Section 8.7, exercise 9. Find the Maclaurin series for $f(x) = \sinh x$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

Hyperbolic sine (written sinh) and hyperbolic cosine (written cosh) are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

It is easy to check that these functions are derivatives of each other:

$$\frac{d}{dx} \sinh x = \cosh x, \qquad \frac{d}{dx} \cosh x = \sinh x.$$

Furthermore, we have

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0, \qquad \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1.$$

By definition, the Maclaurin series for a function $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots.$$

We make a table of the derivatives of $f(x) = \sinh x$ and these derivatives evaluated at 0.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
5	$\cosh x$	1
$\vdots$	$\vdots$	$\vdots$

So we see that the Maclaurin series for $f(x) = \sinh x$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= \frac{0}{0!} x^0 + \frac{1}{1!} x^1 + \frac{0}{2!} x^2 + \frac{1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \\
&= \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

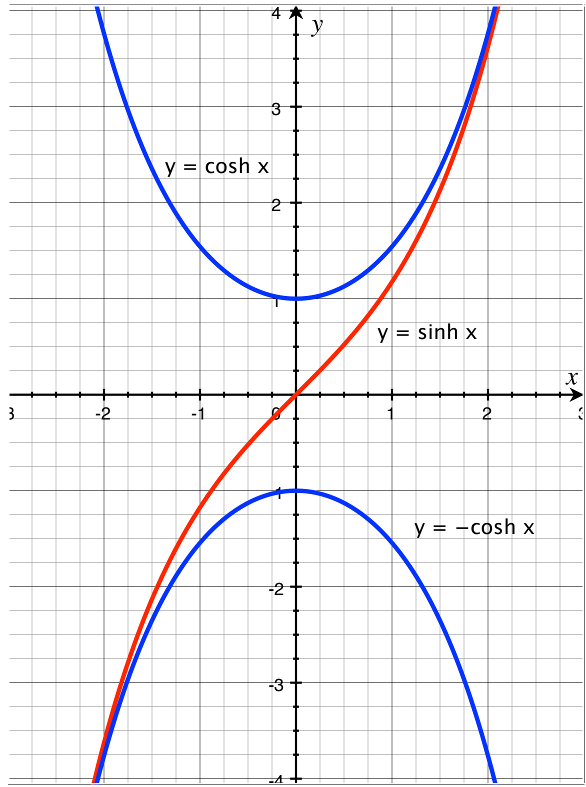
To find the radius of convergence, we can use the Ratio Test. Let $a_n = x^{2n+1}/(2n+1)!$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!} \right| \\
&= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} \\
&= x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \\
&= x^2 \cdot 0 \\
&= 0,
\end{aligned}$$

which is certainly less than 1 (regardless of the value of x), so the Ratio Test implies that the series converges for all x. Hence the radius of convergence is $R = \infty$. □

Section 8.7, exercise 21. Prove that the series obtained in Exercise 9 represents $\sinh x$ for all x.

We first establish some facts about the functions $\sinh x$ and $\cosh x$. The graphs of $y = \sinh x$, $y = \cosh x$, and $y = -\cosh x$ are shown at the right. These graphs show several important properties of these functions.



First, we note that $\cosh x$ is always positive. This is easy to see from the definition,

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

because the exponential function e^x is always positive (even for negative values of x).

Second, we see that $\cosh x$ is decreasing for $x < 0$ and increasing for $x > 0$. In particular, since $\cosh x$ is continuous (because e^x is continuous), this means that $\cosh x$ is minimized at $x = 0$, with a minimum value of $\cosh 0 = 1$. To verify this, we note that e^x is a strictly increasing function, so we have $e^x < e^y$ if $x < y$. So, if x is positive (so $-x < x$), we have $e^{-x} < e^x$, so

$$\sinh x = \frac{e^x - e^{-x}}{2} > 0.$$

Therefore, $\cosh x$ has a positive derivative for $x > 0$, which means it is increasing there. Similarly, if x is negative (so $-x > x$), we have $e^{-x} > e^x$, so

$$\sinh x = \frac{e^x - e^{-x}}{2} < 0,$$

which means $\cosh x$ is decreasing for $x < 0$.

Third, we see that $\sinh x$ is always between $-\cosh x$ and $\cosh x$. To verify this, we first note that $e^x > 0$ and $e^{-x} > 0$ for all x, so we have

$$\frac{-e^x - e^{-x}}{2} < \frac{e^x - e^{-x}}{2} < \frac{e^x + e^{-x}}{2},$$

which is to say,

$$-\cosh x < \sinh x < \cosh x.$$

In other words, $|\sinh x| < \cosh x$.

Now that we have these facts about $\sinh x$ and $\cosh x$, we can prove that the Maclaurin series we found in Exercise 9 converges to $f(x) = \sinh x$ for all x. By Taylor's Formula, the remainder term in the Maclaurin series is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1},$$

where z is some number between 0 and x. (Note, however, that z depends on n.) We aim to prove that this remainder goes to 0 as $n \to \infty$, which will show that the Maclaurin series converges to $f(x) = \sinh x$.

Depending on whether n is even or odd, the $(n+1)$st derivative $f^{(n+1)}(z)$ is either $\cosh z$ or $\sinh z$. In either case, however, we have $|f^{(n+1)}(z)| \leq \cosh z$, because $|\sinh z| < \cosh z$ (as shown above) and $|\cosh z| \leq \cosh z$ (since $\cosh z$ is always positive). Also, since z is closer to 0 than x is, we have $\cosh z < \cosh x$ (because the function $\cosh x$ is decreasing for $x < 0$ and increasing for $x > 0$). So we see that

$$0 \le |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \le \left| \frac{\cosh z}{(n+1)!} x^{n+1} \right| < \left| \frac{\cosh x}{(n+1)!} x^{n+1} \right|.$$

Now $\lim_{n\to\infty} 0 = 0$, and

$$\lim_{n\to\infty} \left| \frac{\cosh x}{(n+1)!} x^{n+1} \right| = (\cosh x) \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = (\cosh x) \cdot 0 = 0.$$

(Here we used the fact that $\lim_{n\to\infty} x^n/n! = 0$ for every real number x, as shown at the bottom of page 461 in the textbook.) So, by the Squeeze Theorem, we have

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

This implies that

$$\lim_{n\to\infty} R_n(x) = 0$$

(see Theorem 6 on page 414 of the textbook), which was what we wanted to show.

Therefore, for every x, the remainder term in the Maclaurin series goes to 0 as $n \to \infty$, so the series converges to $\sinh x$. □