# Combinatorial Optimization

## Problem set 5: solutions

**1.** Consider the problem of determining the least expensive way to complete a project by a given deadline. When the linear program is formulated, the objective function has a constant term. For instance, if activities A, B, and C have usual times of 8, 5, and 7 days and can be sped up at a cost of \$200, \$150, and \$225 per day, respectively, then the objective function (i.e., the total speedup cost) is

$$200(8 - d_{\mathrm{A}}) + 150(5 - d_{\mathrm{B}}) + 225(7 - d_{\mathrm{C}}),$$

where $d_{\mathrm{A}}$, $d_{\mathrm{B}}$, and $d_{\mathrm{C}}$ are duration variables for the three activities. When expanded, this objective function becomes

$$3925 - 200d_{\mathrm{A}} - 150d_{\mathrm{B}} - 225d_{\mathrm{C}},$$

which has the constant term 3925. Describe a way to handle an objective function with a constant term in the simplex algorithm.

▷ **Solution.** Here are three possible ways to handle an objective function with a constant term in the simplex algorithm. For an illustration, we will use the following maximization LP (because we discussed the simplex algorithm as a maximization algorithm):

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 5x_2 + 17 \\ \text{subject to} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

1. Replace the constant term with a variable $K$ and add an equality constraint to force $K$ to have the correct value:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 5x_2 + K \\ \text{subject to} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & K = 17 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad K \geq 0. \end{aligned}$$

2. In the equation represented by the objective row in the initial simplex tableau, the constant term will appear on the right-hand side. In the example here, we have $z = 3x_1 + 5x_2 + 17$, so the equation represented by the objective row in the initial simplex tableau will be $-3x_1 - 5x_2 + z = 17$. So, when writing the initial simplex tableau, enter the constant term at the top of the RHS column (instead of zero):

| $x_1$ | $x_2$ | $s_1$ | $s_2$ | $z$ | RHS |
|---|---|---|---|---|---|
| $-3$ | $-5$ | 0 | 0 | 1 | 17 |
| 1 | 1 | 1 | 0 | 0 | 8 |
| 2 | 1 | 0 | 1 | 0 | 12 |

3. Ignore the constant term entirely during the simplex algorithm and just remember to add it to the optimal objective value at the end. For the example, solve the following linear program instead, and then add 17 to the optimal objective value:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 5x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

□

**2.** What is the greatest possible number of critical paths in a project with $n$ activities? Describe a family of examples for infinitely many values of $n$ that attain this number of critical paths.

$\triangleright$ **Solution.** Consider a project with $n$ activities. We shall make the following assumptions: First, there are no circular dependencies, so that the CPM network contains no directed cycles (i.e., it is a directed acyclic graph). This is a reasonable assumption, because if a project contains circular dependencies among its activities then it can never be completed. Second, all project durations are positive (specifically, nonzero). This is also a reasonable assumption for real-world projects.

Define an *activity chain* to be a sequence $(a_1, a_2, \ldots, a_r)$ of activities such that $a_i$ is an immediate prerequisite of $a_{i+1}$ for all $1 \le i < r$. Then a critical path is an activity chain such that the total duration of $a_1$, $a_2$, ..., $a_r$ is *maximized*. Observe that because we are assuming that all project durations are positive, no proper subsequence of a critical path is itself a critical path.

Partition the set of activities as follows: Let $A_1$ be the set of all activities having no prerequisites, and for $i \ge 2$ let $A_i$ be the set of all activities not in $\bigcup_{j=1}^{i-1} A_j$ all of whose prerequisites are in $\bigcup_{j=1}^{i-1} A_j$. Let $k$ be the number of nonempty subsets of activities constructed in this way (so $A_1$, $A_2$, ..., $A_k$ is a partition of the set of activities). Because the project has no circular dependencies, every activity is an element of one of these subsets $A_i$. Also, for $i \ge 2$, every activity in $A_i$ has at least one immediate prerequisite in $A_{i-1}$ (otherwise all of its prerequisites would be in $\bigcup_{j=1}^{i-2} A_j$, so it would not be in $A_i$; it would be in $A_j$ for some $j < i$).

**Lemma.** *Let $a_i \in A_i$ and $a_j \in A_j$ for some $j \ge i+2$. If there exists a critical path $P$ in which $a_i$ is immediately followed by $a_j$, then for every $i < h < j$ there exists $a_h \in A_h$ such that no critical path includes both $a_i$ and $a_h$.*

**Proof.** For $h = j-1, j-2, \ldots, i+1$, let $a_h$ be an immediate predecessor of $a_{h+1}$ in $A_{h+1}$. Suppose for the sake of contradiction that there exists $i < h < j$ such that both $a_i$ and $a_h$ are included in some critical path $P'$. But then the portion of $P$ ending with $a_i$, followed by the portion of $P'$ from $a_i$ to $a_h$, followed by $(a_{h+1}, a_{h+2}, \ldots, a_j)$, followed by the portion of $P$ after $a_j$, yields an activity chain, and the total duration of this activity chain is strictly larger than that of $P$ because $P$ is a proper subsequence. This contradicts the fact that $P$ is a critical path. ∎

**Lemma.** *If $a$ is the last activity in a critical path $P$, then no critical path contains $a$ followed by another activity $a'$.*

**Proof.** Otherwise appending $a'$ to $P$ would produce an activity chain with strictly larger total duration. ∎

**Corollary.** *The number of critical paths is bounded above by $|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|$.*

**Proof.** Observe that every critical path has the form of a sequence of activities produced by choosing one activity from $A_1$, followed by zero or one activity from $A_2$, followed by zero or one activity from $A_3$, ..., followed by zero or one activity from $A_k$. (Not all such sequences of activities are critical paths, or even just *paths* in the CPM network, but all critical paths have this form.) So every critical path (along with possibly some other sequences of activities) can be constructed as a sequence $S$ of activities in the following way:

0. Initialize $S$ to be the empty sequence.
1. Choose one activity $a_1 \in A_1$ and append $a_1$ to $S$.
2. Optionally, choose one activity $a_2 \in A_2$ and append $a_2$ to $S$.
3. Optionally, choose one activity $a_3 \in A_3$ and append $a_3$ to $S$.

$\vdots$

$k$. Optionally, choose one activity $a_k \in A_k$ and append $a_k$ to $S$.

There are $|A_1|$ ways to perform step 1. For $2 \le i \le k$, there are $|A_i| + 1$ ways to perform step $i$ (one of which is to skip the step, i.e., choose zero activities from $A_i$). But by the two preceding lemmas, if, given a partially constructed sequence $S$ after steps 1 through $i-1$, there is a way to extend $S$ to a critical path by skipping step $i$, then there exists at least one activity $a_i \in A_i$ such that appending $a_i$ to $S$ *cannot* produce a critical path. Therefore, given any combination of ways to perform the first $i-1$ steps, there are at most $(|A_i|+1)-1 = |A_i|$ ways to perform the $i$th step that will lead to a critical path. Hence there are no more than $|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|$ critical paths in all. ∎

Now observe that $\sum_{i=1}^{k} |A_i| = n$, because each activity is in exactly one of the subsets $A_i$. So the question becomes: How can $n$ be expressed as a sum of positive integers having the greatest possible product?

**Theorem.** *For a fixed positive integer value of $n$, if $\sum_{i=1}^{k} c_i = n$ for positive integers $c_i$, then the maximum value of $\prod_{i=1}^{k} c_i$ is*

$$\begin{cases} 1, & \textit{if } n = 1; \\ 3^m, & \textit{if } n = 3m; \\ 4 \cdot 3^{m-1}, & \textit{if } n = 3m+1 \ge 4; \\ 2 \cdot 3^m, & \textit{if } n = 3m+2. \end{cases}$$

**Proof.** The cases $n \in \{1, 2, 3, 4\}$ are easy to verify by inspection.

If $n \ge 5$, then $2(n-2) = 2n - 4 > n$, so writing $n$ as the single-term sum $n$ is not optimal, because writing it as $2 + (n-2)$ produces a strictly larger product. Suppose $n$ is written as a sum $c_1 + c_2 + \cdots + c_k$ of positive integers, with $k \ge 2$. If any term $c_i$ is greater than 4, then $c_i$ can be replaced by a sum of smaller positive integers whose product is greater than $c_i$ [e.g., $2 + (c_i - 2)$], so such a sum cannot be optimal. If any term $c_i$ is 1, then that term can be removed and another term increased by 1, which will strictly increase the product, so a sum containing the term 1 cannot be optimal either. If the sum contains three terms that are 2, then those terms can be replaced by $3+3$, which will strictly increase the product (because $3 \times 3 > 2 \times 2 \times 2$), so a sum containing three or more 2's also cannot be optimal. Lastly, if any term $c_i$ is 4, then that term can be replaced by $2 + 2$ without changing the product. Therefore, there is an optimal sum consisting solely of 2's and 3's, with no more than two 2's. Consequently, for $n \ge 5$, an optimal sum is

$$\begin{aligned} &\underbrace{3 + \cdots + 3}_{m \text{ terms}}, && \text{if } n = 3m; \\ &2 + 2 + \underbrace{3 + \cdots + 3}_{m-1 \text{ terms}}, && \text{if } n = 3m+1; \\ &2 + \underbrace{3 + \cdots + 3}_{m \text{ terms}}, && \text{if } n = 3m+2. \end{aligned}$$

∎

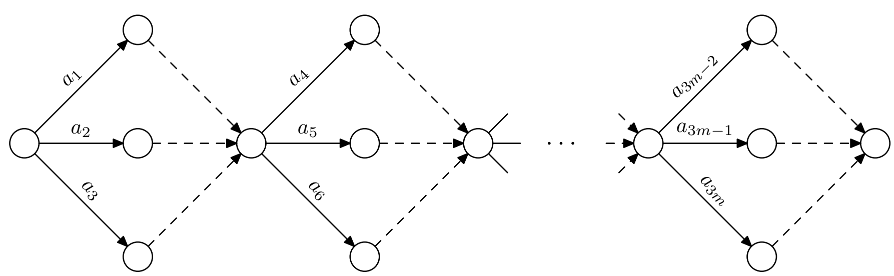
Thus the number of critical paths in a project with $n$ activities is bounded by the expression in this theorem. Moreover, this bound can be achieved as follows.

- For $n = 1$, any project with a single activity (necessarily having no prerequisites) has one critical path.

- For $n = 3m$, a project of the following form has $3^m$ critical paths.

| Activity | Immediate prerequisites | Duration |
|---|---|---|
| $a_1$ | — | 1 |
| $a_2$ | — | 1 |
| $a_3$ | — | 1 |
| $a_4$ | $a_1$, $a_2$, $a_3$ | 1 |
| $a_5$ | $a_1$, $a_2$, $a_3$ | 1 |
| $a_6$ | $a_1$, $a_2$, $a_3$ | 1 |
| $a_7$ | $a_4$, $a_5$, $a_6$ | 1 |
| $a_8$ | $a_4$, $a_5$, $a_6$ | 1 |
| $a_9$ | $a_4$, $a_5$, $a_6$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{3m-2}$ | $a_{3m-5}$, $a_{3m-4}$, $a_{3m-3}$ | 1 |
| $a_{3m-1}$ | $a_{3m-5}$, $a_{3m-4}$, $a_{3m-3}$ | 1 |
| $a_{3m}$ | $a_{3m-5}$, $a_{3m-4}$, $a_{3m-3}$ | 1 |

The CPM network of a project of this form is shown below.

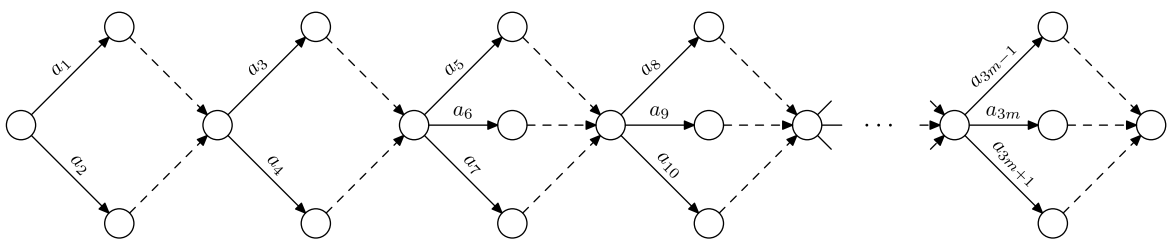



All activities have duration 1, and therefore, as a consequence of the structure of the project, every activity is critical. Hence any path from the beginning of the project to the end is a critical path. There are $m$ "layers" in this CPM network: these are the sets $A_1 = \{a_1, a_2, a_3\}$, $A_2 = \{a_4, a_5, a_6\}$, $\ldots$, $A_m = \{a_{3m-2}, a_{3m-1}, a_{3m}\}$. Each of these layers contains three activities, so there are $3^m$ critical paths.

- For $n = 3m + 1 \geq 4$, a project of the following form has $4 \cdot 3^{m-1}$ critical paths.

| Activity | Immediate prerequisites | Duration |
|---|---|---|
| $a_1$ | — | 1 |
| $a_2$ | — | 1 |
| $a_3$ | $a_1,\ a_2$ | 1 |
| $a_4$ | $a_1,\ a_2$ | 1 |
| $a_5$ | $a_3,\ a_4$ | 1 |
| $a_6$ | $a_3,\ a_4$ | 1 |
| $a_7$ | $a_3,\ a_4$ | 1 |
| $a_8$ | $a_5,\ a_6,\ a_7$ | 1 |
| $a_9$ | $a_5,\ a_6,\ a_7$ | 1 |
| $a_{10}$ | $a_5,\ a_6,\ a_7$ | 1 |
| $a_{11}$ | $a_8,\ a_9,\ a_{10}$ | 1 |
| $a_{12}$ | $a_8,\ a_9,\ a_{10}$ | 1 |
| $a_{13}$ | $a_8,\ a_9,\ a_{10}$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{3m-1}$ | $a_{3m-4},\ a_{3m-3},\ a_{3m-2}$ | 1 |
| $a_{3m}$ | $a_{3m-4},\ a_{3m-3},\ a_{3m-2}$ | 1 |
| $a_{3m+1}$ | $a_{3m-4},\ a_{3m-3},\ a_{3m-2}$ | 1 |

The CPM network of a project of this form is shown below.

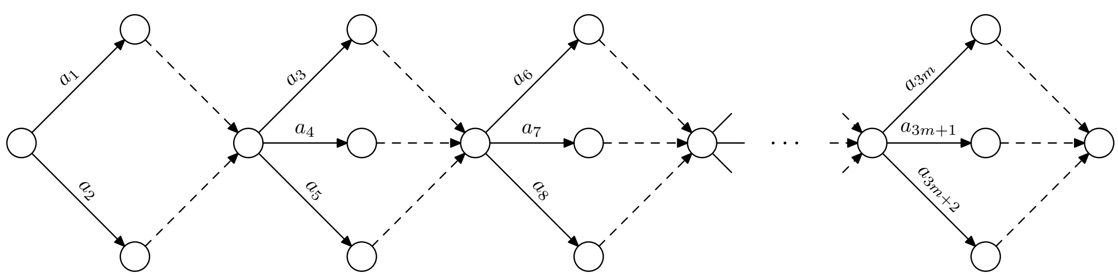



Again, every activity is critical, so any path from the beginning of the project to the end is a critical path. There are $m+1$ "layers" in this CPM network; the first two layers contain two activities each, and the remaining $m-1$ layers contain three activities each, so there are $4 \cdot 3^{m-1}$ critical paths.

- For $n = 3m+2$, a project of the following form has $2 \cdot 3^m$ critical paths.

| Activity | Immediate prerequisites | Duration |
|---|---|---|
| $a_1$ | — | 1 |
| $a_2$ | — | 1 |
| $a_3$ | $a_1$, $a_2$ | 1 |
| $a_4$ | $a_1$, $a_2$ | 1 |
| $a_5$ | $a_1$, $a_2$ | 1 |
| $a_6$ | $a_3$, $a_4$, $a_5$ | 1 |
| $a_7$ | $a_3$, $a_4$, $a_5$ | 1 |
| $a_8$ | $a_3$, $a_4$, $a_5$ | 1 |
| $a_9$ | $a_6$, $a_7$, $a_8$ | 1 |
| $a_{10}$ | $a_6$, $a_7$, $a_8$ | 1 |
| $a_{11}$ | $a_6$, $a_7$, $a_8$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a_{3m}$ | $a_{3m-3}$, $a_{3m-2}$, $a_{3m-1}$ | 1 |
| $a_{3m+1}$ | $a_{3m-3}$, $a_{3m-2}$, $a_{3m-1}$ | 1 |
| $a_{3m+2}$ | $a_{3m-3}$, $a_{3m-2}$, $a_{3m-1}$ | 1 |

The CPM network of a project of this form is shown below.




Again, every activity is critical, so any path from the beginning of the project to the end is a critical path. There are $m+1$ "layers" in this CPM network; the first layer contains two activities, and the remaining $m$ layers contain three activities each, so there are $2 \cdot 3^m$ critical paths. □

**3.** Consider a directed graph $G = (V, E)$ with nonnegative edge weights $c_{ij} \geq 0$ and specified nodes $s, t \in V$. For each node $i \in V$, let $\pi_i$ be the distance of a shortest (directed) path from $i$ to $t$. (Assume that every node $i$ has such a path to $t$.) Show that $\pi$ is an optimal feasible solution to the dual of the node-arc LP formulation for the shortest path problem on $G$ from $s$ to $t$. Is the assumption $c_{ij} \geq 0$ necessary?

▷ **Solution.** The dual of the node-arc LP formulation for the shortest path problem on $G$ from $s$ to $t$ is as follows:

$$
\begin{array}{rll}
\text{maximize} & \pi_s - \pi_t & \\
\text{subject to} & \pi_i - \pi_j \leq c_{ij} & \text{for all arcs } (i,j) \in E \\
& \text{all variables unrestricted.} &
\end{array}
$$

For each node $i \in V$, let $\pi_i$ be the shortest distance from $i$ to $t$. Then for each arc $(i, j) \in E$, the constraint $\pi_i \leq \pi_j + c_{ij}$ is satisfied, because the shortest distance from $i$ to $t$ cannot be greater than the shortest distance from $j$ to $t$ plus the length of the arc joining $i$ and $j$ (by the triangle inequality). In more detail, we know that there exists a directed path $(j = w_0, w_1, w_2, \ldots, w_k = t)$ from $j$ to $t$ of total distance $\sum_{p=0}^{k-1} c_{w_p w_{p+1}} = \pi_j$. So $(i, j, w_1, w_2, \ldots, w_{k-1}, t)$ is a directed walk from $i$ to $t$ of total distance $c_{ij} + \pi_j$. This implies that there is a directed path from $i$ to $t$ of total distance no greater than $c_{ij} + \pi_j$, which means that the *shortest* distance $\pi_i$ from $i$ to $t$ cannot be greater than this. Therefore, all of the constraints are satisfied by this solution $\pi$. Clearly the domains place no restrictions on $\pi$, so $\pi$ is a feasible solution.

Now consider any (directed) path $P = (s = v_0, v_1, v_2, \ldots, v_{l-1}, v_l = t)$ from $s$ to $t$. Add the corresponding constraints:

$$\begin{array}{rcl}
\pi_s - \pi_{v_1} & \le & c_{sv_1} \\
\pi_{v_1} - \pi_{v_2} & \le & c_{v_1v_2} \\
\pi_{v_2} - \pi_{v_3} & \le & c_{v_2v_3} \\
\ddots & & \vdots \\
\pi_{v_{l-1}} - \pi_t & \le & c_{v_{l-1}t} \\
\hline
\pi_s \qquad - \pi_t & \le & \displaystyle\sum_{p=0}^{l-1} c_{v_pv_{p+1}}.
\end{array}$$

The expression on the right-hand side of the resulting inequality is the length of the path $P$. Therefore, the objective value $\pi_s - \pi_t$ cannot be greater than the length of $P$. Since $P$ was an arbitrary path from $s$ to $t$, the value of $\pi_s - \pi_t$ cannot be greater than the length of *any* path from $s$ to $t$; in particular, it cannot be greater than the length of a *shortest* path from $s$ to $t$. So the solution $\pi$ given in this problem, in which $\pi_s$ is the length of a shortest path from $s$ to $t$ and $\pi_t$ is zero, must be optimal.

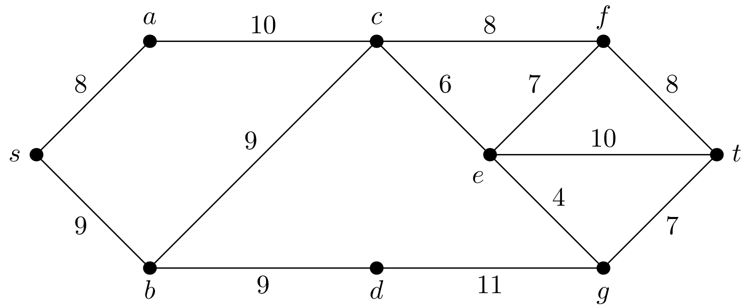
No part of the reasoning above used the assumption that $c_{ij} \ge 0$, so this assumption is not necessary. However, if there is a *cycle* with negative total weight $\big[$say, $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v_0)$ with $\sum_{i=0}^{k-1} c_{v_iv_{i+1}} = C < 0\big]$, then we can add the corresponding constraints to get

$$\begin{array}{rcl}
\pi_{v_0} - \pi_{v_1} & \le & c_{iv_1} \\
\pi_{v_1} - \pi_{v_2} & \le & c_{v_1v_2} \\
\pi_{v_2} - \pi_{v_3} & \le & c_{v_2v_3} \\
\ddots & & \vdots \\
\pi_{v_{k-2}} - \pi_{v_{k-1}} & \le & c_{v_{k-2}v_{k-1}} \\
-\pi_{v_0} \qquad + \pi_{v_{k-1}} & \le & c_{v_{k-1}v_0} \\
\hline
0 & \le & C,
\end{array}$$

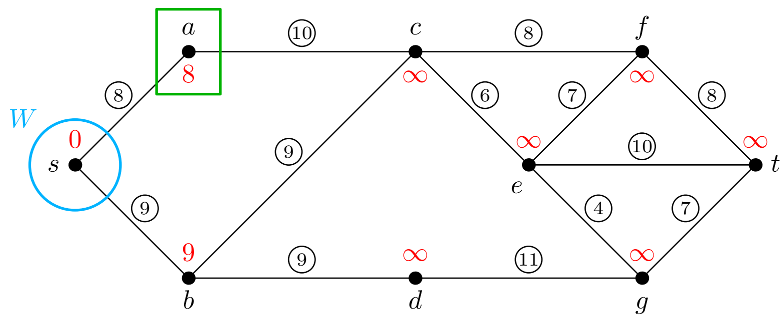
which is a contradiction, so the dual LP is infeasible. (Note that this is a consequence of the fact that the primal is unbounded.)

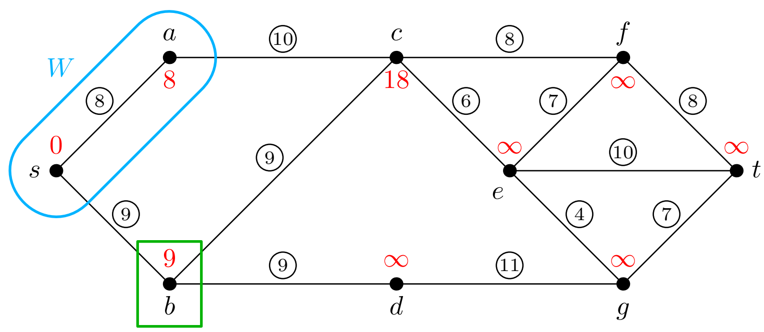
Therefore the statement in the problem is still true and meaningful when the assumption $c_{ij} \ge 0$ is removed, as long as there are no cycles with negative total weight. □

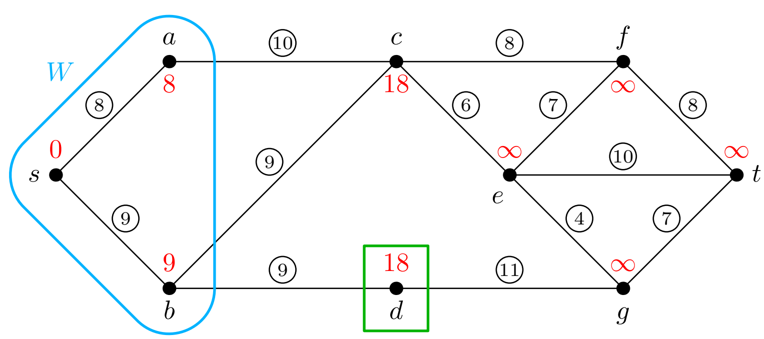
**4.** Use Dijkstra's algorithm (or the primal-dual algorithm) to find a shortest path from $s$ to $t$ in the following undirected graph.



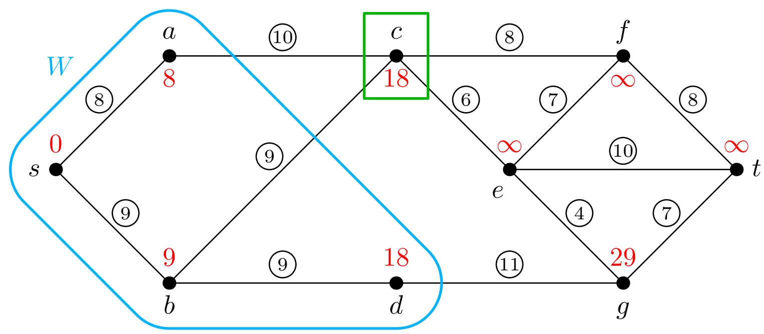
▷ **Solution.** We show the iterations of Dijkstra's algorithm in the figures below. In each figure, the edge weights are circled, the vertex labels (denoting the tentative distance from $s$) are shown in red, the current set $W$ of vertices whose distances from $s$ have definitely been determined is outlined in light blue, and the vertex $x$ outside $W$ with the smallest vertex label is boxed in green.







(Note: At this stage $c$ and $d$ tie for the minimum vertex label. Here we chose $x = d$ arbitrarily, but it is equally valid to choose $x = c$.)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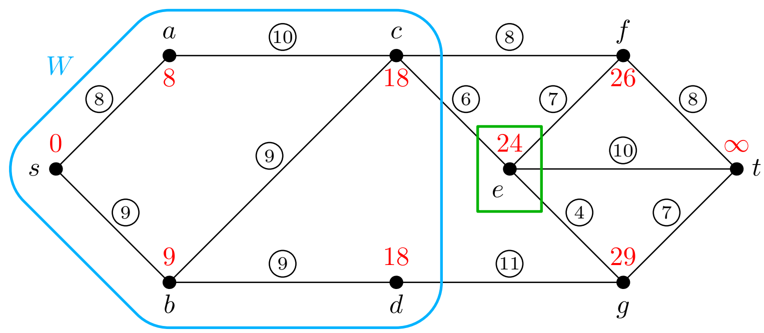
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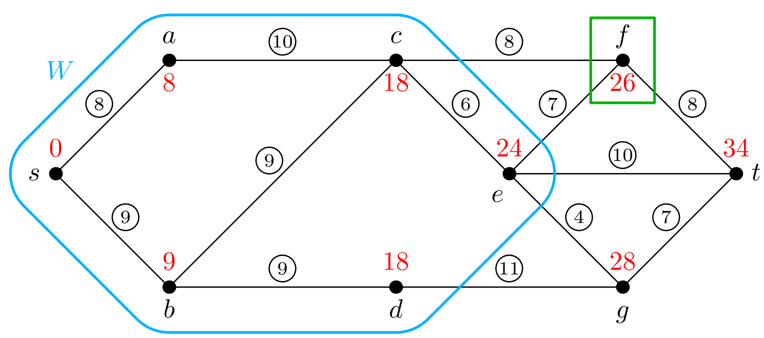
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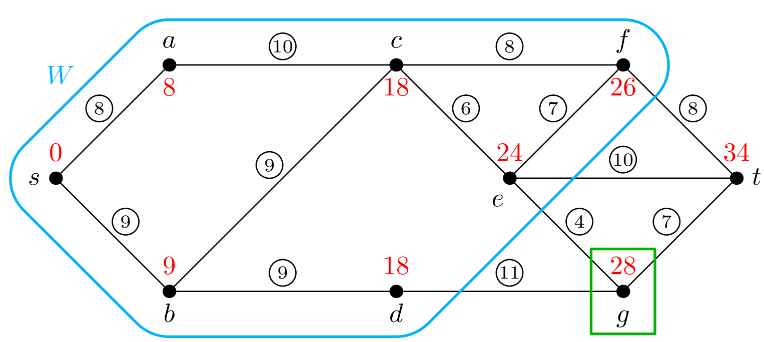
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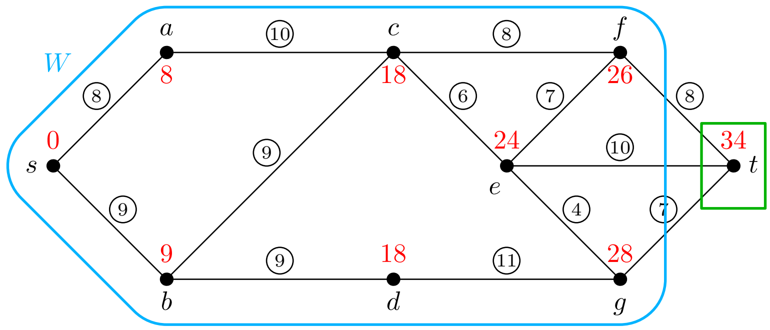
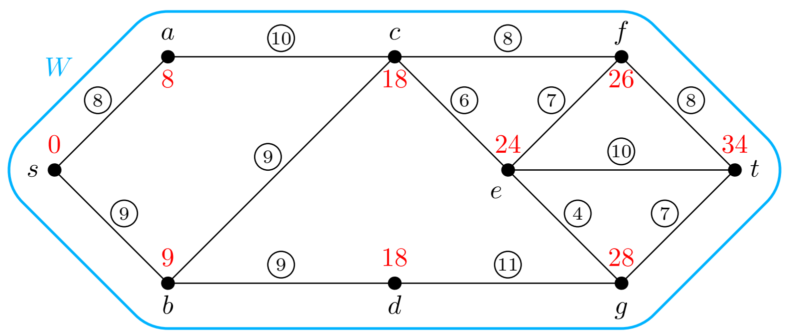
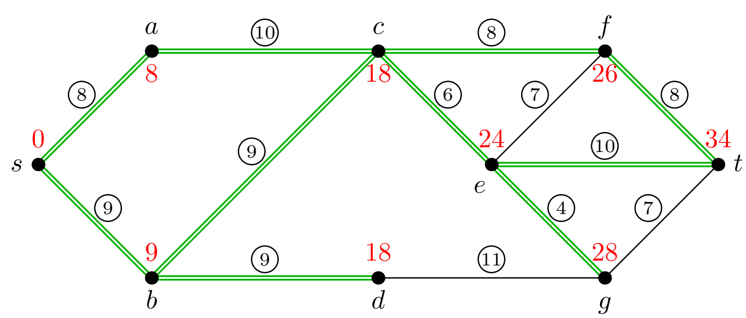


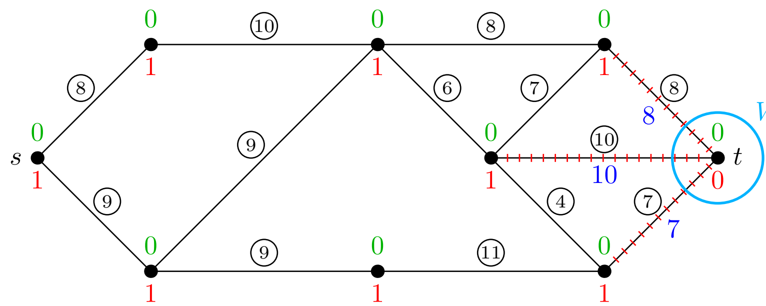




Now all vertices are included in $W$, so the labels give the shortest distance from $s$ for each vertex. To find a shortest path from $s$ to any vertex, we identify the admissible edges, which are those edges whose weight equals the difference of the vertex labels at their endpoints. The admissible edges are shown as double green lines in the figure below.

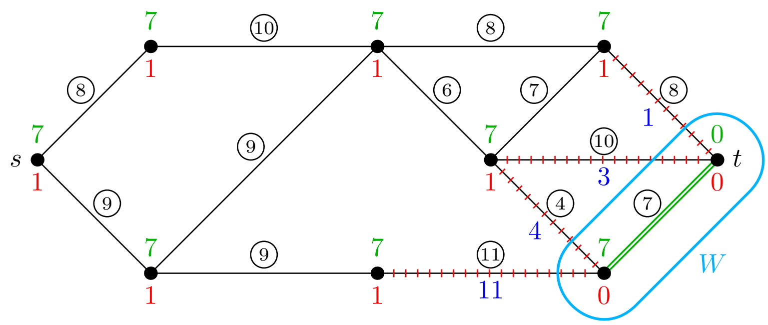



From this figure, we see that there are four shortest paths from $s$ to $t$: $s$–$a$–$c$–$e$–$t$, $s$–$a$–$c$–$f$–$t$, $s$–$b$–$c$–$e$–$t$, and $s$–$b$–$c$–$f$–$t$. Each of these paths has total weight 34.

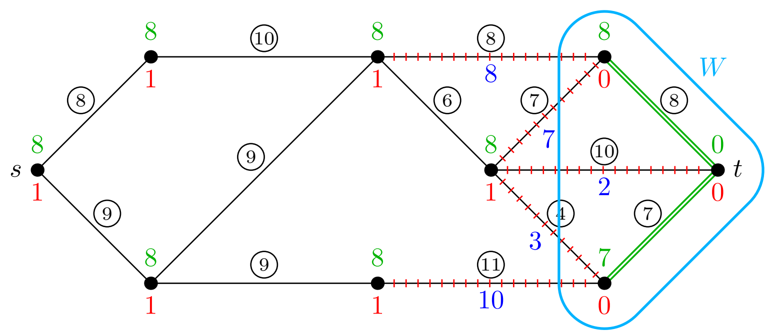
Alternatively, we can use the primal-dual algorithm directly. The figures below show the iterations of the primal-dual algorithm. In each figure, the values of the dual solution $\pi$ are written in green above each vertex, the admissible edges $J$ are drawn as double green lines, the set $W$ of vertices from which $t$ is reachable using only edges in $J$ is outlined in light blue, the values of the optimal solution $\bar{\pi}$ to the dual of the restricted primal are written in red below each vertex, the candidate edges $K$ are drawn with red hash marks, and the values of $c_{ij} - (\pi_i - \pi_j)$ are written in blue below the candidate edges. The value of $\theta_1$, which is the minimum value of $c_{ij} - (\pi_i - \pi_j)$ over all candidate edges, is shown to the lower right of the figure; this is the amount by which the dual values will be increased for all vertices not in $W$.



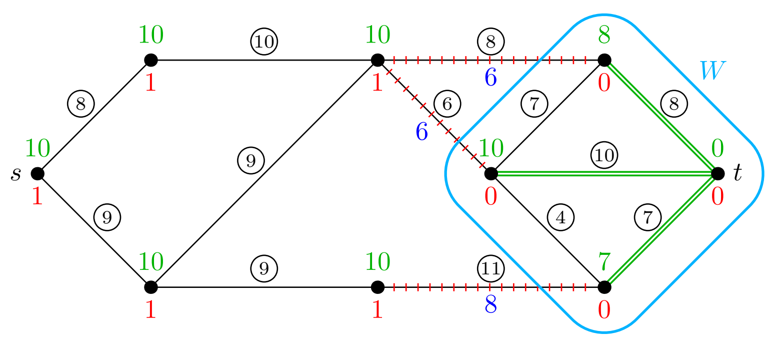

$\theta_1 = 7$



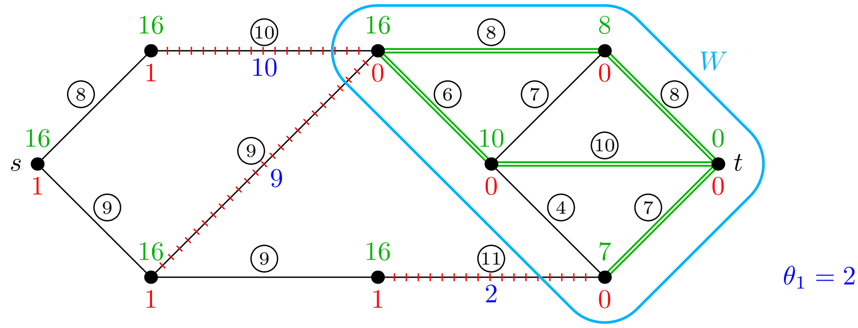

$\theta_1 = 1$



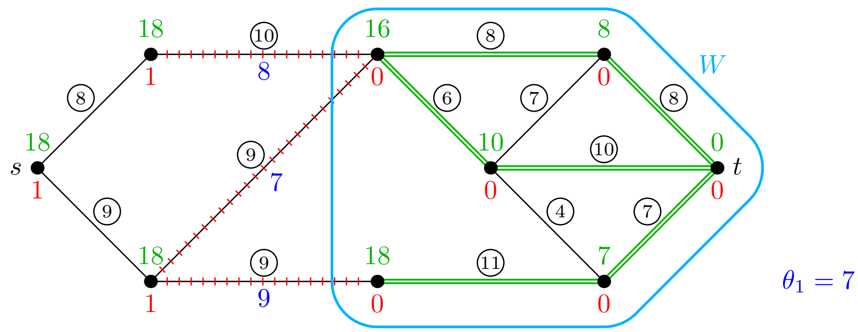

$\theta_1 = 2$



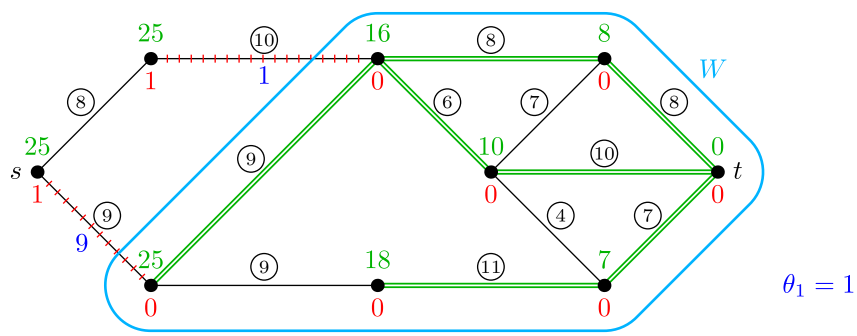

$\theta_1 = 6$



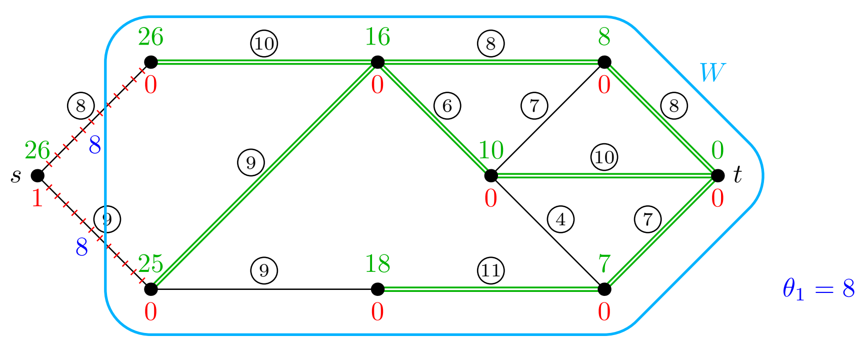
s
t
W
$\theta_1 = 2$

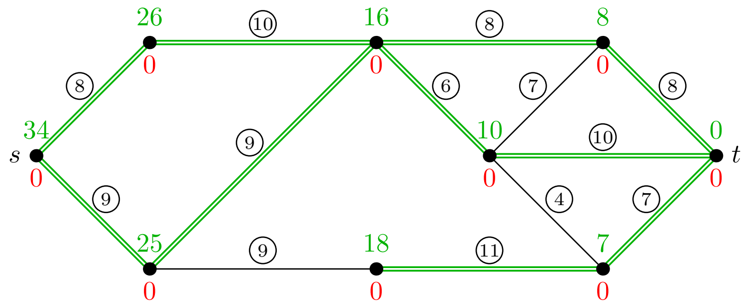
s
t
W
$\theta_1 = 7$

s
t
W
$\theta_1 = 1$

s
t
W
$\theta_1 = 8$

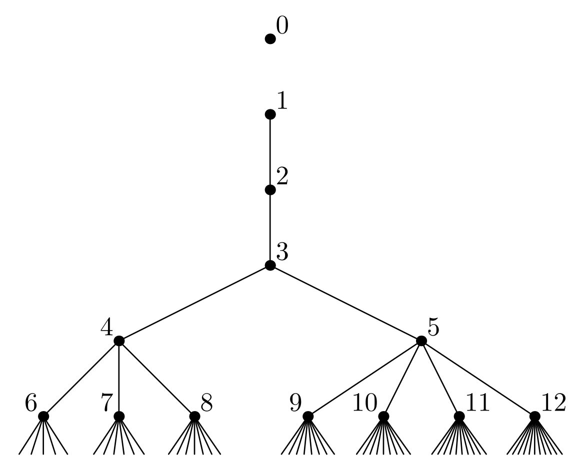



At this stage $t$ is reachable from $s$ using only edges in $J$, so the optimal solution to the dual of the restricted primal is $\bar{\pi} = 0$, which indicates that the current dual solution is optimal. Therefore, the shortest distance from $s$ to $t$ is 34, and any path from $s$ to $t$ using only admissible edges is a shortest path. Again we see that there are four such paths: $s$–$a$–$c$–$e$–$t$, $s$–$a$–$c$–$f$–$t$, $s$–$b$–$c$–$e$–$t$, and $s$–$b$–$c$–$f$–$t$. □

**5.** Give an example of a simple graph with at least two vertices such that no two vertices have the same degree, or explain why this is impossible.

▷ **Solution.** This is impossible for a finite graph. Suppose that $G = (V, E)$ is a simple finite graph with $|V| = n \geq 2$ such that no two vertices have the same degree. The possible vertex degrees in $G$ are 0, 1, 2, ..., $n-1$, so, since all $n$ vertices of $G$ have different degrees, each one of these possible degrees must occur exactly once. Therefore $G$ contains a vertex $u$ of degree 0 and also a vertex $v$ of degree $n-1$. These are not the same vertex, because $n \geq 2$ (so $n - 1 \neq 0$). Now the vertex $v$ must be adjacent to every other vertex, including $u$. On the other hand, the vertex $u$ cannot be adjacent to any other vertex, including $v$. This is a contradiction, so such a graph cannot exist.

However, it *is* possible if we consider graphs with infinitely many vertices. For example, here is a graph on the vertex set $\mathbb{N} = \{0, 1, 2, \ldots\}$ in which each vertex $n$ has degree $n$:




Ignoring the isolated vertex 0, this graph is a rooted tree on the vertex set $\mathbb{N}$ in which 1 is the root and has exactly one child, and every vertex $n$ other than 1 has exactly $n-1$ children.

Here is another construction of a graph on the vertex set $\mathbb{N}$ in which each vertex $n$ has degree $n$; this construction has a very close connection to the golden ratio. Let $\phi = (\sqrt{5} - 1)/2 = 0.618\ldots$ and $\Phi = (\sqrt{5} + 1)/2 = 1.618\ldots$. Note that $\phi$ and $\Phi$ are irrational, and they satisfy the equations $\Phi = \phi + 1$ and $\phi = 1/\Phi$. For $0 \leq x \in \mathbb{R}$, define

$$[\![x]\!] = \begin{cases} \lfloor x \rfloor, & \text{if } x - \lfloor x \rfloor < 1 - \phi; \\ \lceil x \rceil, & \text{if } x - \lfloor x \rfloor > 1 - \phi. \end{cases}$$

In other words, $[\![x]\!]$ rounds $x$ down if the fractional part of $x$ is less than $1-\phi$ and up if the fractional part is greater than $1-\phi$. For our purposes, we need not define what happens if the fractional part is equal to $1-\phi$, because we are going to be applying the function $[\![\cdot]\!]$ only to expressions of the form $\phi n$ and $\Phi n$ for $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ we have $\phi n - \lfloor \phi n \rfloor \neq 1-\phi$, for if $\phi n - \lfloor \phi n \rfloor = 1-\phi$ then $\lfloor \phi n \rfloor = \phi n + \phi - 1 = \phi(n+1) - 1$, and hence $\phi(n+1) \in \mathbb{Z}$, which cannot be true because $\phi \notin \mathbb{Q}$. Likewise, for all $n \in \mathbb{N}$ we have $\Phi n - \lfloor \Phi n \rfloor \neq 1-\phi$, for if $\Phi n - \lfloor \Phi n \rfloor = 1-\phi$ then $\lfloor \Phi n \rfloor = \Phi n + \phi - 1 = \Phi n + \Phi - 2 = \Phi(n+1) - 2$, and hence $\Phi(n+1) \in \mathbb{Z}$, which cannot be true because $\Phi \notin \mathbb{Q}$. Therefore, $[\![\phi n]\!]$ and $[\![\Phi n]\!]$ are well defined for all $n \in \mathbb{N}$.

Next we see that for $m, n \in \mathbb{N}$, $m \geq [\![\phi n]\!]$ if and only if $n \leq [\![\Phi m]\!]$. Note that $[\![x]\!] = k$ if and only if $k - \phi < x < k + 1 - \phi$. So

$$\begin{aligned} m \geq [\![\phi n]\!] &\iff \phi n < m + 1 - \phi \\ &\iff n < \Phi(m + 1 - \phi) \\ &\iff n < \Phi m + \Phi - 1 \\ &\iff n < \Phi m + \phi \\ &\iff n - \phi < \Phi m \\ &\iff n \leq [\![\Phi m]\!]. \end{aligned}$$

Now, for $n \in \mathbb{N}$, define the neighborhood of $n$ to be the set

$$N(n) = \big\{\, m \in \mathbb{N} : [\![\phi n]\!] \leq m \leq [\![\Phi n]\!] \,\big\} \setminus \{n\}.$$

Observe that $\Phi n = (1+\phi)n = n + \phi n$, so the fractional parts of $\phi n$ and $\Phi n$ are equal. Therefore the function $[\![\,\cdot\,]\!]$ rounds $\phi n$ and $\Phi n$ either both up or both down, which means that $[\![\Phi n]\!] - [\![\phi n]\!] = \Phi n - \phi n = (\Phi - \phi)n = n$. The endpoints of the interval $\big[\, [\![\phi n]\!], [\![\Phi n]\!] \,\big]$ are nonnegative integers, so this interval contains $n+1$ elements of $\mathbb{N}$, and then we remove $n$ to form $N(n)$. So $|N(n)| = n$.

Finally, for $m, n \in \mathbb{N}$, we have $m \in N(n)$ if and only if $n \in N(m)$, because for $m \neq n$,

$$m \in N(n) \iff [\![\phi n]\!] \leq m \leq [\![\Phi n]\!] \iff [\![\phi m]\!] \leq n \leq [\![\Phi m]\!] \iff n \in N(m).$$

Therefore, the relation $\sim$ defined on $\mathbb{N}$ by

$$m \sim n \qquad \text{if and only if} \qquad m \in N(n)$$

is symmetric (and antireflexive), so we may define a graph on the vertex set $\mathbb{N}$ by specifying that two vertices $m$ and $n$ are adjacent if and only if $m \in N(n)$. Since $|N(n)| = n$ for all $n$, each vertex $n$ has degree $n$. $\square$

**6.** Let $G = (V, E)$ be a simple undirected graph, and let $n = |V|$. Prove that all of the following statements are equivalent.

(a) $G$ is a tree (that is, $G$ is connected and acyclic).

(b) For any two distinct vertices $u, v \in V$, there exists a unique path in $G$ between $u$ and $v$.

(c) $G$ is minimally connected: $G$ is connected, but if any edge is removed from $G$ then the resulting graph is disconnected.

(d) $G$ is maximally acyclic: $G$ is acyclic, but if any edge is added joining nonadjacent vertices of $G$ then the resulting graph has a cycle.

(e) $G$ is connected and has $n - 1$ edges.

(f) $G$ is acyclic and has $n - 1$ edges.

▷ **Solution.** First we prove the following useful counting lemma.

**Vertices–edges–components lemma.** *If $G = (V, E)$ is a simple undirected graph with $n$ vertices and $m$ edges, then $G$ has at least $n - m$ connected components, with equality if and only if $G$ is acyclic.*

**Proof.** By induction on $m$. If $m = 0$, then $G$ consists of $n$ isolated vertices, so it has $n = n - m$ connected components, and clearly $G$ is acyclic.

Suppose $m \geq 1$. Let $e = \{u, v\} \in E$, and consider the graph $H = (V, E \setminus \{e\})$ formed from $G$ by removing the edge $e$. The graph $H$ has $m - 1$ edges, so by induction $H$ has at least $n - m + 1$ connected components, with equality if and only if $H$ is acyclic.

If $u$ and $v$ are in the same connected component in $H$, then $G$ has the same number of connected components as $H$, i.e., at least $n - m + 1$. Furthermore, there is a path $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v)$ in $H$ between $u$ and $v$, and $k \geq 2$ because $u$ and $v$ are not adjacent in $H$. So $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v, u)$ is a cycle in $G$, which means $G$ is not acyclic.

Otherwise, $u$ and $v$ are in different connected components in $H$, so when the edge $\{u, v\}$ is added to $H$ to form $G$, those two connected components are joined. Hence $G$ has exactly one fewer connected component than $H$, i.e., at least $n - m$ connected components. In this case, the edge $\{u, v\}$ cannot be part of a cycle in $G$ (because this would imply a path between $u$ and $v$ that does not use the edge $\{u, v\}$, which would be a path between $u$ and $v$ in $H$), so $G$ is acyclic if and only if $H$ is acyclic, and therefore $G$ has $n - m$ connected components if and only if $G$ is acyclic. ∎

Now we can begin proving implications among the six statements. More proofs are presented below than are necessary to prove the equivalence of the six statements; all that is necessary is to prove a set of implications such that any statement can be reached from any other statement by following a chain of implications in the set.

(a) $\Longrightarrow$ (b). Suppose $G$ is a tree, i.e., is connected and acyclic. Let $u, v \in V$ be distinct vertices. Then there exists a path between $u$ and $v$, because $G$ is connected. Suppose for the sake of contradiction that this path is not unique, i.e., that there exist two distinct paths $(u = w_0, w_1, w_2, \ldots, w_{p-1}, w_p = v)$ and $(u = x_0, x_1, x_2, \ldots, x_{q-1}, x_q = v)$ for some $p, q \geq 1$. Let $r \geq 1$ be the least positive integer such that $w_r \neq x_r$; such an integer exists because the two paths are distinct (in particular, $1 \leq r \leq \min\{p, q\}$). Let $s \geq r$ be the least integer greater than or equal to $r$ such that $w_s = x_t$ for some $t$; such an integer exists because $w_p = v = x_q$. Then $(w_{r-1}, w_r, w_{r+1}, \ldots, w_{s-1}, w_s = x_t, x_{t-1}, \ldots, x_{r-1} = w_{r-1})$ is a cycle, because $w_r$, $w_{r+1}$, …, $w_{s-1}$ are distinct from all $x_i$'s by the definition of $s$, so no vertices are repeated in this walk (except the beginning and end, $w_{r-1} = x_{r-1}$). Also, the length of this walk is at least 3: we have $s \geq r$, so there are at least two distinct $w_i$'s in the walk; we have $x_t = w_s$, so $x_t \neq w_i = x_i$ for $0 \leq i \leq r - 1$, so there are at least two distinct $x_i$'s in the walk; and $w_r \neq x_r$, so either the walk contains at least three distinct $w_i$'s or it contains at least three distinct $x_i$'s (or both). So $G$ contains a cycle, which is a contradiction. Therefore, for any two distinct vertices $u, v \in V$, there exists a unique path in $G$ between $u$ and $v$.

(a) $\implies$ (c). Suppose $G$ is a tree. Then certainly $G$ is connected. Let $e = \{u, v\} \in E$ be any edge in $G$. Consider the graph $H = (V, E \setminus \{e\})$ formed from $G$ by removing the edge $e$. If there exists a path $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v)$ in $H$ between $u$ and $v$, then its length $k$ must be at least 2 because $\{u, v\}$ is not an edge in $H$; so $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v, u)$ is a cycle in $G$. But this contradicts the fact that $G$ is acyclic. So there can be no path in $H$ between $u$ and $v$, which means that $H$ is disconnected. As the edge $e \in E$ was arbitrary, this shows that $G$ is minimally connected.

(a) $\implies$ (d). Suppose $G$ is a tree. Then certainly $G$ is acyclic. If $G$ has no two nonadjacent vertices, then statement (d) is vacuously true, and there is nothing more to show, so suppose this is not the case. Let $u$ and $v$ be any two nonadjacent vertices in $G$, and consider the graph $H = \big(V, E \cup \big\{\{u, v\}\big\}\big)$ formed from $G$ by adding the edge $\{u, v\}$. Since $G$ is connected, there exists a path $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v)$ in $G$ between $u$ and $v$, and $k \geq 2$ because $u$ and $v$ are not adjacent in $G$. So $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v, u)$ is a cycle in $H$. As $u$ and $v$ were arbitrary nonadjacent vertices in $G$, this shows that $G$ is maximally acyclic.

(a) $\implies$ (e). Suppose $G$ is a tree. Then certainly $G$ is connected. Hence $G$ has exactly one connected component and is acyclic, so by the vertices–edges–components lemma $G$ must have $n - 1$ edges.

(a) $\implies$ (f). Suppose $G$ is a tree. Then certainly $G$ is acyclic. And $G$ has exactly one connected component because it is connected, so by the vertices–edges–components lemma $G$ must have $n - 1$ edges.

The following lemma is used many times to prove the implications in which statement (b) is the hypothesis.

**Lemma.** [Unique paths imply acyclicity.] *If $G = (V, E)$ is a simple undirected graph such that for any two distinct vertices $u, v \in V$ there exists a unique path in $G$ between $u$ and $v$, then $G$ is acyclic.*

**Proof.** Suppose for the sake of contradiction that $G$ contains a cycle $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v_0)$, where $k \geq 3$. Then there are two distinct paths $(v_0, v_1, v_2, \ldots, v_{k-1})$ and $(v_0 = v_k, v_{k-1})$ between $v_0$ and $v_{k-1}$, which is a contradiction. So $G$ is acyclic. ∎

(b) $\implies$ (a). Suppose that for any two distinct vertices $u, v \in V$ there exists a unique path in $G$ between $u$ and $v$. Then $G$ is connected, and by the preceding lemma $G$ is also acyclic.

(b) $\implies$ (c). Suppose that for any two distinct vertices $u, v \in V$ there exists a unique path in $G$ between $u$ and $v$. Then $G$ is connected. Suppose an edge $e = \{u, v\} \in E$ is removed from $G$ to form the graph $H = (V, E \setminus \{e\})$. In $G$, $(u, v)$ was the unique path between $u$ and $v$; this path does not exist in $H$. Any path in $H$ is also a path in $G$, so in $H$ there can be no path between $u$ and $v$, so $H$ is not connected. As the edge $e \in E$ was arbitrary, this shows that the removal of any edge from $G$ produces a disconnected graph. So $G$ is minimally connected.

(b) $\implies$ (d). Suppose that for any two distinct vertices $u, v \in V$ there exists a unique path in $G$ between $u$ and $v$. By the preceding lemma, $G$ is acyclic. If $G$ has no two nonadjacent vertices, then statement (d) is vacuously true, and there is nothing more to show, so suppose this is not the case. Let $u$ and $v$ be any two nonadjacent vertices in $G$, and consider the graph $H = \big(V, E \cup \big\{\{u, v\}\big\}\big)$ formed from $G$ by adding the edge $\{u, v\}$. By assumption, there exists a unique path $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v)$ in $G$ between $u$ and $v$, and the length $k$ of this path must be at least 2 because $\{u, v\}$ is not an edge in $G$. So $(u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v, u)$ is a cycle in $H$. As $u$ and $v$ were arbitrary nonadjacent vertices in $G$, this shows that $G$ is maximally acyclic.

(b) $\implies$ (e). Suppose that for any two distinct vertices $u, v \in V$ there exists a unique path in $G$ between $u$ and $v$. Then $G$ is connected, i.e., it has exactly one connected component. By the preceding lemma, $G$ is acyclic, so by the vertices–edges–components lemma it must have $n - 1$ edges.

(b) $\Longrightarrow$ (f). Exactly the same proof as for (b) $\Longrightarrow$ (e).

A similar lemma is useful to prove the implications in which statement (c) is the hypothesis.

**Lemma.** [Minimal connectivity implies acyclicity.] *If $G = (V, E)$ is a simple undirected minimally connected graph, then $G$ is acyclic.*

**Proof.** Suppose for the sake of contradiction that $G$ has a cycle $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v_0)$ of length $k$, where $k \geq 3$. Let $e$ be the edge $\{v_0, v_1\}$, and consider the graph $H = (V, E \setminus \{e\})$ formed from $G$ by removing the edge $e$. We claim that $H$ is still connected: Let $u, v \in V$ be distinct vertices. Since $G$ is connected, there exists a path $P = (u = w_0, w_1, w_2, \ldots, w_{p-1}, w_p = v)$ of length $p$ in $G$ between $u$ and $v$. If $P$ does not use the edge $\{v_0, v_1\}$, then it is still a path in $H$. Otherwise, let $q \geq 0$ be the least nonnegative integer such that $w_q \in \{v_0, v_1, v_2 \ldots, v_{k-1}\}$, and let $r \leq p$ be the greatest integer no more than $p$ such that $w_r \in \{v_0, v_1, v_2, \ldots, v_{k-1}\}$. We must have $q < r$, because $P$ contains both $v_0$ and $v_1$. Say $w_q = v_s$ and $w_r = v_t$, for some $0 \leq s < k$ and $0 \leq t < k$. We must have $s \neq t$ because $q \neq r$ and the path $P$ contains no repeated vertices. Without loss of generality, assume that $s < t$. If $s \neq 0$, then

$$(u = w_0, w_1, \ldots, w_{q-1}, w_q = v_s, v_{s+1}, \ldots, v_{t-1}, v_t = w_r, w_{r+1}, \ldots, w_{p-1}, w_p = v)$$

is a path in $H$ between $u$ and $v$. Otherwise, $s = 0$, and

$$(u = w_0, w_1, \ldots, w_{q-1}, w_q = v_s = v_0 = v_k, v_{k-1}, \ldots, v_{t+1}, v_t = w_r, w_{r+1}, \ldots, w_{p-1}, w_p = v)$$

is a path in $H$ between $u$ and $v$. As $u$ and $v$ were arbitrary vertices, this shows that $H$ is connected. Therefore, $G$ is not minimally connected, which is a contradiction. So $G$ cannot have a cycle, which means that it is acyclic. ∎

(c) $\Longrightarrow$ (a). Suppose $G$ is minimally connected. Then certainly $G$ is connected. By the preceding lemma, $G$ is also acyclic, and hence a tree.

(c) $\Longrightarrow$ (d). Suppose $G$ is minimally connected. Then it is connected, so it has exactly one connected component, and by the preceding lemma it is acyclic. By the vertices–edges–components lemma, then, $G$ has $n-1$ edges. If an edge is added joining nonadjacent vertices of $G$, then the resulting graph $H$ must still be connected (because adding an edge cannot disconnect a graph), and $H$ has $n$ edges, so by the vertices–edges–components lemma it cannot be acyclic. Therefore $G$ is maximally acyclic.

(c) $\Longrightarrow$ (e). Suppose $G$ is minimally connected. Then it is connected, so it has exactly one connected component, and by the preceding lemma it is acyclic. By the vertices–edges–components lemma, then, $G$ has $n - 1$ edges.

(c) $\Longrightarrow$ (f). Exactly the same proof as for (c) $\Longrightarrow$ (e).

The next lemma is useful when proving the implications in which statement (d) is the hypothesis.

**Lemma.** [Maximal acyclicity implies connectivity.] *If $G = (V, E)$ is a simple undirected maximally acyclic graph, then $G$ is connected.*

**Proof.** Let $u, v \in V$ be two distinct vertices in $G$. If $u$ and $v$ are adjacent, then $(u, v)$ is a path in $G$ from $u$ to $v$. Otherwise, because $G$ is maximally acyclic, adding the edge $e = \{u, v\}$ to $G$ produces the graph $H = (V, E \cup \{e\})$ and $H$ has a cycle $(w_0, w_1, w_2, \ldots, w_{k-1}, w_k = w_0)$. This cycle must include the edge $e$, for otherwise it would be a cycle in $G$, but $G$ is acyclic. Without loss of generality, we may assume that $w_0 = u$ and $w_1 = v$. Then $(u = w_0 = w_k, w_{k-1}, w_{k-2}, \ldots, w_2, w_1 = v)$ is a path from $u$ to $v$ in $G$. As $u$ and $v$ were arbitrary vertices in $G$, this shows that every two distinct vertices in $G$ have a path between them, which is to say, $G$ is connected. ∎

(d) $\implies$ (a). Suppose $G$ is maximally acyclic. Then certainly $G$ is acyclic. By the preceding lemma, $G$ is also connected, and hence a tree.

(d) $\implies$ (e). Suppose $G$ is maximally acyclic. Then certainly $G$ is acyclic, and by the preceding lemma, $G$ is connected, i.e., it has exactly one connected component. Therefore, by the vertices–edges–components lemma, $G$ has $n-1$ edges.

(d) $\implies$ (f). Exactly the same proof as for (d) $\implies$ (e).

(e) $\implies$ (a). Suppose $G$ is connected and has $m = n-1$ edges. Then $G$ has exactly one connected component, and $1 = n-m$, so by the vertices–edges–components lemma, $G$ must be acyclic.

(e) $\implies$ (c). Suppose $G$ is connected and has $n-1$ edges. If any edge is removed from $G$, then the resulting graph $H$ has $n-2$ edges, so by the vertices–edges–components lemma $H$ has at least two connected components, which is to say, $H$ is disconnected. Therefore $G$ is minimally connected.

(e) $\implies$ (d). Suppose $G$ is connected and has $m = n-1$ edges. Then $G$ has exactly one connected component, and $1 = n-m$, so by the vertices–edges–components lemma, $G$ must be acyclic. If any edge is added joining nonadjacent vertices of $G$, then the resulting graph $H$ is still connected (because adding an edge cannot disconnect a graph), so $H$ still has exactly one connected component, but $H$ has $n$ edges, so by the vertices–edges–components lemma, $H$ cannot be acyclic. Therefore $G$ is maximally acyclic.

(e) $\implies$ (f). Exactly the same proof as for (e) $\implies$ (a).

(f) $\implies$ (a). Suppose $G$ is acyclic and has $n-1$ edges. By the vertices–edges–components lemma, $G$ has exactly one connected component, i.e., $G$ is connected.

(f) $\implies$ (c). Suppose $G$ is acyclic and has $n-1$ edges. By the vertices–edges–components lemma, $G$ has exactly one connected component, i.e., $G$ is connected. If any edge is removed from $G$, then the resulting graph $H$ has $n-2$ edges, so by the vertices–edges–components lemma $H$ has at least two connected components, which is to say, $H$ is disconnected. Therefore $G$ is minimally connected.

(f) $\implies$ (d). Suppose $G$ is acyclic and has $n-1$ edges. If any edge is added joining nonadjacent vertices of $G$, then the resulting graph $H$ will have $n$ edges. If $H$ is acyclic, then by the vertices–edges–components lemma it has zero connected components, which is impossible. Thus $H$ has a cycle, and therefore $G$ is maximally acyclic.

(f) $\implies$ (e). Exactly the same proof as for (f) $\implies$ (a). $\square$