classical when $N=e^{-V}$ for a potential $V(x)$ with superlinear growth at infinity ([9]). The change of unknown function to $n\phi$ and $N\phi$ gives the condition $N\phi=e^{-V}$ for $V(x)$ with superlinear growth to ensure the Poincaré inequality. We are not aware of any general condition on d, b and A in this direction.

3.2.2 Time dependent coefficients

In fact the above manipulations are also valid for time dependent coefficients. A situation similar to the Floquet theory and which is therefore useful for periodic coefficients for instance. We now consider solutions to

$$
\begin{cases}
\frac{\partial}{\partial t} N(t,x) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x) \frac{\partial N}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(x) N\big) + d(x) N = 0, \\
\\
\hfill N(t,x) > 0,
\end{cases}
\tag{3.15}
$$

$$
\begin{cases}
\frac{\partial}{\partial t} \phi(t,x) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x) \frac{\partial \phi}{\partial x_j}\big) - \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} \phi + d(x) \phi = 0, \\
\\
\hfill \phi(t,x) > 0.
\end{cases}
\tag{3.16}
$$

Then we have

Lemma 3.3 *For all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to (3.12) with sufficient decay in x to zero at infinity, we have*

$$
\frac{d}{dt} \quad \int \phi(t,x)\ N(t,x)\ H\big(\tfrac{n(t,x)}{N(t,x)}\big) dx
$$

$$
= - \int \phi\ N\ H''\big(\tfrac{n(t,x)}{N(t,x)}\big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{N}\big)\ \frac{\partial}{\partial x_j}\big(\frac{n}{N}\big)\ dx \leq 0.
$$

Again we leave the proof of this variant to the reader.

3.2.3 Scattering equations

To exhibit another class of equation where the General Relative Entropy inequality holds true, let us mention the scattering (also called linear Boltzman) equation

$$
\frac{\partial}{\partial t} n(t,x) + k_T(x)\, n(t,x) = \int_{\mathbb{R}^d} k(x,y)\ n(t,y)\ dy. \tag{3.17}
$$

Here we restrict ourselves to coefficients independent of time for simplicity, but the same inequality holds in the time dependent case as before. We assume that

$$
0 \leq k_T(\cdot) \in L^\infty(\mathbb{R}^d), \qquad 0 \leq k(x,y) \in L^1 \cap L^\infty\big(\mathbb{R}^{2d}\big)\big).
$$

And we do not make special assumption on the symmetry of the cross-section $k(x,y)$.

Again, changing k_T in $k_T + \lambda$ if necessary in order to have a zero first eigenvalue, we assume that there are solutions $N(x)$ and $\phi(x)$ to the stationary equation and its adjoint, namely

$$k_T(x)\, N(x) = \int_{\mathbb{R}^d} k(x,y)\ N(y)\ dy, \qquad N(x) > 0. \tag{3.18}$$

$$k_T(x)\, \phi(x) \ = \int_{\mathbb{R}^d} k(y,x)\ \ \phi(y)\ dy, \qquad \phi(x) > 0. \tag{3.19}$$

These two steady state solutions allow us to derive the General Relative Entropy inequality

Lemma 3.4 *With the above notations, we have*

$$\begin{aligned} &\tfrac{\partial}{\partial t} \ \left[\phi(x)\ N(x)\ H\big(\tfrac{n(x)}{N(x)}\big)\right] \\ &+\int_{\mathbb{R}^d} k(x,y)\big[\phi(y) N(x) H\big(\frac{n(t,x)}{N(x)}\big) - \phi(x) N(y) H\big(\frac{n(t,y)}{N(y)}\big)\big] dy \\ &= \int k(x,y)\phi(x)N(y)\left[H\big(\frac{n(t,x)}{N(x)}\big) - H\big(\frac{n(t,y)}{N(y)}\big)\right. \\ &\qquad\qquad \left. + H'\big(\tfrac{n(t,x)}{N(x)}\big)\big[\tfrac{n(t,y)}{N(y)} - \tfrac{n(t,x)}{N(x)}\big]\right] dy, \end{aligned}$$

and also (after integration in x),

$$\begin{aligned} &\tfrac{d}{dt} \ \int_{\mathbb{R}^d} \big[\phi(x)\ N(x)\ H\big(\frac{n(x)}{N(x)}\big)\big] \\ &= \int k(x,y)\phi(x)N(y)\left[H\big(\frac{n(t,x)}{N(x)}\big) - H\big(\frac{n(t,y)}{N(y)}\big)\right. \\ &\qquad\qquad \left. + H'\big(\tfrac{n(t,x)}{N(x)}\big)\big[\tfrac{n(t,y)}{N(y)} - \tfrac{n(t,x)}{N(x)}\big]\right] dy \\ &\le 0. \end{aligned}$$

Again we leave to the reader the easy computation that leads to this result and just indicate a class of classical examples where N and ϕ can be computed explicitly.

Example 1. We consider the case where the cross-section in the scattering equation is given by

$$k(x,y) = k_1(x) k_2(y).$$

Then we arrive at (up to a multiplicative constant)

$$N(x) = \frac{k_1(x)}{k_T(x)}, \qquad \phi(x) = \frac{k_2(x)}{k_T(x)},$$

and we need the compatibility condition

$$\int_{\mathbb{R}^d} \frac{k_2(x)k_1(x)}{k_T(x)^2} dx = 1.$$

As in the case of Perron-Frobenius thorem in section §3.1.1, this means that 0 is the first eigenvalue, a condition that can always be met changing if necessary k_T in $\lambda + k_T$.

Example 2. We consider the more general case where there exists a function $N(x) > 0$ such that the scattering cross-section satisfies the symmetry condition (usually called detailed balance or microreversibility)

$$k(y,x)N(x) = k(x,y)N(y).$$

Then the choice $k_T(y) = \int_{\mathbb{R}^d} k(x,y)dx$ gives the solutions $N(x)$ to (3.18), and $\phi(x) = 1$ to equation (3.19).

Again we conclude on long time convergence and the possibility to prove exponential time decay to the steady state. As in Lemma 3.1, this follows from a control of entropy by entropy dissipation and thus for the quadratic entropy, from the Poincaré inequality

$$\nu \int \phi(x)N(x)\big(\frac{h}{N}\big)^2 dx \leq \int_{\mathbb{R}^d} k(y,x)\phi(x)N(y) \left[\frac{h(x)}{N(x)} - \frac{h(y)}{N(y)}\right]^2 dy\, dx,$$

whenever

$$\int_{\mathbb{R}^d} \phi(x)h(x)dx = 0.$$

This is not always true but holds whenever there is a function $\psi > 0$ such that

$$\nu_1 = \int N\phi^2/\psi < \infty, \qquad \nu_2\psi(y)N(x) \leq k(x,y), \qquad \nu = (\nu_1\nu_2)^{-1},$$

a condition that is fulfilled for instance if we work on a bounded domain in velocity and k positive (the difficulties in practical examples as cell division equation is that ϕ needs not be bounded in unbounded domains and N can vanish at infinity).

We write, for any function $\psi > 0$, and $\nu_1 = \int N/\psi$,

$$\begin{aligned} \int \phi(x)N(x)\big(\tfrac{h}{N}(x)\big)^2 dx &= \int \phi(x)N(x) \left(\int [\tfrac{h}{N}(x) - \tfrac{h}{N}(y)]\phi(y)N(y)dy\right)^2 dx \\ &\leq \nu_1 \int\int \phi(x)N(x)[\tfrac{h}{N}(x) - \tfrac{h}{N}(y)]^2\psi(y)N(y)dydx \\ &\leq \nu_1\nu_2 \int\int \phi(x)k(x,y)N(y)[\tfrac{h}{N}(x) - \tfrac{h}{N}(y)]^2 dydx \end{aligned}$$

Notice that a large class of the examples above enter this condition but not with the choice $\psi = \phi$.

Chapter 4

The renewal equation: entropy properties

We are still interested in the renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x)\, n(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\ \\ n(t, x=0) = \int B(y) n(t,y) dy, \\ \\ n(t=0,x) = n^0(x). \end{cases} \tag{4.1}$$

Our goal is now to go further than Theorem 2.2 and prove that the renewal problem is well posed in the weak sense, and comes with an entropy structure. To do so we assume that

$$0 \leq d \in L^\infty_{\text{loc}}(\mathbb{R}^+), \quad 0 \leq B \in L^\infty(\mathbb{R}^+), \qquad 0 \leq n^0 \in L^1 \cap L^\infty(\mathbb{R}^+). \tag{4.2}$$

We (still) use the notations

$$\bar{B} = \sup_{x \in \mathbb{R}^+} B(x), D(x) = \int_0^x d(y)\, dy.$$

In preparation, and motivated by the Perron-Froebenius theory, we study the eigenelements and adjoint problem associated to the equation .

4.1 Eigenelements (direct)

The *eigenproblem* associated with the renewal equation consists in finding a solution $(\lambda_0, N(x))$ to

$$\begin{cases} \frac{\partial}{\partial x} N(x) + \big(\lambda_0 + d(x)\big)\, N(x) = 0, \quad x \geq 0, \\ \\ N(x=0) = \int B(y) N(y) dy, \qquad N(x) > 0, \quad . \end{cases} \tag{4.3}$$

We normalize it with two conditions

$$N(0) = \int_0^\infty B(x)N(x)dx = 1, \qquad \int_0^\infty xB(x)N(x)dx < \infty. \tag{4.4}$$

The normalization $\int_0^\infty BN = 1$ is always possible and is used here for uniqueness (because an eigenvector is defined up to multiplication by a real number). The other condition, as we see it later is necessary for the adjoint problem.

This eigenproblem is easy to solve. Direct integration of the equation on N gives $N(x) = e^{-D(x)-\lambda_0 x}$. Once inserted in the boundary condition at $x = 0$, we find

$$1 = \int_0^\infty B(y)e^{-D(y)-\lambda_0 y}\, dy. \tag{4.5}$$

Since the function

$$\lambda \mapsto \int_0^\infty B(y)e^{-D(y)-\lambda y}\, dy$$

is decreasing, we see that there is a unique possible λ_0. However existence is not obvious and depends on B and d. Therefore we introduce the notations and assumptions

$$\begin{cases} \exists \lambda_0, \quad 1 = \int_0^\infty B(y)N(y)\, dy, \quad N(x) = e^{-D(x)-\lambda_0 x}, \\ \qquad\qquad \exists \bar\lambda < \lambda_0, \quad \int_0^\infty B(y)e^{-D(y)-\bar\lambda y}\, dy < \infty. \end{cases} \tag{4.6}$$

Here are three explicit examples where this is obviously possible, still with assumptions (4.2),

Example 1. Demography of type 1

$$B(x) = 0 \text{ for } x \geq x_\sharp, \qquad \text{and } \lambda_0 \text{ has no sign}, \tag{4.7}$$

The difficulty here is that N is not intregrable, which is counter-intuitive in terms of applications.

Example 2. Demography of type 2

$$1 < \int_0^{x_\sharp} e^{-D(x)}B(x)dx < \infty, \qquad \text{and } \lambda_0 > 0, \tag{4.8}$$

This is a better case than type 1 because $\int_0^\infty N$ is finite.

Example 3. Cell cycle type

$$B(x) = 2K\mathbb{1}_{\{x \geq x^*\}}, \quad d(x) = K\mathbb{1}_{\{x\geq x^*\}} \qquad \text{and } \lambda_0 > 0, \tag{4.9}$$

(see section §1.1 for this last example). Compute N and prove that $\int_0^\infty N$ is finite and that $\lambda_0 > 0$ is given implicitely by

$$e^{\lambda_0 x^*} = 2\frac{K}{K+\lambda_0}.$$

Example 4. Individual aging type

$$d(x) \xrightarrow[x\to\infty]{} \infty, \qquad \text{and } \lambda_0 > 0 \text{ can be positive or not,} \tag{4.10}$$

Again $\int_0^\infty N$ is finite. We recall that population aging refers to the increase of $\int x\, n(t,x)dx$ while individual aging refers to increase of death rate $d(x)$ with age x.
Counterexample. $d \equiv 0$, and $B(x) = \frac{\mu}{1+x^2}$.

4.2 Eigenelements (adjoint)

The *adjoint eigenproblem* can easily be computed mimicking the definition $(A(N), \phi) = (N, A^*(\phi))$ with $(u, v) = \int_0^\infty u(x)v(x)dx$. We multiply (4.3) by ϕ and integrate by parts assuming that $N(x)\phi(x) \to 0$, as $x \to \infty$,

$$\int_0^\infty N(x)[-\frac{\partial\phi(x)}{\partial x} + \big(\lambda_0 + d(x)\big)\phi(x)]dx = N(0)\phi(0) = \phi(0)\int_0^\infty B(x)N(x)dx.$$

This has to hold for all N which means that $-\frac{\partial\phi(x)}{\partial x} + \big(\lambda_0 + d(x)\big)\phi(x) = \phi(0)B(x)$.
But we settle more precisely the problem and normalize it

$$\begin{cases} -\frac{\partial}{\partial x}\phi(x) + \big(\lambda_0 + d(x)\big)\,\phi(x) = \phi(0)B(x), \quad x \geq 0, \\ \\ \phi(x) \geq 0, \qquad \int_0^\infty \phi(x)N(x)dx = 1. \end{cases} \tag{4.11}$$

Theorem 4.1 *Assume (4.2) and that (4.6) holds. Then there is a unique solution to (4.11) given by*

$$\phi(x) = \frac{\phi(0)}{N(x)}\int_x^\infty B(y)\; N(y)\; dy,$$

with $\phi(0) = 1/\int_x^\infty y\; B(y)\; N(y)\; dy$.

Proof. The equation on ϕ can also be written

$$\begin{cases} \frac{\partial}{\partial x}N(x)\phi(x) = -\phi(0)B(x)N(x), \quad x \geq 0, \\ \\ \phi(x) \geq 0, \quad \int_0^\infty \phi(x)N(x)dx = 1. \end{cases} \tag{4.12}$$

Therefore $N(x)\phi(x) = C + \phi(0)\int_x^\infty B(y)N(y)dy$ and the constant $C = 0$ is fixed by the value $N(0)\phi(0) = \phi(0)$ in view of $N(0) = \int_0^\infty Bn = 1$, see (4.2). $\square$

4.3 Distribution solutions and entropy

We are now in a position to prove the following improvment of Theorem 2.2. In particular it shows explicitely that the population grows with the rate λ_0 (the so-called *Malthus* parameter).

Theorem 4.2 *(Existence, uniqueness) We assume (4.2) and (4.6),* $0 \leq n^0 \leq C_+ N$, *then there is a unique weak solution* $n \in L^1_{\text{loc}}\big([0,\infty[\times[0,\infty[\big)$ *to the renewal equation (2.1) and it satisfies* $n \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+;\phi(x)dx)\big)$ *and*

$$0 \leq n(t,x)e^{-\lambda_0 t} \leq C_+ N, \tag{4.13}$$

$$e^{-\lambda_0 t}\int_0^\infty \phi(x) n(t,x)dx = \int_0^\infty \phi(x) n^0(x)dx, \tag{4.14}$$

$n \in BV\big([0,\infty[\times[0,\infty[\big)$ *and*

$$e^{-\lambda_0 t}\|\phi \frac{\partial n(t,x)}{\partial t}\|_{M^1(\mathbb{R}^+)} \leq N(n^0), \tag{4.15}$$

with $N(n^0) = \int \phi\big[|\frac{\partial n^0(x)}{\partial x}| + d(x)n^0(x)|\big]dx$,

$$e^{-\lambda_0 t}\|\phi \frac{\partial n(t,x)}{\partial x}\|_{M^1(\mathbb{R}^+\times\mathbb{R}^+)} \leq N(n^0) + e^{-\lambda_0 t}\int_0^\infty \phi(x)\, d(x)\, n(t,x)\, dx, \tag{4.16}$$

and for two initial data n^0 *and* $\widetilde{n}^0$, *we have the contraction principle*

$$e^{-\lambda_0 t}\int_0^\infty \phi(x)|n(t,x) - \widetilde{n}(t,x)|dx \leq \int_0^\infty \phi(x)|n^0(x) - \widetilde{n}^0(x)|dx. \tag{4.17}$$

Notice that we have again defined $\frac{\partial n^0(x)}{\partial x}$ including the the possible jump at $x = 0$ because we define $n^0(0) = \int_0^\infty B(x)n^0(x)dx$.

Theorem 4.3 *(Entropy) The solution satisfies the Generalized Relative Entropy inequality,*

$$\frac{d}{dt}\int_0^\infty \phi(x)N(x)H\big(\frac{n(t,x)e^{-\lambda_0 t}}{N(x)}\big) = -D_H(t) \leq 0,$$

for all convex function $H(\cdot)$ *and, setting* $d\mu(x) = B(x)N(x)dx$, *a probability measure*

$$D_H(t) = H\Big(\int_0^\infty \frac{n(t,x)e^{-\lambda_0 t}}{N(x)}d\mu(x)\Big) - \int_0^\infty H\big(\frac{n(t,x)e^{-\lambda_0 t}}{N(x)}\big)d\mu(x)$$

Proof. (Uniqueness) We begin with the main new point, uniqueness in $L^1_{\text{loc}}\big([0,\infty[\times[0,\infty[\big)$. Consider two possible solutions in $L^1_{\text{loc}}\big([0,\infty[\times[0,\infty[\big)$, by substraction we find a distribution solution such that $n^0 \equiv 0$. Then, we use the Definition 2.1 of distributional solutions with the test function $\psi(t,x)$ built thanks to the time evolution adjoint problem (see Section §2.1). We find

$$\int_0^\infty \int_0^\infty n(t,x)\; S(t,x) dx\; dt = 0,$$

for all smooth function $S(t,x)$ with compact support. This implies indeed that $n \equiv 0$. □

This proof is standard and just expresses the 'HUM' (Hilbert Uniqueness Method); existence for the dual equation implies uniqueness for the direct problem.

Proof. (Existence) This follows from the limiting procedure described in the Chapter 2, departing from the discrete model (2.12). But the estimates have to be done with the correct weights coming from the eigenelements. We just indicate the idea. We use the notations

$$d_i = \frac{1}{h}\int_{x_{i-1}}^{x_i} d(x)\; dx, \qquad B_i = \frac{1}{h}\int_{x_{i-1}}^{x_i} B(x)\; dx.$$

First of all we truncate the indices by a finite I_h, with $x_I = hI \xrightarrow[h\to 0]{} \infty$ to ensure the property (see Example 3 in Section 4.1)

$$\bar{d} = \max_{1\le i\le I} d_i < \infty, \qquad h\;\bar{d} \le 1.$$

We also use the notations of Section 2.4 for d_h, B_h and for instance (see below what is N_i)

$$N_h(x) = \sum_{1\le i\le I} N_i \mathbb{1}_{\{x_{i-1}\le x\le x_i\}}$$

And at this stage a semi-discrete model will be easier to handle, that is a vector function $n \in C^1(\mathbb{R}^+;\mathbb{R}^I)$ solving

$$\begin{cases} h\frac{d}{dt} n_i(t) + n_i(t) - n_{i-1}(t) + hd_i n_i(t) = 0, \qquad 1 \le i \le I, \\[2ex] n_0(t) = h \displaystyle\sum_{1\le i\ge I} B_i\; n_i(t). \end{cases} \tag{4.18}$$

The discrete version of the solution (λ_0, N) to (4.3) is $(\lambda_h, (N_i)_{1\le i\le I})$, given by

$$\begin{cases} N_i - N_{i-1} + h(\lambda_h + d_i) N_i = 0, \qquad 1 \le i \le I, \\[2ex] N_0 = 1 = h \displaystyle\sum_{1\le i\ge I} B_i\; N_i. \end{cases} \tag{4.19}$$

It is easy to check that, following the arguments in Section 2.4,

$$N_h \text{ is bounded in } BV_{\text{loc}}, \qquad N_h \xrightarrow[h\to 0]{} N \text{ in } L^p_{\text{loc}},$$
$$\lambda_h \xrightarrow[h\to 0]{} \lambda_0.$$

The adjoint problem is given by the adjoint of the matrix associated to (4.19), namely

$$\begin{cases} -\phi_i + \phi_{i-1} + h(\lambda_h + d_{i-1})\phi_{i-1} = hB_i, \qquad 1 \le i \le I, \\ \displaystyle\sum_{1\le i\le I} N_i\, \phi_i = 1, \qquad \phi_i \ge 0. \end{cases} \tag{4.20}$$

Again it is easy to check that

$$\phi_h \text{ is bounded in } BV_{\text{loc}}, \qquad \phi_h \xrightarrow[h\to 0]{} \phi \text{ in } L^p_{\text{loc}}.$$

We may apply the Generalized Relative Entropy method and find

$$0 \le n_i^0 \le C_+ N_i \implies 0 \le n_i(t) e^{-\lambda_h t} \le C_+ N_i, \quad \forall t \ge 0,$$
$$\frac{d}{dt} \sum_{1\le i\le I} e^{-\lambda_h t} n_i(t) \phi_i = 0,$$

after approximation of the initial data in the continous problem, we recover the first uniform estimates (4.13), (4.14).

Differentiating in time the equation (2.12), we recover the same equation. Therefore

$$\frac{d}{dt} \sum_{1\le i\le I} e^{-\lambda_h t} \left| \frac{dn_i(t)}{dt} \right| \phi_i \le 0,$$

and thus, with $n_0^0 = h \displaystyle\sum_{1\le i\ge I} B_i\, n_i^0$,

$$\begin{aligned} e^{-\lambda_h t} h \sum_{1\le i\le I} \left| \frac{dn_i(t)}{dt} \right| \phi_i &\le \sum_{1\le i\le I} \left| n_i^0 - n_{i-1}^0 + h d_i n_i^0 \right| \phi_i \\ &\le \sum_{1\le i\le I} \left[\left| n_i^0 - n_{i-1}^0 \right| + h d_i n_i^0 \right] \phi_i. \end{aligned}$$

Passing to the limit we find (4.15).

The estimate (4.16) follows directly from (4.15) and the equation (4.2) itself.

Finally, (4.17) is a variant of (4.13) as in the case of Theorem 2.2. □

4.4 Long time asymptotic: exponential decay

To simplify notations, we set, for a solution to (4.1) , $\widetilde{n} = ne^{-\lambda_0 t}$.

Theorem 4.4 *Under assumption (4.7), and*

$$\exists \mu_0 > 0, \quad s.t. \qquad B(x) \geq \mu_0 \frac{\phi(x)}{\phi(0)}, \tag{4.21}$$

the solution to (4.1) satisfies

$$\int |\widetilde{n}(t,x) - \rho N(x)| \phi(x) dx \leq e^{-\mu_0 t} \int |n^0(x) - \rho N(x)| \phi(x) dx \tag{4.22}$$

with $\rho = \int_0^\infty n^0(x)\phi(x)dx$ *a conserved quantity, see (4.14).*

The assumption (4.21) is restrictive if we have in mind that B can vanish for $x \approx 0$ and be positive afterwards because $\phi(x) > 0$ on the convex hull of the support of B and $\{0\}$. But for x large, or close to the end point of the support of B, in general the quantity $\phi(x)$ vanishes faster than B. See Exercise 5 below.

Proof. We define

$$h(t,x) = \widetilde{n}(t,x) - \rho N(x).$$

By linearity it still satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t} h(t,x) + \frac{\partial}{\partial x} h(t,x) + \lambda_0 \, h(t,x) = 0, \quad t \geq 0, \, x \geq 0, \\ \\ h(t, x=0) = \int B(y) h(t,y) dy. \end{cases}$$

Using the dual equation (4.11), we have again by a simple combination of these two equations,

$$\begin{cases} \frac{\partial}{\partial t} \big[h(t,x)\phi(x) \big] + \frac{\partial}{\partial x} \big[h(t,x)\phi(x) \big] = -\phi(0) B(x) h(t,x), \quad t \geq 0, \, x \geq 0, \\ \\ \phi(0) h(t, x=0) = \phi(0) \int B(y) h(t,y) dy. \end{cases}$$

Therefore

$$\begin{cases} \frac{\partial}{\partial t} \big[|h(t,x)|\phi(x) \big] + \frac{\partial}{\partial x} \big[|h(t,x)|\phi(x) \big] = -\phi(0) B(x) |h(t,x)|, \quad t \geq 0, \, x \geq 0, \\ \\ \phi(0) |h(t, x=0)| = \phi(0) | \int B(y) h(t,y) dy|. \end{cases}$$

After integration in x, we obtain since $\int \phi(x) h(t,x) dx = 0$,

$$\begin{aligned} \frac{d}{dt} \int |h(t,x)| \phi(x) dx \quad &= -\phi(0) \int B(x) |h(t,x)| dx + \phi(0) | \int B(x) h(t,x) dx| \\ \\ &= -\phi(0) \int B(x) |h(t,x)| dx + | \int [\phi(0) B(x) - \mu_0 \phi(x)] h(t,x) dx| \\ \\ &\leq -\phi(0) \int B(x) |h(t,x)| dx + \int [\phi(0) B(x) - \mu_0 \phi(x)] |h(t,x)| dx \\ \\ &= -\mu_0 \int |h(t,x)| \phi(x) dx. \end{aligned}$$

We conclude using the Gronwall lemma. □

Remark 4.1 *Following Lemma 3.1 , exponential rates follow from a Poincaré inequality relating the entropy dissipation to the entropy. Here this reads, for the square entropy and for all functions h such that $\int_0^\infty \phi h = 0$,*

$$\nu \int_0^\infty \phi N \big(\frac{h}{N}\big)^2 \leq \int_0^\infty \big(\frac{h}{N}\big)^2 B(x)N(x)dx - \Big(\int_0^\infty \frac{h}{N} B(x)N(x)dx\Big)^2 .$$

This holds true with assumption (4.21) (in teh prove above we have used the entropy $H(u) = |u|$). But when B vanishes close to 0, this is obviously completely wrong.

Nevertheless, there is always exponential rate of convergence to the steady state ([4, 7]).

4.5 Exercises

Exercise 1. Consider the solution ϕ to the adjoint problem in Section 4.2. In the case of examples 1 of Section 4.1, show that ϕ vanishes for $x \geq x_\sharp$. In the case of examples 2, what is the behavior of ϕ for x large.

Exercise 2. Let $x_0 > 0$ and consider for $0 \leq x \leq x_0$ the equation

$$\begin{cases} \frac{\partial}{\partial x}[(x - x_0)N] + \lambda_0 N = 0, \\ \\ N(0) = 1 = \int_0^{x_0} B(x)N(x)dx. \end{cases}$$

If $\int_0^{x_0} B(x)dx > 1$, show that there is a unique solution with $\lambda_0 > 0$. Give the adjoint equation and compute its solution.

Exercise 3. Let $v(x) \geq v_m > 0$ and consider the equation

$$\begin{cases} \frac{\partial}{\partial x}[v(x)N] + (\lambda_0 + d(x))N = 0, \\ \\ v(0)N(0) = 1 = \int_0^{x_0} B(x)N(x)dx. \end{cases}$$

In the case of the examples of Section §4.1, examine the existence of solutions.

Exercise 4. (Complete cell cycle) Consider the system, for $i = 1, 2, ..., I$

$$\begin{cases} \frac{\partial}{\partial t} n_i(t,x) + \frac{\partial}{\partial x}[v_i(x)n_i(t,x)] + [d_i(x) + K_{i\to i+1}(x)]n_i(t,x) = 0, \\ \\ v_i(0)n_i(t, x = 0) = \int_{x' \geq 0} K_{i-1\to i}(x')\, n_{i-1}(t,x')\, dx', \qquad 2 \leq i \leq I, \\ \\ v_1(0)n_1(t, x = 0) = 2\int_{x' \geq 0} K_{I\to 1}(x')\, n_I(t,x')\, dx', \end{cases} \tag{4.23}$$

1. Study the direct eigenelement associated with this problem and give a condition ensuring $\lambda_0 > 0$.

2. Write the adjoint equation.
3. Write the GRE principle for this equation.

Exercise 5. Take $d \equiv 0$ and

$$B(x) = \nu\, e^{-\mu x}, \qquad \nu > \mu,$$

1. Show that assumption (4.7) is satisfied.
2. Show that

$$\lambda_0 = \nu - \mu, \quad \phi(x) = \phi(0) e^{-\mu x}, \quad N(x) = e^{-\lambda_0 x}.$$

3. Show that the assumption (4.21) is satisfied with $\mu_0 = \nu$.

Exercise 6. Consider the eigenvalue problem for the size structured model

$$\begin{cases} \frac{\partial}{\partial x} N(x) + (\lambda_0 + B(x))N(x) = 4B(2x)N(2x), \qquad x \geq 0, \\ \\ N(0) = 0, \qquad N(x) \geq 0. \end{cases}$$

Assume that the 'largest mother cell is smaller than twice the smallest daughter cell', i.e., $B(x)$ vansihes for $x \leq x_-$ and $x \geq x_+$ with $x_+ \leq 2x_-$.
1. Write a system of two functions $N_-(x)$ for $\frac{x_-}{2} \leq x \leq x_-$, and $N_+(x)$ for $x_- \leq x \leq x_+$.
2. Show that $N_-(x_-/2) = 0$ and compute $N_-(x_-)$ as a function of $N_+(x)$.
3. Show that $N_+(x)$ satisfies a renewal equation.
4. Prove that there is a unique possible λ_0 in (4.5).
5. Give a condition on x_-, x_+ and B such that $\int N < \infty$.

Exercise 7. Consider the age structured structured

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n + d(x) n = 0, \qquad x \geq 0, \\ \\ \frac{\partial}{\partial t} v(t) + \mu v = \int B(x) n(t,x) dx, \\ \\ n(t,0) = v(t). \end{cases}$$

Compute the eigenelements and express the General Relative Entropy principle. Study the large time behavior of the quantities $(n(t,x), v(t))$.

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