Combinatorial Optimization

Problem set 3

Assigned Thursday, June 4, 2015. Due Monday, June 8, 2015.

For your convenience, a tool to help with pivoting simplex tableaux is available online at `http://www.math.cmu.edu/~bkell/pivot.html`.

1. In class (and on the "Analysis of a simplex tableau" handout), I claimed that if a simplex tableau (for a non-degenerate linear program) contains a column having a negative entry in the objective row and no positive entries below, then the linear program is unbounded. Prove this claim.

2. Here is a shortcut for determining the entries in the artificial objective row in an initial (two-phase) simplex tableau:

1. Fill in the entries in the objective row and the rows that come from the constraints.
2. In the columns for artificial variables, enter zeroes in the artificial objective row. (Also, if you are including the $-\xi$ column, enter 1 in that column in the artificial objective row.)
3. In every other column, compute the sum of the entries in the rows that come from constraints having artificial variables (i.e., the rows that come from $\geq$ and $=$ constraints). Negate this sum and enter it in the artificial objective row.

Justify this shortcut.

3. Solve the following linear program by hand, using the two-phase simplex algorithm.

$$\begin{aligned} \text{maximize} \quad & 3x_1 - 8x_2 + 10x_3 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 \leq 40 \\ & 5x_1 - x_3 \geq 10 \\ & 2x_1 - x_2 + x_3 = 12 \\ & x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ unrestricted.} \end{aligned}$$

4. In the description of the two-phase simplex algorithm in class, I omitted one possibility that may occur at the end of Phase I: the value of ξ is 0, but at least one artificial variable remains in the basis (having the value 0). If this happens, then we need to "drive the artificial variable out of the basis" in order to get a basis consisting solely of non-artificial variables, so that we can begin Phase II. Papadimitriou and Steiglitz discuss this case (which they call "Case 3") in Section 2.8, on page 56. Give an example of a linear program for which this case occurs, and go through the full two-phase simplex algorithm to solve your example.

5. Write the dual of the following linear program.

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 \geq 0 \\ & 4x_1 + 3x_2 - 4x_3 - 2x_4 \leq 3 \\ & -x_1 - x_2 - 2x_3 + x_4 = 1 \\ & x_1 \text{ unrestricted}, \quad x_2 \geq 0, \quad x_3 \leq 0, \quad x_4 \text{ unrestricted.} \end{aligned}$$

The optimal solution of the linear program above has $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 7/2$. Use complementary slackness to determine the optimal solution to the dual. Verify that the two solutions are both feasible for their respective linear programs and that they have the same objective value.

6. Describe (at least) two essentially different ways to use the (maximizing) simplex algorithm to solve a minimization linear program. What are the comparative advantages and disadvantages of each? Given a minimization linear program, what characteristics would indicate that one method or the other may be a better approach?