28 May.

"Anatomy" of an LP.

Objective → $\boxed{\max \ \boxed{40p + 120w}}$ ← Objective function

Constraints → s.t.
$$\boxed{\begin{aligned} p + w &\le 100 \\ p + 4w &\le 160 \\ 10p + 20w &\le 1100 \end{aligned}}$$

Variable domains → $\boxed{p \geq 0, \quad w \geq 0}$

Terminology

- Solution: An assignment of values to variables.
- Feasible solution: A solution that satisfies all constraints (and domains).
- Feasible region (a.k.a. feasible set): The set of all feasible solutions.
- Objective value: The value of the objective function corresponding to a given solution.
- Optimal (feasible) solution: A feasible solution whose objective value is at least as "good" as that of any other feasible solution.
- Optimal objective value: The objective value of an optimal feasible solution.

More terminology.

- **Feasible LP**: An LP with at least one feasible solution.
- **Infeasible LP**: One that is not feasible.
- **Unbounded LP**: A feasible LP with no optimal feasible solution.

Solving an LP means attempting to find an optimal feasible **solution** (not just the optimal objective value!).

Possible outcomes:

- LP is infeasible.
- LP is feasible.
  - LP is unbounded.
  - LP has an optimal feasible solution.
    - LP has a **unique** optimal feasible solution.
    - LP has a **nonunique** optimal feasible solution (i.e., at least two).

29 May. Feasible region for Farmer Brown's problem.

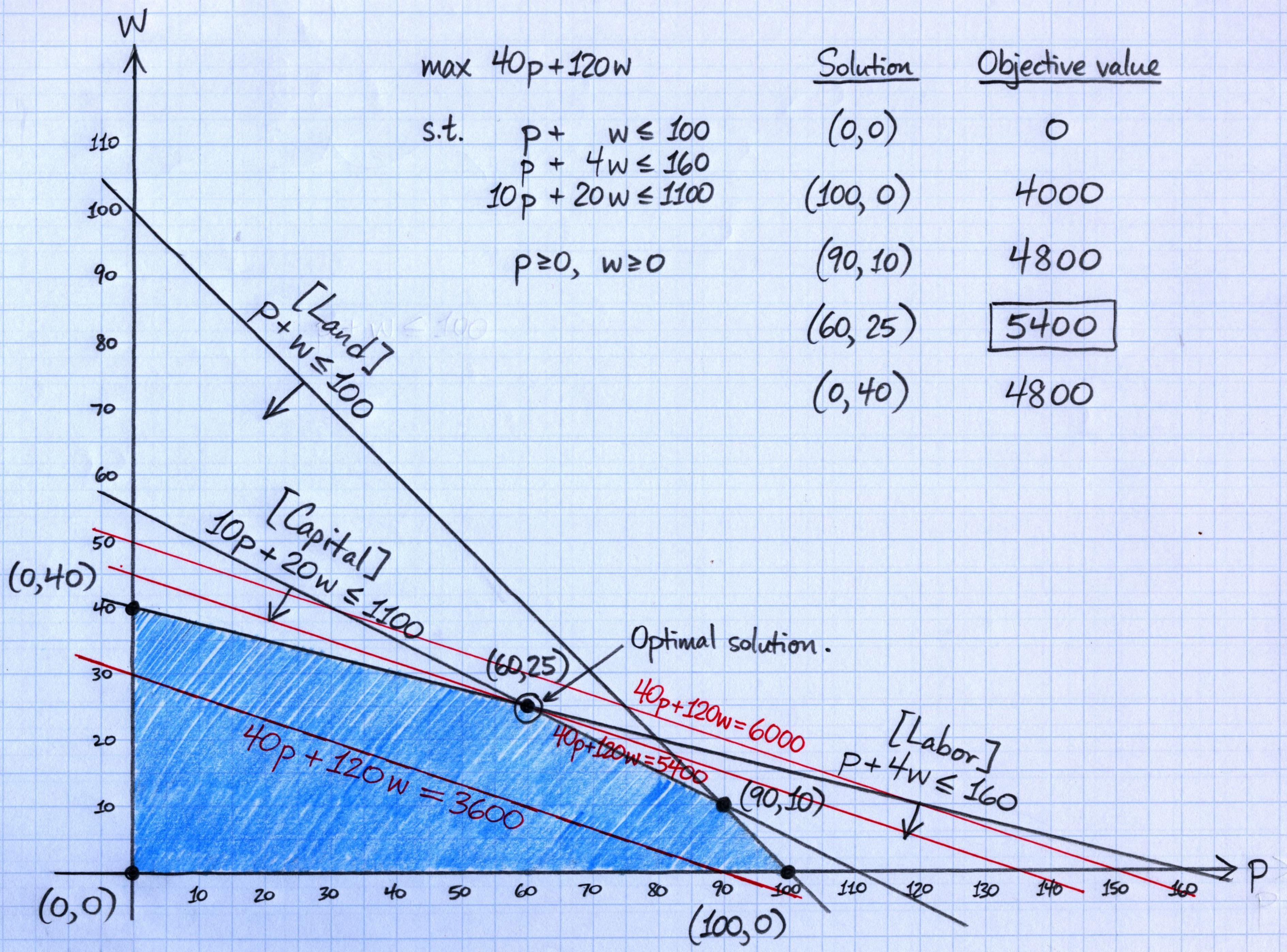
max $40p + 120w$

s.t.

$$\begin{aligned} p + w &\le 100 \\ p + 4w &\le 160 \\ 10p + 20w &\le 1100 \end{aligned}$$

$p \ge 0, \ w \ge 0$

Solution	Objective value
(0,0)	0
(100, 0)	4000
(90, 10)	4800
(60, 25)	$\boxed{5400}$
(0, 40)	4800



29 May.

<u>Defn.</u> A <u>level curve</u> of a function f is the curve defined by $f = K$ for some constant K.

Graphical solution process (for an LP with two variables):

1. Draw constraints.

2. Determine feasible region, and find coordinates of corners.

3. Evaluate objective function at each corner.

4. Choose the best solution(s).

<u>Warning:</u> If the feasible region is unbounded, the LP may also be unbounded. Be sure to consider this possibility. Drawing a couple level curves may help.

— Note that the feasible region of a two-variable LP is a <u>polygon</u> (if bounded).
In higher dimensions (i.e., more variables), feasible region is a <u>polytope</u>.

29 May.

Matrix form of LP. [P&S §2.1]

Farmer Brown:

$$
\begin{aligned}
\max\ & 40p + 120w \\
\text{s.t.}\ & p + w \le 100 \\
& p + 4w \le 160 \\
& 10p + 20w \le 1100 \\
& p \ge 0,\ w \ge 0.
\end{aligned}
$$

We can write this LP in terms of matrices as follows:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t.}\ & Ax \le b \\
& x \ge 0
\end{aligned}
$$

where

$$
c = \begin{bmatrix} 40 \\ 120 \end{bmatrix}, \quad x = \begin{bmatrix} p \\ w \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 10 & 20 \end{bmatrix}, \quad b = \begin{bmatrix} 100 \\ 160 \\ 1100 \end{bmatrix}.
$$

Note that $c^T x = 40p + 120w$,

$$
Ax = \begin{bmatrix} p + w \\ p + 4w \\ 10p + 20w \end{bmatrix}.
$$

Vector inequality $Ax \le b$ (and $x \ge 0$) is to be interpreted <u>componentwise</u>.

Slack variables and standard form.

We can turn inequalities into equalities by adding slack variables to fill up the gap between the two sides:

$$
\begin{aligned}
\max\quad & 40p + 120w \\
\text{s.t.}\quad & p + w + s_1 = 100 \\
& p + 4w + s_2 = 160 \\
& 10p + 20w + s_3 = 1100 \\
& p \geq 0,\ w \geq 0,\ \underbrace{s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0}_{\text{To ensure original inequalities are satisfied.}}
\end{aligned}
$$

Therefore, we may convert any LP to standard form:

$$
\begin{aligned}
\max\quad & c^T x \\
\text{s.t.}\quad & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Note: – The x vector here includes any necessary slack variables.

– P&S prefers writing LPs as minimization problems. For certain reasons, I will write LPs as maximization problems instead. Conversion between the two is easy: negate objective function.

29 May.

Basic feasible solutions. [P&S §2.2, 2.3]

Suppose the coefficient matrix A of an LP in standard form is an $m \times n$ matrix (i.e., m rows, n columns).

<u>Assumption:</u> A has m linearly independent columns (i.e., A has rank m).

- Intuitively, this is equivalent to saying the LP has no "redundant" constraints — no (LHS of) constraint is a linear combo of any others.

<u>Defn.</u> A <u>basis</u> of A is a linearly independent collection $\mathcal{B}$ of m columns of A:

$$\mathcal{B} = \{ A_{j_1}, A_{j_2}, \ldots, A_{j_m} \}.$$

Alternatively, we can think of $\mathcal{B}$ as an $m \times m$ nonsingular matrix $B = [A_{j_i}]$ formed by choosing m linearly independent columns of A.

The <u>basic solution</u> corresponding to $\mathcal{B}$ is a vector $x \in \mathbb{R}^n$ such that

<u>nonbasic</u> variables $\longrightarrow$ $x_j = 0$ for $A_j \notin \mathcal{B}$;

<u>basic</u> variables $\longrightarrow$ $x_{j_k} = k$th component of $B^{-1}b$, $k = 1, \ldots, m$.

So, to find a basic solution x:

1. Choose a basis $\mathcal{B}$, a set of m linearly independent columns of A.

2. Set all components of x corresponding to columns not in $\mathcal{B}$ equal to zero. (These are the <u>nonbasic</u> variables.)

3. Solve the m resulting equations to determine the remaining components of x. (These are the <u>basic</u> variables.)

<u>Example.</u> $\max\ 40p + 120w$

$$\begin{aligned} \text{s.t.}\quad p + w + s_1 &= 100 \\ p + 4w + s_2 &= 160 \\ 10p + 20w + s_3 &= 1100 \\ p \geq 0,\ w \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 &\geq 0 \end{aligned}$$

Obvious basis is $\mathcal{B} = \{A_3, A_4, A_5\}$, corresponding to basic variables $\{s_1, s_2, s_3\}$, because these columns of A form the identity matrix.

This yields basic solution $(p, w, s_1, s_2, s_3) = (0, 0, 100, 160, 1100)$.

— Note this is a <u>corner</u> of the feasible region.

Another basis is $\mathcal{B} = \{A_1, A_3, A_4\}$, corresponding to basic variables $\{p, s_1, s_2\}$. This yields the basic solution $(p, w, s_1, s_2, s_3) = (110, 0, -10, 50, 0)$.

— Note that this solution is <u>not feasible</u> because $s_1 < 0$.

<u>Defn.</u> If a basic solution is feasible, it is a <u>basic feasible solution</u> (bfs).