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Duality [P&S Chap. 3]

(Based on an example in Chapter 5 of Linear Programming: Foundations and Extensions, third edition, by Robert J. Vanderbei.)

Consider the following maximization problem:

$$\begin{array}{lrl} \max & 8x_1 + 5x_2 + 6x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 \le 8 & (1) \\ & 4x_1 + 2x_2 - x_3 \le 7 & (2) \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 & \end{array}$$

Let z^* denote the optimal objective value for this LP. Suppose that we want to determine upper and lower bounds for z^*.

Lower bounds are easy: Every feasible solution demonstrates a lower bound for z^*.
For example, the feasible solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$$

yields the objective value $8(1)+5(2)+6(3) = 36$, so we know that $z^* \ge 36$.

Different feasible solutions produce different lower bounds. E.g., $x_1 = 0$, $x_2 = 5$, $x_3 = 3$ yields the objective value $8(0)+5(5)+6(3) = 43$, so we can conclude $z^* \ge 43$, which is a better lower bound than $z^* \ge 36$.

But finding lower bounds this way is like trying to solve the maximization problem by trial and error. Better method: Use the simplex algorithm to solve the maximization problem. This produces the (optimal) feasible solution $x_1 = 3$, $x_2 = 0$, $x_3 = 5$, which yields the objective value $8(3)+5(0)+6(5) = 54$. This feasible solution demonstrates that $z^* \geq 54$.

In fact we know that the simplex algorithm produces optimal solutions so we now know that z^* actually equals 54. But how could we easily convince a skeptic of this conclusion without having to go through all the steps of the simplex algorithm? We would like an easily verifiable certificate of optimality. For this we need upper bounds for z^*.

Upper bounds for z^* come from linear combinations of the constraints in the maximization problem. For example, we can multiply constraint (1) by 8 and constraint (2) by 2 and add them together to get the following inequality:

$$\begin{array}{rl} 8(x_1 + x_2 + x_3) & \leq 8(8) \\ + \; 2(4x_1 + 2x_2 - x_3) & \leq 2(7) \\ \hline 16x_1 + 12x_2 + 6x_3 & \leq 78. \end{array}$$

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Duality example — 2

Since all of the variables x_1, x_2, x_3 are nonnegative, and each of the coefficients in $16x_1 + 12x_2 + 6x_3$ is at least as large as the corresponding coefficient in the objective function $8x_1 + 5x_2 + 6x_3$, we can see that

$$z = 8x_1 + 5x_2 + 6x_3 \le 16x_1 + 12x_2 + 6x_3 \le 78.$$

So $z^* \le 78$.

The inequality $16x_1 + 12x_2 + 6x_3 \le 78$ is a direct consequence of the constraints of the maximization problem, and we could show this inequality to a skeptic (by providing the multipliers 8 and 2) in order to prove that the objective value can be no greater than 78.

But can we do better than $z^* \le 78$? Can we get, say, $z^* \le 60$? We can find other linear combinations of the constraints by trying other multipliers, and as long as the coefficients of the inequality we get are at least as large as the corresponding coefficients in the objective function, we will get an upper bound for z^*.

Searching for upper bounds this way, though, is again a trial-and-error process. We should be more methodical. So let's assign variable names to the multipliers we use. If we multiply constraint (1) by y_1 and multiply constraint (2) by y_2 and add them together, we get the inequality

$$\begin{array}{rrrrl} y_1(& x_1 + & x_2 + & x_3) & \le y_1(8) \\ + \; y_2(& 4x_1 + & 2x_2 - & x_3) & \le y_2(7) \\ \hline \end{array}$$
$$(y_1+4y_2)x_1 + (y_1+2y_2)x_2 + (y_1-y_2)x_3 \le 8y_1+7y_2.$$

This will give us an upper bound for z^* as long as the coefficients in this new inequality are at least as large as the corresponding coefficients in the objective function. So we need

$$\begin{aligned} y_1 + 4y_2 &\ge 8, \\ y_1 + 2y_2 &\ge 5, \\ y_1 - y_2 &\ge 6. \end{aligned}$$

We also need $y_1 \ge 0$, $y_2 \ge 0$ so that when we multiply the constraints by these values we don't flip the direction of the inequalities.

If these conditions are satisfied, then we can reason that

$$\begin{aligned} z = 8x_1+5x_2+6x_3 &\le (y_1+4y_2)x_1+(y_1+2y_2)x_2+(y_1-y_2)x_3 \\ &\le 8y_1+7y_2, \end{aligned}$$

So we can conclude that $z^* \le 8y_1 + 7y_2$.

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Duality example — 3

To make this upper bound as good as possible, we want to minimize the value of $8y_1 + 7y_2$. This gives us the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & 8y_1 + 7y_2 \\ \text{subject to} \quad & y_1 + 4y_2 \geq 8 \\ & y_1 + 2y_2 \geq 5 \\ & y_1 - y_2 \geq 6 \\ & y_1 \geq 0,\ y_2 \geq 0. \end{aligned}$$

This minimization problem is the dual of the original maximization problem.

↳ Note that the dual is a linear program.

If we solve this minimization problem, we get the optimal solution $y_1 = 6.4$, $y_2 = 0.4$. Using these multipliers for the constraints of the original maximization problem, we obtain the inequality

$$\begin{array}{rl} 6.4(x_1 + x_2 + x_3) & \leq 6.4(8) \\ +\ 0.4(4x_1 + 2x_2 - x_3) & \leq 0.4(7) \\ \hline 8x_1 + 7.2x_2 + 6x_3 & \leq 54. \end{array}$$

This proves that $z^* \leq 54$, because

$$z = 8x_1 + 5x_2 + 6x_3 \leq 8x_1 + 7.2x_2 + 6x_3 \leq 54.$$

This upper bound, together with the feasible solution $x_1 = 3$, $x_2 = 0$, $x_3 = 5$ that actually gives us an objective value of 54, provides a proof that the optimal objective value of the original maximization problem is indeed 54.

— Note that you can prove this fact to a skeptic even if the skeptic has no knowledge or understanding of the simplex algorithm at all. The skeptic doesn't have to trust that you performed the simplex algorithm correctly to get the optimal feasible solution and multipliers — the skeptic need not understand where those numbers came from at all! You can just say, "Look, the solution $x_1 = 3$, $x_2 = 0$, $x_3 = 5$ is feasible" (and the skeptic easily verifies this and agrees), "and it yields the objective value 54" (which the skeptic easily verifies), "so $z^* \geq 54$. Moreover, if you multiply the first constraint by 6.4 and the second constraint by 0.4 and add them together, you get $8x_1 + 7.2x_2 + 6x_3 \leq 54$" (the skeptic easily verifies this), "and each of these coefficients is at least as large as the corresponding coefficient in the objective function" (the skeptic easily verifies this), "so

$z = 8x_1 + 5x_2 + 6x_3 \leq 8x_1 + 7.2x_2 + 6x_3 \leq 54$,

so $z^* \leq 54$. Therefore $z^* = 54$. Q.E.D."

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"Anatomy" of the dual (LPs in canonical form)

PRIMAL MAX LP

$$\begin{array}{ll} \max & \boxed{8x_1+5x_2+6x_3} \\ \text{s.t.} & \boxed{\begin{array}{l} x_1+x_2+x_3 \\ 4x_1+2x_2-x_3 \end{array}} \begin{array}{l} \le \\ \le \end{array} \boxed{\begin{array}{c} 8 \\ 7 \end{array}} \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{array}$$

DUAL MIN LP

$$\begin{array}{ll} \min & \boxed{8y_1+7y_2} \\ \text{s.t.} & \boxed{\begin{array}{l} y_1+4y_2 \\ y_1+2y_2 \\ y_1-y_2 \end{array}} \begin{array}{l} \ge \\ \ge \\ \ge \end{array} \boxed{\begin{array}{c} 8 \\ 5 \\ 6 \end{array}} \\ & y_1 \ge 0,\ y_2 \ge 0. \end{array}$$

Observe:

- Coefficients in objective function in the primal become the right-hand sides of the constraints in the dual.
- Right-hand sides of the constraints in the primal become the coefficients in the objective function in the dual.
- Coefficient matrix in the primal is transposed to become the coefficient matrix in the dual.

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The form of the dual for general linear programs.

From the preceding example, we see that the dual of the LP

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

is

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0. \end{aligned}$$

What about other constraints and variable domains?

Recall that our motivation is to find a linear combination of the constraints in the max LP that will produce a $\leq$ inequality which provides an upper bound for z^*.

- If we have a $\geq$ constraint in the max LP, then its coefficient in this linear combination should be <u>nonpositive</u> so that the inequality flips.
  - So a $\geq$ constraint in the max LP should correspond to $y_j \leq 0$ in the dual min LP.

- If we have an = constraint in the max LP, then its coefficient in the linear combination can be any real number — it will remain an equation regardless.
  - So an = constraint in the max LP should correspond to y_j unrestricted in the dual min LP.

Recall also that the way we got an upper bound for z^* as a linear combination $y^T Ax \leq y^T b$ of the constraints in the max LP was to ensure that the vector $A^T y$ of coefficients of x on the left-hand side of the inequality was componentwise at least as large as the coefficients c in the objective function of the max LP:

$$A^T y \geq c.$$

(That's the set of constraints in the dual.)
Then we were able to conclude that

$$z = c^T x \leq (A^T y)^T x = y^T A x \leq y^T b.$$

But this inequality was using the assumption that $x \geq 0$.

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General dual — 2

How does this reasoning change if some of the components of x are not nonnegative?

Example. Suppose $x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

The objective function in the max LP is
$z = c_1 x_1 + c_2 x_2 + c_3 x_3$.

From some linear combination of the constraints in the max LP, we get the inequality

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \leq \beta.$$

What is required in order ~~for~~ this to enable us to deduce an upper bound for z^*?

This was our reasoning before.

- Since $x_1 \geq 0$, if $\alpha_1 \geq c_1$ then $\alpha_1 x_1 \geq c_1 x_1$.
- Since $x_2 \leq 0$, if $\alpha_2 \leq c_2$ then $\alpha_2 x_2 \geq c_2 x_2$.
- Since x_3 is unrestricted, we need $\alpha_3 = c_3$ in order to guarantee $\alpha_3 x_3 \geq c_3 x_3$.

If all of these are met, then we can make the conclusion

$$z = c_1 x_1 + c_2 x_2 + c_3 x_3 \leq \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \leq \beta.$$

Therefore:

- If we have a <u>nonpositive</u> variable in the max LP, then the corresponding constraint in the dual min LP should have $\leq$.

- If we have an <u>unrestricted</u> variable in the max LP, then the corresponding constraint in the dual min LP should have $=$.

<u>Summary.</u> Dualization table:

<u>In primal max LP</u>		<u>In dual min LP</u>
$\leq$ constraint	$\longleftrightarrow$	nonnegative variable (≥ 0)
$\geq$ constraint	$\longleftrightarrow$	nonpositive variable (≤ 0)
$=$ constraint	$\longleftrightarrow$	unrestricted variable
nonnegative variable (≥ 0)	$\longleftrightarrow$	$\geq$ constraint
nonpositive variable (≤ 0)	$\longleftrightarrow$	$\leq$ constraint
unrestricted variable	$\longleftrightarrow$	$=$ constraint

[Mnemonic: "Natural" variable domains (always ≥ 0) correspond to "natural" constraints ($\leq$ in max LP, $\geq$ in min LP).]
(Canonical)

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General dual – 3

Exercise: Go through this reasoning starting with a <u>min</u> LP as the primal, and work to find a linear combination of constraints that yields a <u>lower</u> bound on the optimal objective value.

What you should find is that the dualization table works right-to-left too!

So:

Theorem. The dual of the dual is the primal.

Example. Writing a dual of an LP:

PRIMAL:
$$\begin{aligned} \max\quad & 10x_1 + 8x_2 + 9x_3 \\ \text{s.t.}\quad & 2x_1 + 3x_2 - x_3 \le 40 \\ & x_1 + x_3 \ge 22 \\ & x_1 - 2x_2 + 3x_3 = 15 \\ & 5x_2 + 7x_3 \le 38 \\ & x_1 \ge 0,\quad x_2 \le 0,\quad x_3 \text{ unrestricted} \end{aligned}$$

DUAL:
$$\begin{aligned} \min\quad & 40y_1 + 22y_2 + 15y_3 + 38y_4 \\ \text{s.t.}\quad & 2y_1 + y_2 + y_3 \ge 10 \\ & 3y_1 - 2y_3 + 5y_4 \le 8 \\ & -y_1 + y_2 + 3y_3 + 7y_4 = 9 \\ & y_1 \ge 0,\quad y_2 \le 0,\quad y_3 \text{ unrestricted},\quad y_4 \ge 0 \end{aligned}$$

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Example. The dual of the dual is the primal, so if we want to find lower bounds for a (dual) min LP using linear combinations of constraints, the best multipliers to use are the optimal values of the primal max LP. Example from earlier:

PRIMAL

$$\max\ 8x_1+5x_2+6x_3$$
$$\text{s.t.}\quad x_1+x_2+x_3\le 8$$
$$4x_1+2x_2-x_3\le 7$$
$$x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0$$

DUAL

$$\min\ 8y_1+7y_2$$
$$\text{s.t.}\quad y_1+4y_2\ge 8$$
$$y_1+2y_2\ge 5$$
$$y_1-y_2\ge 6$$
$$y_1\ge 0,\ y_2\ge 0$$

Optimal primal solution:
$x_1=3,\ x_2=0,\ x_3=5$

Optimal objective value:
$8(3)+5(0)+6(5)=54$

Optimal dual solution:
$y_1=6.4,\ y_2=0.4$

Optimal objective value:
$8(6.4)+7(0.4)=54$

Upper bound using optimal dual solution as multipliers:

$$
\begin{array}{rl}
6.4(x_1+x_2+x_3) & \le 6.4(8)\\
+0.4(4x_1+2x_2-x_3) & \le 0.4(7)\\
\hline
8x_1+7.2x_2+6x_3 & \le 54
\end{array}
$$

so $z^*\le 54$.

Lower bound using optimal primal solution as multipliers:

$$
\begin{array}{rl}
3(y_1+4y_2) & \ge 3(8)\\
0(y_1+2y_2) & \ge 0(5)\\
+\ 5(y_1-y_2) & \ge 5(6)\\
\hline
8y_1+7y_2 & \ge 54
\end{array}
$$

so $w^*\ge 54$ (where w^* is the optimal objective value to the dual).

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"Anatomy" of the dual (LPs in general form)

PRIMAL MAX LP

$$\boxed{\max} \ \boxed{10x_1 + 8x_2 + 9x_3}$$

$$\text{s.t.} \quad \begin{array}{rcl} 2x_1 + 3x_2 - x_3 & \leq & 40 \\ x_1 + x_3 & \geq & 22 \\ x_1 - 2x_2 + 3x_3 & = & 15 \\ 5x_2 + 7x_3 & \leq & 38 \end{array}$$

$$\boxed{x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ unrestricted}}$$

DUAL MIN LP

$$\boxed{\min} \ \boxed{40y_1 + 22y_2 + 15y_3 + 38y_4}$$

$$\text{s.t.} \quad \begin{array}{rcl} 2y_1 + y_2 + y_3 & \geq & 10 \\ 3y_1 - 2y_3 + 5y_4 & \leq & 8 \\ -y_1 + y_2 + 3y_3 + 7y_4 & = & 9 \end{array}$$

$$\boxed{y_1 \geq 0, \ y_2 \leq 0, \ y_3 \text{ unrestricted}, \ y_4 \geq 0}$$

- The dual of a max LP is a min LP, and vice versa.
- The coefficients in the objective function in the primal become the right-hand sides of the constraints in the dual.
- The right-hand sides of the constraints in the primal become the coefficients in the objective function in the dual.
- The coefficient matrix in the primal is <u>transposed</u> to become the coefficient matrix in the dual.
- The relation symbols ($\leq$, $\geq$, or $=$) in the constraints in the primal determine the variable domains in the dual.
- The variable domains in the primal determine the relation symbols in the constraints in the dual.

IN PRIMAL MAX LP		IN DUAL MIN LP
$\leq$ constraint	$\longleftrightarrow$	nonnegative variable (≥ 0)
$\geq$ constraint	$\longleftrightarrow$	nonpositive variable (≤ 0)
$=$ constraint	$\longleftrightarrow$	unrestricted variable
nonnegative variable (≥ 0)	$\longleftrightarrow$	$\geq$ constraint
nonpositive variable (≤ 0)	$\longleftrightarrow$	$\leq$ constraint
unrestricted variable	$\longleftrightarrow$	$=$ constraint