IP formulation examples
June 24, 2015

1. **Machine shop reopening.** A small machine shop is reopening after a fire had forced it to close for extensive repairs. The shop has three product lines: plates, gears, and housings. Each product line requires specialized equipment, and because of inactivity and possible damage all equipment must be serviced before it is used.

   The shop plans to open on a limited basis for the first two weeks, employing only three workers, each for 40 hours per week. It has available 2800 units of metal and can purchase additional metal for \$2 per unit. The per-unit labor, metal, overhead costs, and selling prices for their products are shown below.

	Labor (minutes)	Metal (units)	Overhead (dollars)	Selling price (dollars)
Plates	10	4	6	24
Gears	30	1	9	32
Housings	20	6	8	30

   The existing backlog of orders for gears includes mostly orders for large quantities. Therefore, management does not believe that it would be useful to make gears during the first two weeks unless the shop can produce at least 200 of them.

   The servicing costs are \$600 for the plate equipment, \$900 for the gear equipment, and \$700 for the housing equipment. The shop does not expect to use all equipment in the first two weeks.

   Management has \$2000 remaining from its fire insurance settlement, and plans to spend that sum on the necessary service and possibly additional metal stock. The large backlog of orders that accumulated while the shop was closed indicates that they can sell any products they make. The overhead is charged against the selling price.

   Management's goal for the first two weeks is to maximize the profit so that they can afford to reopen full operations as quickly as possible.

2. **Utility connections.** Five major electrical consumers A, B, C, D, and E (e.g., manufacturing plants, hospitals, and housing developments) are to be added in a region served by three power plants X, Y, and Z. The connection costs to the new consumers (in millions of dollars) are given in the table below. The objective is to connect the new consumers to the generating plants in the most economical way possible.

	A	B	C	D	E
X	2	2	3	1	8
Y	3	7	2	6	4
Z	5	4	4	3	6

   The needs of the new consumers are 12, 10, 15, 16, and 15, respectively. The available capacities of the generating plants are 40, 32, and 30, respectively.

   Two of the new consumers, A and B, are hospitals. In order to lessen the possibility that both hospitals could be without power simultaneously, they cannot both be connected to the same power plant.

   Determine which new consumers should be connected to which power plant in order to minimize the connection costs.

3. **Product introduction.** Esquire Products will produce four new product lines in the next month. The respective per-unit profits on the lines are \$200, \$220, \$185, and \$190. They are basically testing the market and do not wish to produce more than 700 of any one line. The respective fixed start-up costs for the products are \$4000, \$5000, \$3000, and \$3500. Each item produced will require a part called an autorhombulator. The supplier of the autorhombulators charges according to the following schedule:

$$\begin{cases} \$50 & \text{ordering charge,} \\ \$9 & \text{each for the first 100 units,} \\ \$6 & \text{each for all additional units.} \end{cases}$$

   Esquire has budgeted \$20,500 for the start-up costs and the purchase of the autorhombulators. Product lines 1 and 2 require a half hour of production time per item while lines 3 and 4 require 0.4 hour per item. There will be 800 hours of production time available during the month. How many of each line should they produce in order to maximize their profit?

4. **Boolean satisfiability.** Determine truth values for the Boolean variables x_1, x_2, and x_3 so that the propositional formula

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3)$$

   is satisfied (i.e., evaluates to true), or determine that the formula is unsatisfiable. Here $\wedge$ denotes AND, $\vee$ denotes OR, and $\bar{x}_i$ denotes NOT x_i.

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Machine shop reopening

Variables and domains:

$p \geq 0$: number of plates to produce
$g \geq 0$: number of gears to produce
$h \geq 0$: number of housings to produce
$m \geq 0$: number of units of additional metal to buy
$s_p \in \{0,1\}$: whether to service plate equipment
$s_g \in \{0,1\}$: whether to service gear equipment
$s_h \in \{0,1\}$: whether to service housing equipment

Objective: Maximize profit (selling price less overhead).

$$\max\ 18p + 23g + 22h$$

Constraints:

— Resource constraints for labor and metal.

$$10p + 30g + 20h \leq 14400 \quad [\text{labor}]$$

$\nwarrow$ $3 \times (40\ \text{hr/wk}) \times (2\ \text{wk}) \times (60\ \text{min/hr})$

$$4p + g + 6h \leq 2800 + m \quad [\text{metal}]$$

— Budget for servicing costs and additional metal.

$$600 s_p + 900 s_g + 700 s_h + 2m \leq 2000 \quad [\text{budget}]$$

— Must service equipment in order to use it.

$$p \leq 10000\, s_p$$
$$g \leq 10000\, s_g$$
$$h \leq 10000\, s_h$$

[If $s_p = 0$, then p must be 0. If $s_p = 1$, then $p \leq 10000$, which is sufficiently large not to constrain p.]

— If any gears are made, then at least 200 must be made.

$$g \geq 200\, s_g \qquad \left[\begin{array}{l} \text{If } s_g = 0, \text{ this is } \overbrace{g \geq 0}^{\text{no constraint really}}. \\ \text{If } s_g = 1, \text{ this is } g \geq 200. \end{array}\right]$$

(This is a <u>minimum batch size</u> constraint.)

Full formulation:

$$\begin{array}{lll}
\max & 18p + 23g + 22h & \text{[profit]} \\
\text{s.t.} & 10p + 30g + 20h \leq 14400 & \text{[labor]} \\
& 4p + g + h \leq 2800 + m & \text{[metal]} \\
& 600\, s_p + 900\, s_g + 700\, s_h + 2m \leq 2000 & \text{[budget]} \\
& \left.\begin{array}{l} p \leq 10000\, s_p \\ g \leq 10000\, s_g \\ h \leq 10000\, s_h \end{array}\right\} & \text{[service requirements]} \\
& g \geq 200\, s_g & \text{[min batch size]} \\
& p \geq 0,\ g \geq 0,\ h \geq 0,\ m \geq 0,\ s_p \in \{0,1\},\ s_g \in \{0,1\}, & \\
& s_h \in \{0,1\}. &
\end{array}$$

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Utility connections

Variables and domains:

For $i \in \{A,B,C,D,E\}$ and $j \in \{X,Y,Z\}$:

$x_{ij} \in \{0,1\}$: whether to connect consumer i to plant j.

Objective: Minimize total connection cost.

$$\begin{aligned} \min\ & 2x_{AX} + 2x_{BX} + 3x_{CX} + x_{DX} + 8x_{EX} \\ & + 3x_{AY} + 7x_{BY} + 2x_{CY} + 6x_{DY} + 4x_{EY} \\ & + 5x_{AZ} + 4x_{BZ} + 4x_{CZ} + 3x_{DZ} + 6x_{EZ} \end{aligned}$$

Constraints:

– Total demand connected to each plant cannot exceed available capacity.

$$12x_{AX} + 10x_{BX} + 15x_{CX} + 16x_{DX} + 15x_{EX} \le 40 \quad [\text{capacity of } X]$$
$$12x_{AY} + 10x_{BY} + 15x_{CY} + 16x_{DY} + 15x_{EY} \le 32 \quad [\text{capacity of } Y]$$
$$12x_{AZ} + 10x_{BZ} + 15x_{CZ} + 16x_{DZ} + 15x_{EZ} \le 30 \quad [\text{capacity of } Z]$$

– Every consumer must be connected to (exactly) one plant.

$$x_{AX} + x_{AY} + x_{AZ} = 1$$
$$x_{BX} + x_{BY} + x_{BZ} = 1$$
$$x_{CX} + x_{CY} + x_{CZ} = 1$$
$$x_{DX} + x_{DY} + x_{DZ} = 1$$
$$x_{EX} + x_{EY} + x_{EZ} = 1$$

- A and B cannot be connected to the same plant.

$$X_{AX} + X_{BX} \leq 1$$
$$X_{AY} + X_{BY} \leq 1$$
$$X_{AZ} + X_{BZ} \leq 1$$

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Product introduction

Variables and domains:

$$\left.\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \\ x_4 \geq 0 \end{array}\right\}$$ Numbers of units of each of the four lines to produce.

$$\left.\begin{array}{l} s_1 \in \{0,1\} \\ s_2 \in \{0,1\} \\ s_3 \in \{0,1\} \\ s_4 \in \{0,1\} \end{array}\right\}$$ Whether to pay start-up costs for each of the four lines.

$a \geq 0$: Number of autorhombulators to buy.

[... and some other, less obvious variables later.]

Objective: Maximize profit.

$$\max\ 200x_1 + 220x_2 + 185x_3 + 190x_4$$

Constraints:

— Production limit: No more than 700 of any one line.

$$x_1 \leq 700$$
$$x_2 \leq 700$$
$$x_3 \leq 700$$
$$x_4 \leq 700$$

— Resource constraints for production time and autorhombulators.

$$0.5x_1 + 0.5x_2 + 0.4x_3 + 0.4x_4 \le 800 \quad \text{[prod. time]}$$

$$x_1 + x_2 + x_3 + x_4 \le a \quad \text{[autorhombulators]}$$

— Must pay start-up costs in order to produce each line.

$$x_1 \le 10000\, s_1$$
$$x_2 \le 10000\, s_2$$
$$x_3 \le 10000\, s_3$$
$$x_4 \le 10000\, s_4$$

(Actually these can be combined with the production limit constraints from earlier: $x_1 \le 700\, s_1$, etc. Then both sets of requirements are handled together.)

— Autorhombulator purchase cost.
This is tricky, because this cost is not linear — it is <u>piecewise linear.</u> Here is how to handle a piecewise linear cost.

First, write a piecewise-defined mathematical function giving the cost of purchasing $a \ge 0$ autorhombulators.

<u>Case 1:</u> $a = 0$. Easy, cost is \$0.

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Case 2: $1 \le a \le 100$. We have to pay the \$50 ordering charge, plus \$9 for each unit. So the cost is $50 + 9a$.

Case 3: $a \ge 100$. We have to pay the \$50 ordering charge, plus \$900 to buy the first 100 units, plus \$6 for each of the other $a-100$ units. So the cost is $950 + 6(a-100) = 350 + 6a$.

Therefore, the cost $C(a)$ of purchasing $a \ge 0$ autorhombulators is

$$C(a) = \begin{cases} 0, & \text{if } a = 0; & \text{[case 1]} \\ 50 + 9a, & \text{if } 1 \le a \le 100; & \text{[case 2]} \\ 350 + 6a, & \text{if } a \ge 100. & \text{[case 3]} \end{cases}$$

Note that cases 2 and 3 overlap at $a = 100$, but that's OK because both formulas give the same value there.

We introduce $\{0,1\}$ variables representing which case we are in.

$c_1 \in \{0,1\}$: whether we are in case 1
$c_2 \in \{0,1\}$: whether we are in case 2
$c_3 \in \{0,1\}$: whether we are in case 3

We also introduce variables a_1, a_2, a_3, which will be the value of a if we are in the corresponding case, or zero otherwise.

$a_1 \geq 0$: the value of a if we are in case 1, or else zero.

$a_2 \geq 0$: the value of a if we are in case 2, or else zero.

$a_3 \geq 0$: the value of a if we are in case 3, or else zero.

[Note: these are not artificial variables. ☺]

Now we need some constraints.

We must be in exactly one case:

$$c_1 + c_2 + c_3 = 1$$

We need to enforce the intended meanings of the a_i's, and put them in the correct ranges:

$$0c_1 \leq a_1 \leq 0c_1$$

(This is pretty silly for this particular example, because case 1 is $0 \leq a \leq 0$.)

$$1c_2 \leq a_2 \leq 100c_2$$

(Case 2: $1 \leq a \leq 100$)

$$100c_3 \leq a_3 \leq \underbrace{10000c_3}$$

(Case 3: $100 \leq a \leq 10000$)

This part is necessary in order to force a_3 to be zero if $c_3 = 0$, but it shouldn't constrain a_3 if $c_3 = 1$, so the coefficient 10000 is chosen to be sufficiently large that $a \leq 10000$ will never be a serious restriction.

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We need to tie a_1, a_2, a_3 to the previously defined variable a:

$$a = a_1 + a_2 + a_3$$

Now the expression for the cost $C(a)$ is

$$\underbrace{0c_1}_{\text{case 1}} + \underbrace{50c_2 + 9a_2}_{\text{case 2}} + \underbrace{350c_3 + 6a_3}_{\text{case 3}}.$$

So the budget constraint for start-up and autorhombulator purchase costs is

$$\begin{aligned} &4000 s_1 + 5000 s_2 + 3000 s_3 + 3500 s_4 \\ &\quad + 50 c_2 + 9a_2 + 350 c_3 + 6a_3 \leq 20500. \end{aligned}$$

Full formulation: [removing a_1 because it's always zero]

$$\max \; 200x_1 + 220x_2 + 185x_3 + 190x_4 \quad \text{[profit]}$$

s.t.

$$\left.\begin{aligned} x_1 &\le 700\,s_1 \\ x_2 &\le 700\,s_2 \\ x_3 &\le 700\,s_3 \\ x_4 &\le 700\,s_4 \end{aligned}\right\} \text{[production limits and must pay start-up costs]}$$

$$0.5x_1 + 0.5x_2 + 0.4x_3 + 0.4x_4 \le 800 \quad \text{[production time]}$$

$$x_1 + x_2 + x_3 + x_4 \le a \quad \text{[autorhombulators]}$$

$$c_1 + c_2 + c_3 = 1 \quad \text{[in exactly one case for piecewise function]}$$

$$\left.\begin{aligned} c_2 \le a_2 &\le 100\,c_2 \\ 100\,c_3 \le a_3 &\le 10000\,c_3 \end{aligned}\right\} \text{[ranges in piecewise function]}$$

$$a = a_2 + a_3 \quad \text{[connect a to piecewise variables]}$$

$$4000\,s_1 + 5000\,s_2 + 3000\,s_3 + 3500\,s_4 + 50\,c_2 + 9a_2 + 350\,c_3 + 6a_3 \le 20500 \quad \text{[budget]}$$

$x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$, $x_4 \ge 0$, $s_1 \in \{0,1\}$, $s_2 \in \{0,1\}$, $s_3 \in \{0,1\}$, $s_4 \in \{0,1\}$, $a \ge 0$, $c_1 \in \{0,1\}$, $c_2 \in \{0,1\}$, $c_3 \in \{0,1\}$, $a_2 \ge 0$, $a_3 \ge 0$.

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Boolean satisfiability

Terminology used in the Boolean satisfiability (SAT) problem:

- Variable: $x_1, x_2, \ldots$
- Literal: a variable or its negation (e.g., x_3, $\overline{x_5}$)
- Disjunction: things connected with $\vee$ (OR)
- Conjunction: things connected with $\wedge$ (AND)

A propositional formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals. In this case the disjunctions of literals are called clauses.

Theorem. Every propositional formula can be rewritten in conjunctive normal form.

Proof. Exercise.

The given propositional formula is in CNF. So we need to satisfy each clause.

Variables and domains:

$x_1 \in \{0,1\}$

$x_2 \in \{0,1\}$ [0 means false, 1 means true]

$x_3 \in \{0,1\}$

Constraints: Satisfy each clause.

$$\begin{array}{ll} x_1 + (1-x_2) \geq 1 & [x_1 \vee \bar{x}_2] \\ (1-x_2)+(1-x_3) \geq 1 & [\bar{x}_2 \vee \bar{x}_3] \\ x_1 + x_2 + (1-x_3) \geq 1 & [x_1 \vee x_2 \vee \bar{x}_3] \\ (1-x_1) + x_3 \geq 1 & [\bar{x}_1 \vee x_3] \end{array}$$

Objective: We are not really optimizing anything here — just seeking a feasible solution. So we can just use a constant function as the objective function, e.g.,

$$\max 0$$

Then the possible outcomes of the solution process are "optimal objective value is 0" (so the formula is satisfiable, and the "optimal" values of the variables give a satisfying assignment of truth values) or else "infeasible."