How to integrate $\int \sin^6 x \cos^2 x\, dx$

I do not know of an easy way to evaluate this integral. But here's one way that isn't too bad. It repeatedly uses the reduction formula

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx \qquad \text{where } n \geq 2 \text{ is an integer,}$$

which is proved as Example 6 in Section 6.1.

$$\begin{aligned}
\int \sin^6 x \cos^2 x\, dx &= \int \sin^6 x(1-\sin^2 x)\, dx && [\,\text{using } \sin^2 x + \cos^2 x = 1\,] \\
&= \int \sin^6 x\, dx - \int \sin^8 x\, dx \\
&= \int \sin^6 x\, dx - \left(-\tfrac{1}{8}\cos x \sin^7 x + \tfrac{7}{8}\int \sin^6 x\, dx\right) && [\,\text{reduction formula}\,] \\
&= \int \sin^6 x\, dx + \tfrac{1}{8}\cos x \sin^7 x - \tfrac{7}{8}\int \sin^6 x\, dx \\
&= \tfrac{1}{8}\cos x \sin^7 x + \tfrac{1}{8}\int \sin^6 x\, dx && [\,\text{combining like integrals}\,] \\
&= \tfrac{1}{8}\cos x \sin^7 x + \tfrac{1}{8}\left(-\tfrac{1}{6}\cos x \sin^5 x + \tfrac{5}{6}\int \sin^4 x\, dx\right) && [\,\text{reduction formula}\,] \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x + \tfrac{5}{48}\int \sin^4 x\, dx \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x + \tfrac{5}{48}\left(-\tfrac{1}{4}\cos x \sin^3 x + \tfrac{3}{4}\int \sin^2 x\, dx\right) \\
& && [\,\text{reduction formula}\,] \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x - \tfrac{5}{192}\cos x \sin^3 x + \tfrac{5}{64}\int \sin^2 x\, dx \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x - \tfrac{5}{192}\cos x \sin^3 x + \tfrac{5}{64}\cdot\tfrac{1}{2}\int (1-\cos 2x)\, dx \\
& && [\,\text{using } \sin^2 x = \tfrac{1}{2}(1-\cos 2x)\,] \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x - \tfrac{5}{192}\cos x \sin^3 x + \tfrac{5}{128}\left(x - \tfrac{1}{2}\sin 2x\right) + C \\
& && [\,\text{mental substitution } u = 2x\,] \\
&= \tfrac{1}{8}\cos x \sin^7 x - \tfrac{1}{48}\cos x \sin^5 x - \tfrac{5}{192}\cos x \sin^3 x + \tfrac{5}{128}x - \tfrac{5}{256}\sin 2x + C.
\end{aligned}$$

Compare Example 3 in Section 6.2, which shows how to integrate $\sin^2 x$ (the same method is used above).