21-257

In-Class Activity
Friday, November 21, 2014

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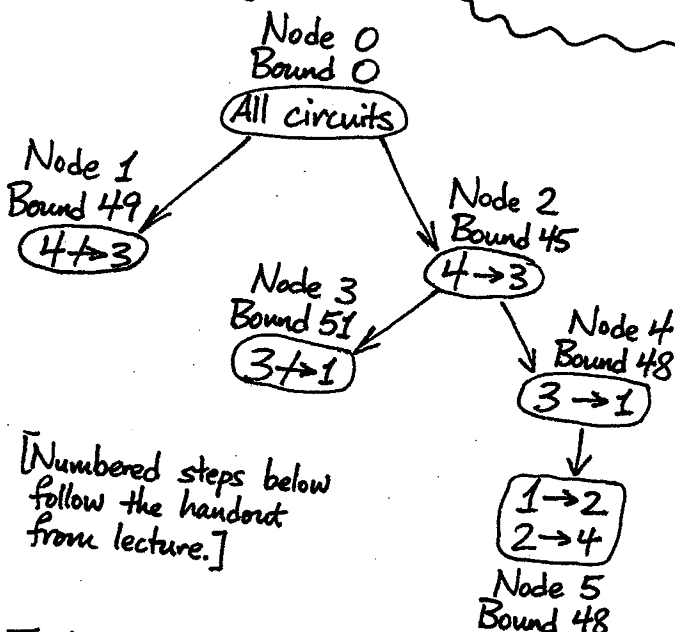
Solve the traveling salesman problem with distances in the following table.

From \ To	1	2	3	4
1	M	15	22	18
2	9	M	11	14
3	12	17	M	20
4	14	8	7	M

Branch-and-bound tree:
(Constructed as we go through the algorithm)



[Numbered steps below follow the handout from lecture.]

Initialize root node: Bound is 0, incoming matrix is original distance matrix (with $L=0$).

Process Node 0:

Incoming matrix:

	1	2	3	4
1	M	15	22	18
2	9	M	11	14
3	12	17	M	20
4	14	8	7	M

$L=0$.

1. Compute opportunity matrix.
   (a) Set $L=0$.
   (b) Ensure every row and column has exactly one M: ✓
   (c) Ensure every column contains at least one zero. Subtract smallest number in each column from all numbers in the column, and add that number to L.

	1	2	3	4
1	M	7	15	4
2	0	M	4	0
3	3	9	M	6
4	5	0	0	M

$$L = \underset{\text{previous value of } L}{0} + \underbrace{9+8+7+14}_{\text{numbers we subtracted from columns}} = 38$$

   (d) Ensure every row contains at least one zero. Rows 2 and 4 already contain zeroes; don't need to change them. Subtract 4 from row 1 and 3 from row 3.

	1	2	3	4
1	M	3	11	0
2	0	M	4	0
3	0	6	M	3
4	5	0	0	M

$$L = \underset{\text{previous value of } L}{38} + \underbrace{4+3}_{\text{numbers we subtracted from rows}} = 45$$

This is the opportunity matrix for Node 0. Its L-value is 45!

2. [First special case does not apply; opportunity matrix is not 2×2.]
3. [Second special case does not apply; there is no zero in the opportunity matrix that is the only number in its row or the only number in its column.]

(continued) →

(Processing Node 0)

4. Compute regrets:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & 3 \\ 2 & 0 & & & 0 \\ 3 & 3 & & & \\ 4 & & 3 & 4 & \end{array}$$

Largest.

5. Choose largest regret: (Corresponds to the link $4 \to 3$.)

6. Create child nodes, Nodes 1 and 2.

(a) Right child (Node 2): Label is $4 \to 3$. Bound is 45 (L-value of opp. matrix).

Incoming matrix for Node 2:

[Delete row 4 and column 3.]

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & M & 3 & 0 \\ 2 & 0 & M & 0 \\ 3 & 0 & 6 & 3 \end{array} \qquad L = 45.$$

(b) Left child (Node 1): Label is $4 \not\to 3$. Bound is $45 + 3 = 48$. (L-value of opp. matrix plus largest regret.)

Incoming matrix for Node 1:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & M & 3 & 11 & 0 \\ 2 & 0 & M & 4 & 0 \\ 3 & 0 & 6 & M & 3 \\ 4 & 5 & 0 & N & M \end{array} \qquad L = 45.$$

N here. Not 48!

7. Done processing Node 0.

Process Node 2 next, because it has the smallest bound.

<u>Process Node 2:</u>

Incoming matrix:

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & M & 3 & 0 \\ 2 & 0 & M & 0 \\ 3 & 0 & 6 & 3 \end{array} \qquad L=45.$$

1. Compute opportunity matrix.

(a) Set $L=45$.

(b) Ensure every row and column has exactly one M:

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & M & 3 & 0 \\ 2 & 0 & M & 0 \\ 3 & 0 & 6 & M \end{array}$$

M here. (L does not change.)

(c) Ensure every column has at least one zero:

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & M & 0 & 0 \\ 2 & 0 & M & 0 \\ 3 & 0 & 3 & M \end{array} \qquad L = \underbrace{45}_{\text{previous value of } L} + \underbrace{3}_{\text{number subtracted from column 2}} = 48.$$

(d) Every row has at least one zero: ✓

So the matrix above is the opportunity matrix for Node 2.

2. [First special case does not apply.]

3. [Second special case does not apply.]

4. Compute regrets:

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & & 3 & 0 \\ 2 & 0 & & 0 \\ 3 & 3 & & \end{array}$$

5. Choose largest regret. There is a tie between $1 \to 2$ and $3 \to 1$. Go back to <u>original</u> distance matrix and compare those distances: $1 \to 2$ is 15, $3 \to 1$ is 12. So choose $3 \to 1$ because it's smaller.

6. Create child nodes, Nodes 3 and 4.

(a) Right child (Node 4): Label is $3 \to 1$. Bound is 48.

Incoming matrix for Node 4:

$$\begin{array}{c|cc} & 2 & 4 \\ \hline 1 & 0 & 0 \\ 2 & M & 0 \end{array} \qquad L = 48.$$

(b) Left child (Node 3): Label is $3 \not\to 1$. Bound is $48 + 3 = 51$.

Incoming matrix for Node 3:

$$\begin{array}{c|ccc} & 1 & 2 & 4 \\ \hline 1 & M & 0 & 0 \\ 2 & 0 & M & 0 \\ 3 & N & 3 & M \end{array} \qquad L = 48.$$

N here. Not 51!

7. Done processing Node 2.

(continued) →

Process Node 4 next, because it has the smallest bound.

Process Node 4: Incoming matrix: $\begin{array}{c|cc} & 2 & 4 \\ \hline 1 & 0 & 0 \\ 2 & M & 0 \end{array}$ $L = 48$.

1. Compute opportunity matrix.
   (a) Set $L = 48$.
   (b) Ensure every row and column has exactly one M:

$$\begin{array}{c|cc} & 2 & 4 \\ \hline 1 & 0 & M \\ 2 & M & 0 \end{array}$$

← M here. (L does not change.)

   (c) Every column has at least one zero: ✓
   (d) Every row has at least one zero: ✓

   So the matrix above is the opportunity matrix for Node 4. L-value is 48.

2. First special case applies, because opportunity matrix is 2×2.
   (a) Opportunity matrix is $\begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix}$. ✓
   (b) Create one child node, Node 5. Label is $1 \to 2$, $2 \to 4$
   (these are the links corresponding to zeroes in opportunity matrix).
   Bound is 48 (L-value of 2×2 opportunity matrix).
   (c) Child node represents a complete circuit:

$4 \to 3$
$3 \to 1$
$1 \to 2$
$2 \to 4$

i.e.,

$$\begin{tikzcd} 1 \arrow[r, "(15)"] & 2 \arrow[d, "(14)"] \\ 3 \arrow[u, "(12)"] & 4 \arrow[l, "(7)"] \end{tikzcd}$$

← Distances from original distance matrix

   (d) Check that total length of circuit equals child node bound: $15 + 14 + 7 + 12 = 48$. ✓
   (e) Done processing Node 4. Skip steps 3 through 7.

Now, a complete circuit has been found. Is it optimal?

Yes, because there is no unexplored non-terminal node with a smaller bound than the length of this circuit (both 49 and 51 are greater than 48).

So stop: The circuit above is optimal.