Introduction to Modal Logic

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Propositional Logic

First, let me review some ideas from basic Propositional Logic (logic without the quantifiers $\forall$ and $\exists$)

Alphabet

Our alphabet will be composed of the following symbols:

1. $p_0, p_1, \ldots$ — variables
2. $\neg, \rightarrow, \wedge, \vee$ — connectives
3. $(,)$ — precedence symbols
4. $\perp$ — false

We write $\mathscr{P} := \{ p_0, p_1, \ldots \}$.

Formulas

We define the set of propositional formulas, $\mathscr{F}$ by:

- For every $p \in \mathscr{P}$, $p \in \mathscr{F}$ and $\bot \in \mathscr{F}$
- If $\varphi \in \mathscr{F}$ then $\neg\varphi \in \mathscr{F}$.
- If $\varphi, \psi \in \mathscr{F}$
  - $(\varphi \wedge \psi) \in \mathscr{F}$
  - $(\varphi \vee \psi) \in \mathscr{F}$
  - $(\varphi \rightarrow \psi) \in \mathscr{F}$

Example

- $((p \wedge (q \vee r)) \rightarrow s) \in \mathscr{F}$
- $(p\wedge) \wedge \vee q \notin \mathscr{F}$

Truth

What does it mean for a formula to be true?
There are two approaches to showing that a formula is true: Syntactically and Semantically. We will begin with semantics.

Semantics

Note, we desire a way of deciding the truth of a statement.

Definition

A **truth assignment** is a function $v : \mathscr{P} \to \{T, F\}$.
We then extend v to a function $\bar{v} : \mathscr{F} \to \{T, F\}$ called a **valuation** in the way you'd expect, ie. by consulting a truth table.
For example, if $v(p) = T$ and $v(q) = T$ then $\bar{v}(p \wedge q) = T$.
and so on for other connectives.

Tautologies

Definition

We say a truth assignment v **models** a formula φ (written $v \models \varphi$) if $\bar{v}(\varphi) = T$.

We say a formula φ is **satisfiable** if there is a truth assignment v such that $v \models \varphi$.

We say a formula φ is a **tautology** if for every truth assignment v, $v \models \varphi$.

Examples

Example

The sentence $\varphi = P \vee \neg P$ is a tautology; for any truth assignment this statement is sent to T. (This is called the **law of the excluded middle**)

The statement $\psi = P \implies Q$ is not a tautology; consider the truth assignment $P \mapsto T$ and $Q \mapsto F$. Then ψ is sent to F by the valuation.

ψ is valid however. The truth assignment v where $P \mapsto F$, we have $v \models \psi$.

Syntax

Another avenue for deciding whether a formula φ is true is whether we can prove φ from a list of axioms.
Here is a list of axioms:

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$
- $\varphi \to (\psi \to \varphi \wedge \psi)$
- $\varphi \wedge \psi \to \varphi$
- $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$
- $\psi \to \varphi \vee \psi$
- $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$
- $\perp \to \varphi$
- $\varphi \vee \neg\varphi$

Inference

There is one rule of inference: Modus Ponens. That says if we can prove $\varphi \to \psi$ and we can prove φ then we can infer ψ.

Definition

If there is a proof of φ then we write $\vdash \varphi$.

Soundness

Theorem

If φ is provable, then φ is true under all truth assignments. In symbols, $\vdash \varphi$ implies $\models \varphi$.

Proof.

You need only check that the axioms and the rule of modus ponens is valid with respect to truth assignments. It is! $\square$

Completeness

Theorem

If φ is true under all truth assignments, then φ is provable.
In symbols, $\models \varphi$ implies $\vdash \varphi$.

Proof.

Out of our scope! $\square$

Modal Logic

We have now seen the propositional calculus. We wish to extend it to make it a bit more expressive.
To do this, we add two unary operators to our alphabet: $\Box$ and $\Diamond$, which we read as *necessarily* and *possibly*.

Formulas

The set of modal formulas $\mathscr{F}_M$ is defined to be:

- If $\varphi \in \mathscr{F}$ then $\varphi \in \mathscr{F}_M$, ie. all propositional formulas are modal formulas.
- If $\varphi \in \mathscr{F}_M$ then $\Box\varphi \in \mathscr{F}_M$.
- If $\varphi \in \mathscr{F}_M$ then $\Diamond\varphi \in \mathscr{F}_M$.

Example

A typical modal formula may look like:

$$\Box(A \to (\Diamond B \vee A))$$

odes bind tight.

Truth

As before, we now have a set of formulas. We need to make sense of what it means for a formula to be true.

Modal Models

Definition

A **model** $\mathscr{M} = \langle W, R, V \rangle$ is a triple, where:

- W is a nonempty set. W is called our **universe** and elements of W are called **worlds**
- R is a relation on W. R is called our **accessibility relation**. The interpretation is if w_1 is R-related to w_2 then w_1 "knows about" w_2 and must consider it in making decisions about whether something is possible or necessary.
- V is a function mapping the set of propositional variables $\mathscr{P}$ to $\mathcal{P}(W)$. The interpretation is the if P is mapped into a set contain w then w thinks that the variable P is true.

Models

Definition

Fix $\mathscr{M} = \langle W, R, V \rangle$. We will define now what it means for $\mathscr{M}$ to model a modal formula φ at some world w.

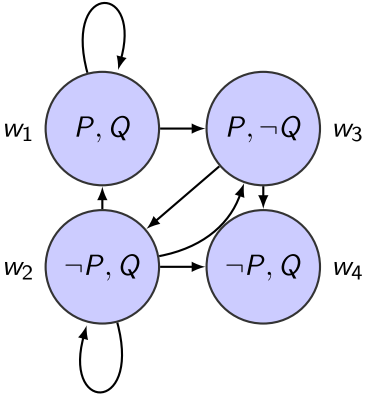
- $\mathscr{M} \models_w P$ if and only if $w \in V(P)$.
- $\mathscr{M} \models_w \neg P$ if and only if $\mathscr{M} \not\models_w P$.
- We decide if $\mathscr{M} \models_w \varphi$ where $\varphi = \psi \wedge \theta$, $\varphi = \psi \vee \theta$, or $\psi \rightarrow \theta$ by looking it up in the truth table.
- $\mathscr{M} \models_w \Box\varphi$ if and only if for every $w' \in W$ such that wRw' we have $\mathscr{M} \models_{w'} \varphi$; ie. every world that w is "accessible" to via R thinks that φ is true.
- $\mathscr{M} \models_w \Diamond\varphi$ if and only if there is $w' \in W$ such that wRw' we have $\mathscr{M} \models_{w'} \varphi$; ie. there's some world that w is "accessible" to via R thinks that φ is true.

More on Models

Definition

For a formula φ and a model $\mathscr{M}$ we say $\mathscr{M} \models \varphi$ if $\mathscr{M} \models_w \varphi$ for every world w.
We say $\models \varphi$ if $\mathscr{M} \models \varphi$ for every model $\mathscr{M}$.

An Example



1. $\mathcal{M} \models_{w_1} P \wedge \Box P$
2. $\mathcal{M} \models_{w_1} Q \wedge \Diamond Q$
3. $\mathcal{M} \models_{w_1} \neg \Box Q$
4. $\mathcal{M} \models_{w_2} Q \wedge \Diamond \neg Q$
5. $\mathcal{M} \models_{w_3} P$
6. $\mathcal{M} \models_{w_3} \Box \neg P$
7. $\mathcal{M} \models_{w_4} (\Box P) \wedge \neg(\Diamond P)$

Unexpected behavior!

Notice, some things happen that we didn't really want. For example, you would expect:

1. $\Box P \to P$
2. $P \to \Diamond P$
3. $\Box P \to \Diamond P$
4. $\Box P \to \Box\Box P$
5. $P \to \Box\Diamond P$
6. $\Diamond P \to \Box\Diamond P$

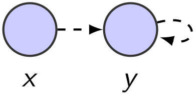
Relations

Too see why, let's first talk about some special properties of relations.

Serial

Definition

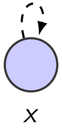
Let R be a relation on W. We say R is **serial** if for every $x \in W$ there is some $y \in W$ such that xRy.



Reflexive

Definition

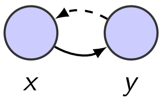
Let R be a relation on W. We say R is **reflexive** if for every $x \in W$ we have xRx.



Symmetric

Definition

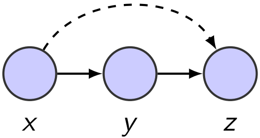
Let R be a relation on W. We say R is **symmetric** if for every $x, y \in W$ if xRy then yRx.



Transitive

Definition

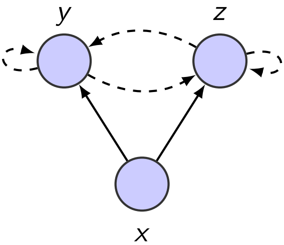
Let R be a relation on W. We say R is **transitive** if for every $x, y, z \in W$ if xRy and yRz then xRz.



Euclidean

Definition

Let R be a relation on W. We say R is **euclidean** if for every $x, y, z \in W$ if xRy and xRz then yRz.



Properties of the accessibility relation will tell us about axioms that hold in our models.

Some Axioms

Here are some axioms:

- **N** $\Box\psi$ for all propositional tautologies ψ
- **K** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- **T** $\Box\varphi \to \varphi$
- **D** $\Box\varphi \to \Diamond\varphi$
- **4** $\Box\varphi \to \Box\Box\varphi$
- **B** $\varphi \to \Box\Diamond\varphi$
- **5** $\Diamond\varphi \to \Box\Diamond\varphi$

Theorem

N and K hold in all models.

Proof.

If ψ is an axiom, then ψ holds in every model, so clearly $\Box\psi$ holds in every model.
Assume $\Box(\varphi \to \psi)$. Want to show $\Box\varphi \to \Box\psi$. Assume $\Box\varphi$. Fix a world w. Then for every world related to w, φ holds and $\varphi \to \psi$ holds. So ψ holds. So $\Box\psi$ holds in w. □

Corollary

The axioms N and K are sound for all models.

Axiom D fails

There is a model $\mathscr{M}$ such that $\Box P \to \Diamond P$ fails.

Proof.

$\square$

Problem: The relation is not serial!

Serial implies Axiom D

Theorem

If a the accessibility relation is serial, then

$$\mathscr{M} \models \Box\varphi \to \Diamond\varphi$$

Proof.

By seriality, for every world w there is w' such that wRw'. If $\Box\varphi$ holds at w, then φ holds in w', and thus $\Diamond\varphi$ holds in w. □

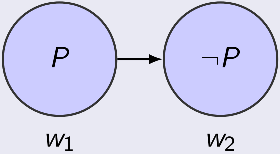
Corollary

The axiom D is sound for all models with serial accessibility relations.

Axiom T fails

The is a model $\mathscr{M}$ where $\Box\varphi \to \varphi$ fails.

Proof.



□

Problem: The relation is not reflexive!

Reflexive implies Axiom T

Theorem

If a the accessibility relation is reflexive, then

$$\mathscr{M} \models \Box\varphi \rightarrow \varphi$$

Proof.

If $\Box\varphi$ holds at w, then φ holds in w as wRw by reflexivity. $\square$

Soundness of T and D

Lemma

Reflexive implies Serial

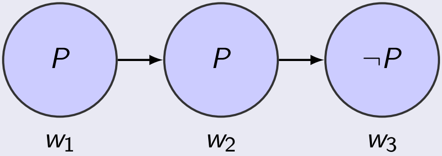
Corollary

The axioms T and D are sound for all models with reflexive accessibility relations.

Axiom 4 fails

The is a model $\mathscr{M}$ where $\Box\varphi \to \Box\Box\varphi$ fails.

Proof.



$\Box$

Problem: The relation is not transitive!

Soundness of Axiom 4

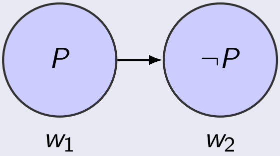
Corollary

Axiom 4 is sound for all models with transitive accessibility relations.

Axiom B fails

The is a model $\mathscr{M}$ where $\varphi \to \Box\Diamond\varphi$ fails.

Proof.



$\square$

Problem: The relation is not symmetric!

Soundness of Axiom B

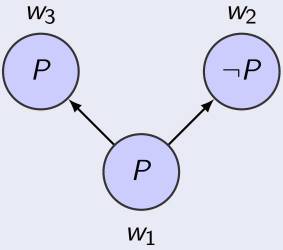
Corollary

Axiom B is sound for all models with symmetric accessibility relations.

Axiom 5 fails

The is a model $\mathscr{M}$ where $\Diamond\varphi \to \Box\Diamond\varphi$ fails.

Proof.



□

Problem: The relation is not euclidean!

Soundness of Axiom 5

Corollary

Axiom 5 is sound for all models with euclidean accessibility relations.

Soundness of S5

Lemma

TFAE:

1. *Equivalence Relation*
2. *Reflexive, Symmetric, Transitive*
3. *Serial, Symmetric, Transitive*
4. *Euclidean, Reflexive*

Let the Axiom S be defined as K+N+T.

Corollary (S5 is sound)

If we can prove φ using the axioms S5 then every model with its accessibility relation an equivalence relation models φ, ie. this system is sound.

Completeness of S5

Theorem (S5 is complete)

If every model $\mathscr{M}$ with its accessibility relation an equivalence relation models φ then we can prove φ using the axioms S5, ie. this system is complete.