21-110: Problem Solving in Recreational Mathematics

Homework assignment 5

Assigned Monday, March 15, 2010. Due Wednesday, March 24, 2010.

Work at least **FOUR** of the following problems, at least one of which must be from Part B. All problems are of equal weight. If you submit solutions for more than four problems, you will get credit for your best four (with the proviso that you will get credit for at most three problems from Part A).

You are welcome to work with other students, but the solutions you hand in should be written in your own words. You are not allowed to see the paper another student is going to hand in. If you do collaborate with other students, list their names. If you use other sources, cite them. Give credit where credit is due. See the syllabus for more information about academic integrity.

Hints are encrypted with a *Caesar cipher*, in which each letter is replaced by the letter three places ahead in the alphabet, wrapping around to the beginning if necessary. For example, the letter A is encrypted as D, and Y is encrypted as B. To decrypt the hints, move each letter backward three places.

— Part A —

Problem 1. Make an addition table and a multiplication table for the integers modulo 7 (in other words, for remainders after division by 7).

Problem 2. The country of Euclidia has an awkward currency system. The basic unit of currency there is called the yakblat. There are only two kinds of coins: the pizpaz, which is worth 24 yakblats, and the burbox, which is worth 57 yakblats. (In particular, there is no such thing as a one-yakblat coin.)

When I visited Euclidia, I went into a shop to buy a piece of bubble gum, which cost 15 yakblats. Paying for such an item in Euclidia requires the making of change: I significantly overpaid when I handed the cashier my coins, and then the cashier handed me the proper amount of change so that my net purchase came out to 15 yakblats. Fortunately the cashier and I both had big buckets of coins, so we had as many pizpazzes and burboxes as we required.

Your goal is to find a way to pay exactly 15 yakblats in such a transaction.

(a) Write a linear Diophantine equation representing this situation. You will need to introduce appropriate variables. Explain, carefully and precisely, what quantities your variables represent.

(b) Find a solution to the linear Diophantine equation you wrote in part (a). You might use the extended Euclidean algorithm described on the handout given in class, or the method described on pages 127–137 of *Problem Solving Through Recreational Mathematics*. Or you can use a different method.

(c) Interpret your solution from part (b). In other words, what coins should I hand the cashier, and what coins should the cashier give me as change?

Problem 3. Let

$$\begin{aligned} A &= \{1,2,3,4,5\}, \\ B &= \{\, x \mid x \text{ is even} \,\}, \\ C &= \{\, x \mid x \text{ is a multiple of } 3 \,\}, \text{ and} \\ D &= \{\, x \mid x \text{ is prime} \,\}, \end{aligned}$$

where the universal set is the set of all positive integers no greater than 20.

(a) What is $A \cup C$? What is the cardinality of $A \cup C$?

(b) What is $B \cap C$? What is the cardinality of $B \cap C$?

(c) What are $A \setminus D$ and $D \setminus A$? What are the cardinalities of these sets?

(d) What is $\overline{A}$? What is the cardinality of $\overline{A}$?

(e) What is $(B \cup D) \cap \overline{A}$? What is the cardinality of this set?

Problem 4. Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{3\}$, and $D = \{1, 2, 3\}$.

(a) Use set-builder notation to describe D.

(b) Which of these sets are subsets of which other of these sets? In other words, list all true statements of the form $X \subseteq Y$, where X and Y are to be replaced with A, B, C, or D.

— Part B —

Problem 5. Write a short story that explains or demonstrates the ideas of modular arithmetic (in particular, how addition and multiplication work). Ideally, this should be a story that could be illustrated and made into a children's book. Be creative.

Problem 6. Prove that the Diophantine equation $n^2 = 3k + 2$ has no integer solutions.

Hint: Wklqn derxw wkh srvvleoh uhpdlqghuv zkhq q lv glylghg eb wkuhh. Frqvlghu hdfk ri wkh wkuhh srvvlelolwlhv vhsdudwhob. Zkdw grhv wkh uljkw-kdqg vlgh ri wkh htxdwlrq lpsob?

Problem 7. Let d be a fixed positive integer, with $d \geq 2$. We saw in class that remainders after division by d can be added, subtracted, and multiplied; in other words, arithmetic modulo d makes sense. Mathematicians use the word *ring* to describe a "place" where we can add, subtract, and multiply. So another way to express the fact that remainders after division by d can be added, subtracted, and multiplied is to say that the integers modulo d form a ring.

What about division? A ring in which division also makes sense (except division by zero) is called a *field*. Does it make sense to divide remainders after division by d? In other words, do the integers modulo d form a field? This is the question we will explore in this problem.

What does division mean, anyway? Division should be the opposite of multiplication, so the division problem $a \div b \equiv c \pmod{d}$ should mean that $b \times c \equiv a \pmod{d}$. This is how you should think about division for the purposes of this problem.

(a) Part of Problem 1 is to make a multiplication table for the integers modulo 7. (If you haven't done Problem 1 yet, it is probably worthwhile to do it now.) Let's consider the division problem $a \div b$, where a and b are integers modulo 7 and $b \neq 0$. (Division by zero is always bad.) If $a \div b$ is to make sense, there should be some value for c (an integer modulo 7) such that $b \times c \equiv a \pmod{7}$. Using the multiplication table from Problem 1, show that, no matter which values are chosen for a and b (as long as $b \neq 0$), there is exactly one value for c such that $b \times c \equiv a \pmod{7}$. Explain why this means that division makes sense modulo 7.

   Hint: Orrn dw wkh urzv dqg wkh froxpqv ri wkh pxowlsolfdwlrq wdeoh.

(b) What is $3 \div 5 \pmod{7}$?

   Hint: Xvh wkh pxowlsolfdwlrq wdeoh.

(c) Now make a multiplication table for the integers modulo 6. Show that there is no value for c such that $3 \times c \equiv 5 \pmod{6}$, and explain why this means that the division problem $3 \div 5 \pmod{6}$ does not have an answer.

These examples show that the integers modulo d form a field for *some* values of d, but not all. In fact, it turns out that the integers modulo d form a field if and only if the number d is prime.

Problem 8. Let A, B, and C be sets.

(a) Write an expression for $A \cap B$ without using the $\cap$ symbol. (You can use unions, complements, set differences, and parentheses. Be sure to use parentheses if you need them to avoid ambiguity.)

(b) Write an expression for $A \cap B \cap C$ without using the $\cap$ symbol.

Hint: Xvh Yhqq gldjudpv.