Integration Review

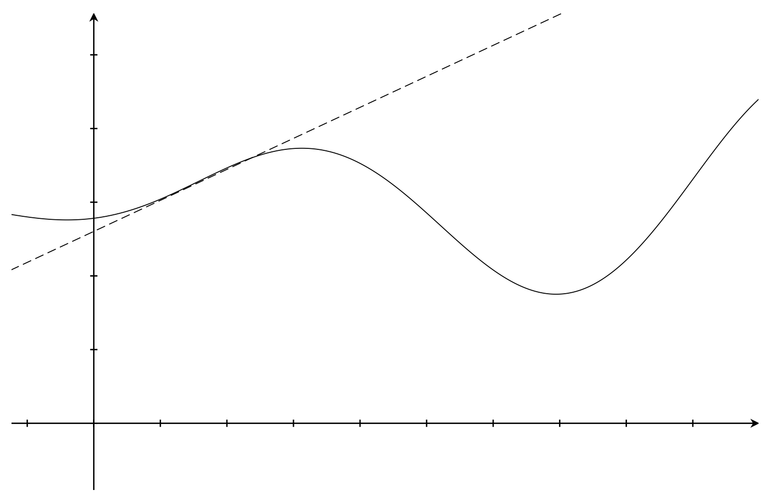
May 11, 2013

Goals:

- Review the fundamental theorem of calculus.
- Review u-substitution.
- Review integration by parts.
- Do lots of integration examples.

1 Fundamental Theorem of Calculus

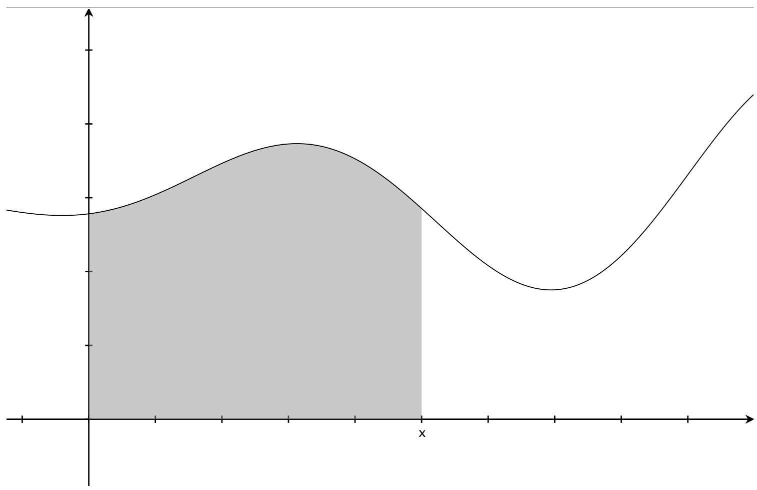
In Calculus 1, much focus was placed on the idea of the *derivative*, or instantaneous rate of change of a function. The instantaneous rate of change of a function f at point x, denoted $f'(x)$, is the slope of the tangent line of f at point x, as pictured:



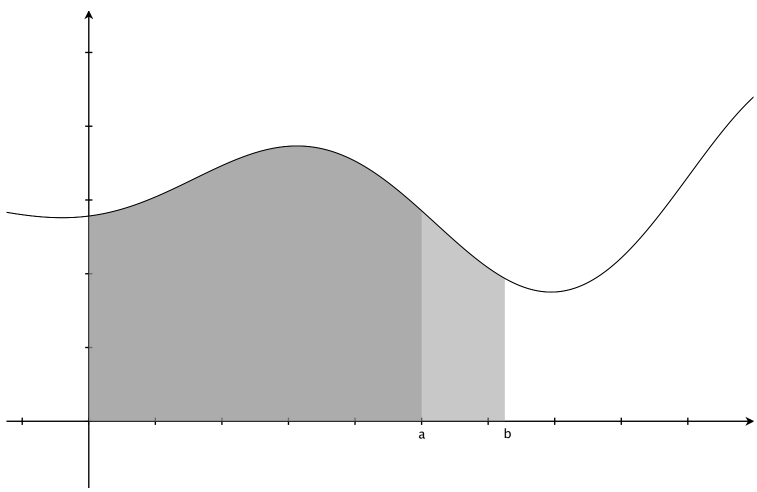
At the end, a tangent (pardon the pun) was taken, and another concept was discussed. We asked the question: how do we calculate the area under the curve of a function f from a to b? We denote this $\int_a^b f(x)\,dx$. Let's briefly explore this problem.

Observation 1: We only need to be able to calculate the area from 0 to x

Why is this? Well, let $g(x)$ be the function which calculates the area under f from 0 to x, as pictured.



Then, notice that the area under f from a to b is just $g(b) - g(a)$.



There's really nothing special about 0 either. We could have made g calculate the area under f from k to x as well, where k is any fixed number.

Observation 2: $g'(x) = f(x)$

Let's pretend we have the function g above, even though we have not yet found it. We can ask: how does g change as x changes? We expect that a little change in the function should equal a little change in the area equal to how high the function is. This is hand wavy, but of course we have not addressed what the area under a curve is formally yet anyway. We, however, conclude that we expect $g'(x) = f(x)$.

Observation 3: If we find a g such that $g'(x) = f(x)$ then we are done.

The above equation is called a *differential equation.* We are trying to find a y such that $y' = f(x)$. Such a solution y is called an *anti-derivative* of f, for obvious reasons. We say 'an' anti-derivative rather than 'the' anti-derivative because it is easy to see there are many. For instance, if y satisfies the above and is an anti-derivative of $f(x)$, then $y + 1$ is as well. This is because differentiation is *linear* so we can take the derivative of each part of the sum separately. Also, constants are constant, and therefore do not change. Therefore, $\frac{d}{dx}(y+1) = \frac{d}{dx}(y) + \frac{d}{dx}(1) = y'$

The beginning part of this course will be to investigate how to find solutions of $y' = f(x)$; that is, we will find the anti-derivative of functions. The main application of this is obvious: anti-derivatives give us the tool to calculate areas by the above 3 observations.

We call the fundamental link between the finding the area under the curve and the derivative the *fundamental theorem of calculus.* States formally, it says

$$\frac{d}{dx}\int_k^x f(z)dz = f(x)$$

Said aloud, it says the rate of change with respect to x of finding the area under f from some constant to x is exactly the value that f has at x.

We call the area under the curve functional described above the *integral.* We often say integral instead of anti-derivative. For the *most general anti-derivative* of f, we write

$$\int f(x)\;dx$$

This is a function of x which, when you take it's derivative, you get f. Note, there is a uniqueness issue as there are many anti-derivatives.

Theorem 1. *If F and G are both anti-derivatives of f then $F - G = k$ where k is a constant.*

Proof. $\frac{d}{dx}(F - G) = f - f = 0$. Therefore, the derivative of the difference is 0. It isn't hard to see (or, believe) that constant functions are the only functions with derivatives 0. Therefore, their difference must be constant. □

Therefore, if F is an anti-derivative of f then we write:

$$\int f(x)\;dx = F(x) + C$$

Clearly anything in the above form is an anti-derivative, and by the theorem any anti-derivative has the above form for some constant C.

2 Easy Integrals

We begin with a few easy integrals which we solve essentially by inspection.

Theorem 2 (The Power Rule for Integrals)**.** *If $n \neq -1$ then*

$$\int x^n\;dx = \frac{1}{n+1}x^{n+1} + C$$

Proof. By the definition of anti-derivative, it suffices to show that the derivative of the right hand side is x^n. This is trivia by the power rule for derivatives. □

By a similar method, we can determine the following integrals:

$$\int cf(x)\;dx = c\int f(x)\;dx \qquad \int (f(x) + g(x))\;dx = \int f(x)\;dx + \int g(x)\;dx$$

$$\int x^n\;dx = \frac{1}{n+1}x^{n+1} + C \quad [n \neq -1] \qquad \int x^{-1}\;dx = \ln|x| + C$$

$$\int \sin(x)\;dx = -\cos(x) + C \qquad \int \cos(x)\;dx = \sin(x) + C$$

$$\int e^x\;dx = e^x + C$$

3 u-Substitution

Recall that the chain rule for derivatives:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

This tells us that:

$$\int f'(g(x))g'(x)\;dx = f(g(x)) + C$$

This is the principle of u-substitution. If we can identify a $g(x)$ as above, and set $u = g(x)$, then we can do a substitution of u for $g(x)$ and do a trade of $du = g'(x)\,dx$. So, the above looks like

$$\int f'(g(x))g'(x)\,dx = \int f'(u)du = f(u) + C = f(g(x)) + C$$

This trick, in these cases, is to identify a $g(x)$ that works. Given any integral, one should also first begin by trying to identify easy substitutions.

Example 1. We calculate the following integral:

$$\int \frac{\ln(x)}{x}\,dx$$

Set $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$. Then

$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= \int \overbrace{\ln(x)}^{u}\overbrace{\frac{1}{x}\,dx}^{du}\\ &= \int u \cdot du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\left(\ln(x)\right)^2 + C\end{aligned}$$

Warning: Remember that you have to go back to 'x world' after doing a u-substitution. If the problem was given to you in terms of x, and it's a little rude to give the answer in terms of u, right?

4 Integration by Parts

Recall the product rule for derivatives:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

This tells us:

$$f(x)\cdot g(x) - \int g'(x)\cdot f(x)\,dx = \int f'(x)\cdot g(x)\,dx$$

Or, in the more relevant direction:

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx$$

The is the principle of integration by parts. If we identify two parts of an integral, we can essentially integrate one part and take the derivative of the other to calculate the integral.

In practice, we usually write this in terms of the letters For example, to calculate:

$$\int f'(x)\cdot g(x)\,dx$$

We would set:

$$\begin{aligned} u &= g(x) & dv &= f'(x)\,dx\\ du &= g'(x)\,dx & v &= f(x)\end{aligned}$$

And get:

$$\int f'(x)\cdot g(x)\,dx = \int udv = uv - \int vdu = g(x)\cdot f(x) - \int g'(x)\cdot f(x)\,dx$$

Parts is particularly useful when there is one part with a much simpler function as it's derivative (like $\ln(x)$ or the inverse trig functions) or when there is a polynomial time some other function which is not too difficult to integrate (like xe^x or $x^3\sin(x)$) as polynomials "vanish" after successive derivatives.

Example 2. We calculate the following integral:

$$\int x \cdot e^x \, dx$$

Observe that x becomes simpler when you take the derivative, and e^x does not become too complicated when you integrate. Therefore, we do parts.

$$\begin{aligned} u &= x & dv &= e^x \, dx \\ du &= \, dx & v &= e^x \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \int x \cdot e^x &= \overbrace{x}^{u} \cdot \overbrace{e^x}^{v} - \int \overbrace{e^x}^{v} \overbrace{dx}^{du} \\ &= x \cdot e^x - e^x + C \end{aligned}$$

Hint: It is much easier to take derivatives than to take integrals. Therefore, it is in your interest to check the answer if it only takes a few seconds to do so. You see that the above when you take the derivative becomes:

$$\overbrace{e^x + xe^x}^{\frac{d}{dx}(xe^x)} - \overbrace{e^x}^{\frac{d}{dx}(e^x)} + \overbrace{0}^{\frac{d}{dx}C}$$

Which is, of course, xe^x which is what we wanted.

5 Examples

Now, let's do lots of examples of finding integrals of functions. For all the following integrals, the only knowledge that you need is how to calculate derivative of all the usual functions, the chain rule and product rule for derivatives, and u-substitution and integration by parts described above.

1. $\displaystyle\int x \sin(x) \, dx$

2. $\displaystyle\int (x+3)^{12} x^2 \, dx$

3. $\displaystyle\int x \ln(x^2) \, dx$

4. $\displaystyle\int e^x \sin(x) \, dx$

5*. $\displaystyle\int \frac{e^{2x}}{1+e^x} \, dx$

6. $\displaystyle\int \sin^2(x) \cos^3(x) \, dx$

7*. $\displaystyle\int \frac{1}{x+\sqrt{x}} \, dx$

8. $\displaystyle\int \sec^2(x) \, dx$

9. $\displaystyle\int \tan(x) \, dx$

10. $\displaystyle\int \frac{4x+2}{x^2+x+9}$