21-110: Problem Solving in Recreational Mathematics

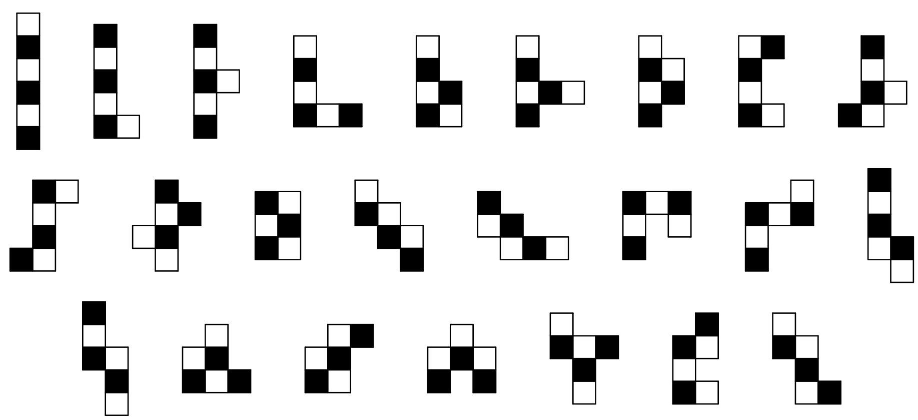
Homework assignment 2 solutions

Problem 1. Suppose you have one copy of each free hexomino. Prove that it is impossible to arrange them into a rectangle.

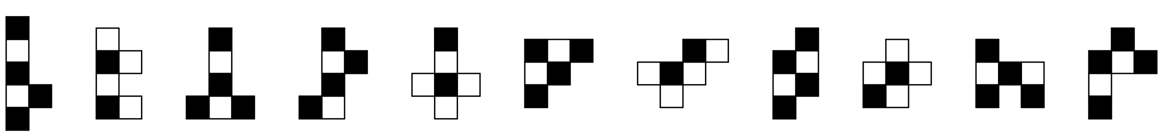
Solution. There are 35 free hexominoes. Each hexomino consists of six squares, so the total area of all 35 free hexominoes is $35 \times 6 = 210$ squares. The prime factorization of 210 is $2 \times 3 \times 5 \times 7$, so the possible dimensions of a rectangle having an area of 210 are 1×210, 2×105, 3×70, 5×42, 6×35, 7×30, 10×21, and 14×15. We can rule out the 1×210 and 2×105 rectangles right away, because many of the hexominoes cannot possibly fit in a rectangle having a width less than 3.

In all of the remaining cases, we observe that one of the dimensions of the rectangle is even (since 2 is a factor of 210), and so, if we color the squares of the rectangle like a checkerboard, there must be the same number of light and dark squares: along the even dimension of the rectangle, adjacent rows of squares alternate colors, so these rows can be divided into pairs, with each pair having the same number of light and dark squares. Therefore, in any of the possible rectangles colored checkerboard-style, there must be 105 light squares and 105 dark squares.

Now we color the free hexominoes with light squares and dark squares in a checkered pattern, as if they had been cut from a checkerboard. Most of the hexominoes (24 of them) have three dark squares and three light squares. Let's call these the "odd" hexominoes, since they cover an odd number of squares of each color. These are the odd hexominoes:



The remaining 11 hexominoes cover two dark squares and four light squares, or four dark squares and two light squares (depending on how we choose to color each one). Let's call these the "even" hexominoes, since they cover an even number of squares of each color. These are the even hexominoes:



There are an even number of odd hexominoes and an odd number of even hexominoes. Since "even $\times$ odd = even" and "odd $\times$ even = even," the 35 free hexominoes will always cover an even number of squares of each color in all, no matter how they are arranged on a (large) checkerboard. But all of the possible rectangles have 105 squares of each color, which is odd. So there is no way to arrange the 35 free hexominoes into a rectangle. □

[This proof is yet another example of the use of *parity*, that is, evenness or oddness. The mouse-in-a-maze problem discussed in class, and the fly-on-a-cube problem from the last homework, also used parity in their solutions. Parity is a simple but often surprisingly useful tool.]

Problem 2. There are exactly 11 free hexominoes that can be folded to make a cube. These hexominoes are called *nets* of the cube.

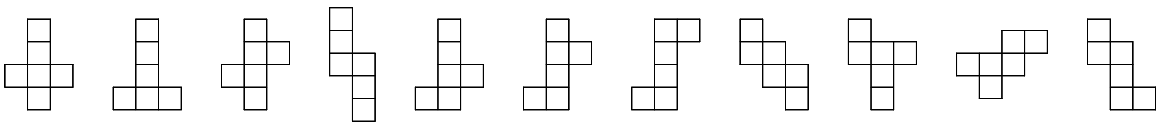
(a) Draw the 11 nets of the cube.

(b) Explain why each of the other free hexominoes is not a net of the cube.

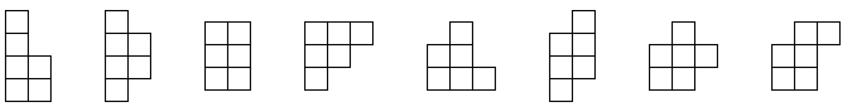
(c) All of the nets of the cube have the same perimeter. What is this common perimeter? Why is it not the same as the number of edges of a cube?

Solution.

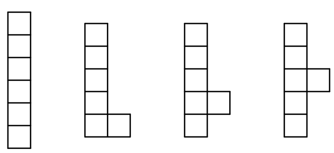
(a) The 11 nets of the cube are shown below.



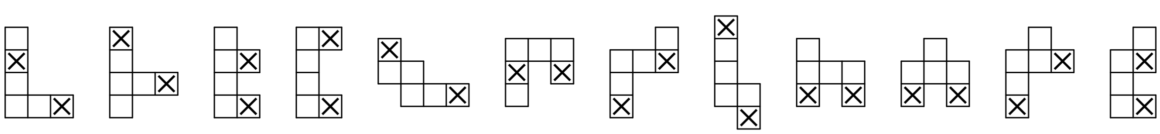
(b) First we can rule out any hexomino that has four squares in a 2×2 arrangement, since such an arrangement makes it impossible to fold the hexomino so that each square forms a face of a cube. Therefore the following eight hexominoes are not nets of the cube:



We can also rule out the hexominoes that have five or more squares in a straight line, because attempting to fold such a hexomino into a cube would result in overlapping squares (and therefore we would not be able to form all six faces of the cube). This eliminates the following four hexominoes:



Similarly, each of the remaining hexominoes has at least one pair of squares that would overlap if we attempted to fold the hexomino into a cube. One such pair is marked for each of the 12 hexominoes below, showing that they cannot be nets of the cube.



(c) The nets of the cube all have perimeter 14 (this is easy to check). A cube has 12 edges. The reason these numbers are not the same is that the 14 "outer edges" of a net of the cube do not directly produce the edges of the cube in a one-to-one fashion. Rather, each edge of the cube is formed in one of two ways: either as a fold on an "inner edge" of the net (that is, a dividing line between two adjacent squares in the hexomino), or as a seam between two "outer edges" of the net. Therefore the perimeter of the net, having length 14, contributes exactly seven of the edges of the cube, and the five "inner edges" of the net contribute the remaining five edges of the cube (note that every net of the cube has exactly five "inner edges"). This accounts for all 12 of the edges of the cube. □

Problem 3. The following questions refer to two-dimensional shapes in the plane.

(a) Draw a shape that has reflection symmetry but not rotation symmetry. Show the line of symmetry.

(b) Draw a shape that has rotation symmetry but not reflection symmetry. Describe the rotation symmetry (how many degrees do you need to rotate the shape in order to have it look the same?).

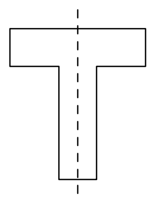
(c) Draw a shape that has neither reflection symmetry nor rotation symmetry. Explain why this shape has neither of these symmetries.

(d) Draw a shape that has both reflection symmetry and rotation symmetry. Show the line of symmetry and describe the rotation symmetry.

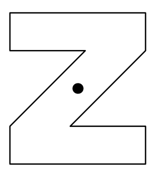
(e) Is there a shape that has *two* lines of symmetry but does *not* have rotation symmetry? If so, give an example. If not, why not?

Solution. There are many possible shapes that can be drawn, but simple examples can be found among the letters of the alphabet (or among the polyominoes!).

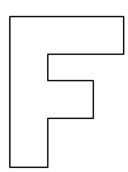
(a) The letter T has reflection symmetry about a vertical line, but it does not have rotation symmetry.



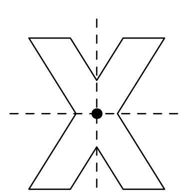
(b) The letter Z has rotation symmetry of 180° about its center, but it does not have reflection symmetry.



(c) The letter F has neither reflection symmetry nor rotation symmetry. There is no line about which it can be reflected and look the same as before, nor is there any angle by which it can be rotated and look the same (other than a trivial 360° rotation).



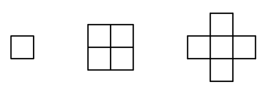
(d) The letter X has both reflection symmetry, about both a horizontal and a vertical line, and rotation symmetry about its center by 180°.



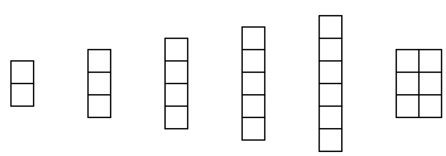
(e) If a shape has two lines of reflection symmetry, it must also have rotation symmetry. Imagine a cutout of such a shape that is painted red on one side and blue on the other, with the red side initially face-up. Mark the "top" edge of the shape somehow, so that we can detect a rotation. Now "reflect" it about one of its lines of symmetry by flipping it over (this causes the blue side to come up), and then reflect it a second time about its *other* line of symmetry (this causes the red face to come up again). We see that the end result is as if we had simply rotated the original shape by some angle. (Try this with a piece of paper.) But the shape looks the same as it did originally, except for the markings we made, because each of the reflections kept the shape the same. So the shape must have rotation symmetry as well as reflection symmetry. (The moral of the story is that two reflections are equivalent to one rotation.) □

Problem 4. Consider all of the free polyominoes up to area 6 (that is, the monomino, the domino, the trominoes, the tetrominoes, the pentominoes, and the hexominoes). Classify them according to their symmetry.

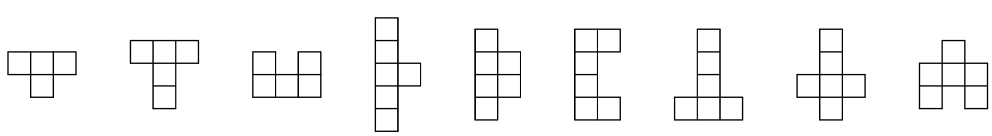
Solution. There are 56 of these free polyominoes in all: one monomino, one domino, two trominoes, five tetrominoes, 12 pentominoes, and 35 hexominoes. Out of all of these, there are three that have the greatest symmetry possible for shapes made up of connected squares, namely, four lines of symmetry (one horizontal, one vertical, and two diagonal) and rotational symmetry by 90°. These three polyominoes are shown below.



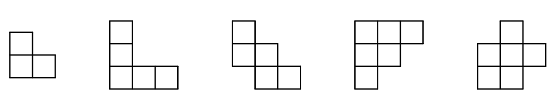
The next most symmetrical of these polyominoes are those in the shape of (non-square) rectangles. These polyominoes have two lines of symmetry (one horizontal and one vertical) and rotational symmetry by 180°. There are six such polyominoes, shown below.



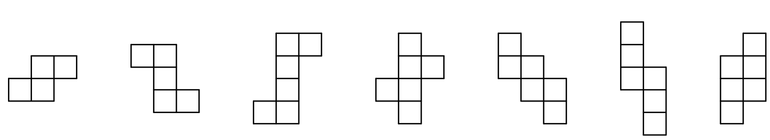
Next, there are nine of these polyominoes having only one line of symmetry, either horizontal or vertical (depending on orientation), and no rotational symmetry.



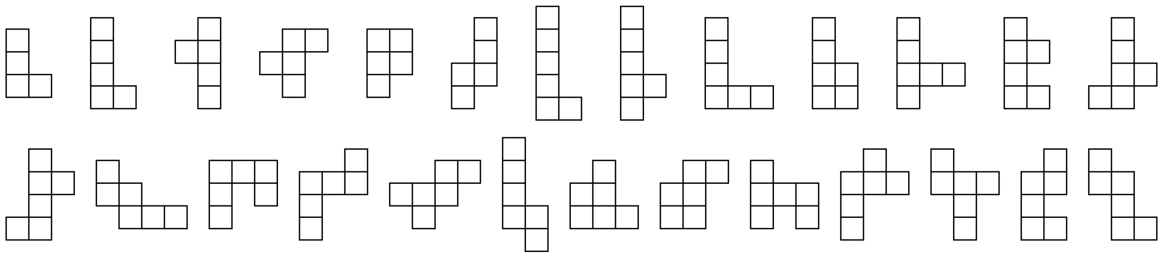
In addition, five of the polyominoes up to area 6 have one diagonal line of symmetry and no rotational symmetry. These might be considered to have the same symmetry as the previous nine—after all, they have the same number of lines of symmetry and the same (i.e., no) rotational symmetry—but we have listed them separately here because they seem to have a different "flavor" than the others. (This is not a mathematically precise statement.)



Then there are seven of these polyominoes that have 180° rotational symmetry but no lines of reflection symmetry (not even diagonally, as you will see if you examine them carefully):



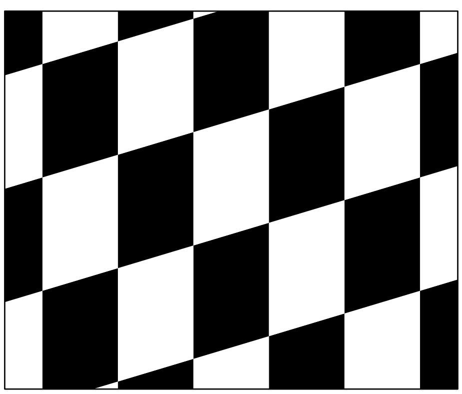
The remaining 26 polyominoes with area no greater than 6 have neither reflection symmetry nor rotational symmetry (other than trivial 360° rotational symmetry). They are listed below.

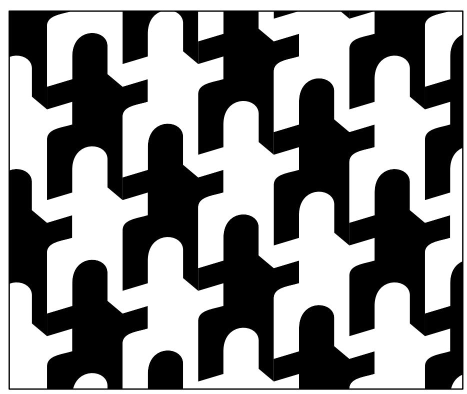


There are at least two potential types of symmetry that are not exhibited by any of these 56 “small” polyominoes: two lines of *diagonal* symmetry and 180° rotational symmetry, but no horizontal or vertical symmetry; and 90° rotational symmetry but no reflection symmetry. The smallest polyomino having the first type of symmetry has area 7, and the smallest having the second type of symmetry has area 8. Can you find them? □

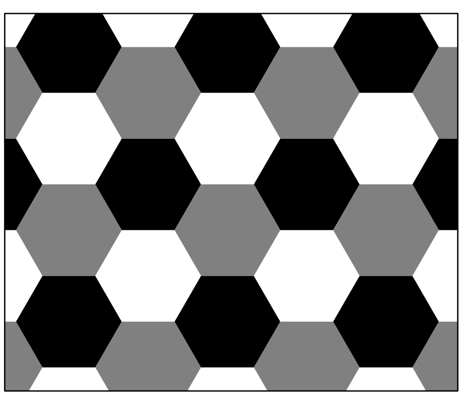
Problem 5. Design at least two interesting tessellations, based on different fundamental tilings. (For example, one may be based on the regular tiling of squares, and the other may be based on a semiregular tiling of hexagons and triangles.) Present your tessellations carefully—neatness is important. I suggest cutting out a template from heavy paper or cardboard to trace, or cutting out many individual tiles to paste together. Use color to highlight the repeating nature of your tessellations and explain any other interesting features. Be creative!

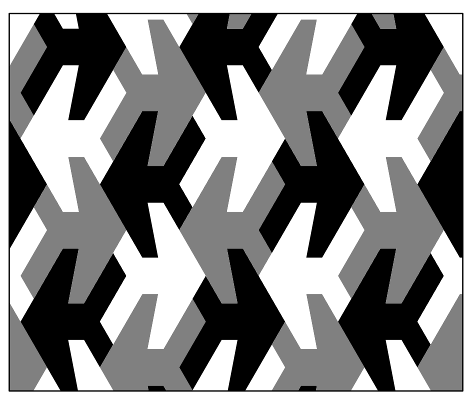
Solution. Here’s a modification of the “jumping man” tiling I sketched in class. It’s based on the tiling of parallelograms shown on the left.





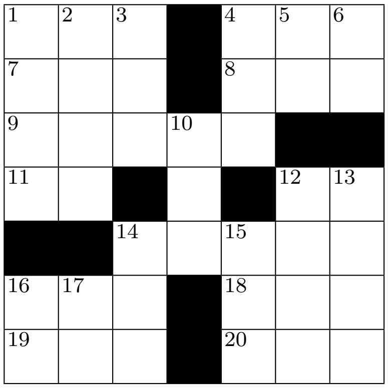
I tried to make a tessellation of airplanes from the regular tiling of hexagons, but I had a hard time getting the airplanes to look right. Here’s the best I could come up with.





(This tessellation was harder to work with than the first one, because the tiles aren’t all facing the same direction.) □

Problem 6. Solve the following crossword. Every numbered word in the puzzle must appear in the given word list, and no word may be used more than once. Explain the process you used.



Word list: an, as, at, do, eh, in, it, lo, me, no, oh, so, to; ago, aim, ale, all, and, ant, ash, ate, awe, din, dot, elk, end, has, hat, ice, ink, key, kid, lie, lot, mow, new, not, ode, owe, pat, pin, set, ski, spa, tag, tea, toe, ton, vat, wed, who, win, won, woo, zap, zip; also, apes, hall, have, head, iris, iron, noon, peas, soot, tend, urns, used, vase, zone; adore, aside, astir, hooks, legal, shark, shear, snort, stare, stars, start, syrup, voted.

Solution. The solution to this puzzle is *unique*, meaning that there is only one solution. We shall proceed in a very careful and systematic manner to fill in the words, entering each word into the puzzle only when we have logically deduced that it is the only possibility. By doing so, not only will we have found a solution, but we will also have proved that the solution is unique. (The following description is a somewhat polished form of my reasoning, so I have edited out some of the dead ends I tried, red herrings I chased, and false steps I took.)

A helpful first step is to write down the words of each length in columns on graph paper, to make it easier to find, say, all of the three-letter words whose middle letter is A. We will also want to cross off the words as we use them in the puzzle, so that we avoid using any word more than once.

It will be helpful to have some notation for referring to an entry in the puzzle that is partially filled in (and, by extension, the set of words that can fit there). Let's use underscores (_) to represent blanks, so "_A__" represents a four-letter word whose second letter is A. Sometimes we won't know exactly what letter goes in a certain square, but we will know it has to be one of only a few choices. In this situation, we'll use a list of letters in square brackets []. So, for instance, "_[AER]T" represents a three-letter word whose second letter is either A, E, or R, and whose last letter is T.

The five-letter words seem to be a good place to start—there are only 13 of them in the word list (whereas there are 43 three-letter words), and they will cross many other words, making it more likely that we will be able to eliminate some possibilities.

Let's start by considering the possibilities for the five-letter word at 9 Across. By examining the crossing words, we see that we can eliminate ADORE, because it would require the word at 2 Down to be __D_, and no four-letter word in the word list has D as its third letter. Similarly, we can eliminate ASTIR, because there is no three-letter word of the form __R for 4 Down. Going through the whole list of five-letter words, we can further eliminate HOOKS (there is no __H_ for 1 Down); SHARK and SHEAR (there is no __H_ for 2 Down); SNORT, STARE, STARS, and START (there is no R__ for 10 Down); and SYRUP (there is no __Y_ for 2 Down). This leaves only ASIDE, LEGAL, and VOTED as possible words for 9 Across.

Likewise, we can reject the following words for 14 Across: ASIDE (no _D__ for 12 Down), ASTIR (no _I__ for 12 Down), HOOKS (no _K__ for 12 Down), LEGAL (no G__ for 15 Down), SHARK (no _K__ for 13 Down), SNORT and START (no _T__ for 13 Down), SYRUP (no R__ for 15 Down), and VOTED (no _D__ for 13 Down). This leaves only ADORE, SHEAR, STARE, and STARS as possibilities for 14 Across.

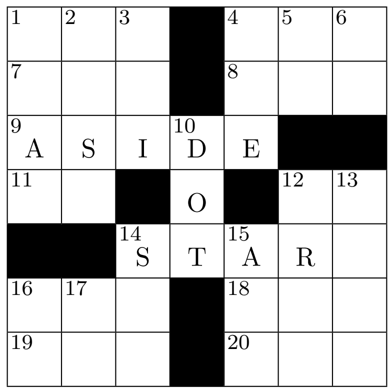
So the fourth letter of 10 Across (the first letter of 10 Down) must be either D, A, or E, and the second letter of 14 Across (the third letter of 10 Down) must be either D, H, or T. In other words, 10 Down must look like [ADE]_[DHT]. There are only five three-letter words like this in the word list: AND, ANT, ASH, DOT, and END. Fitting these together, we see that there are just seven possibilities for the combination of words at 9 Across, 10 Down, and 14 Across: ASIDE, DOT, STARE; ASIDE,

DOT, STARS; LEGAL, AND, ADORE; LEGAL, ANT, STARE; LEGAL, ANT, STARS; LEGAL, ASH, SHEAR; and VOTED, END, ADORE.

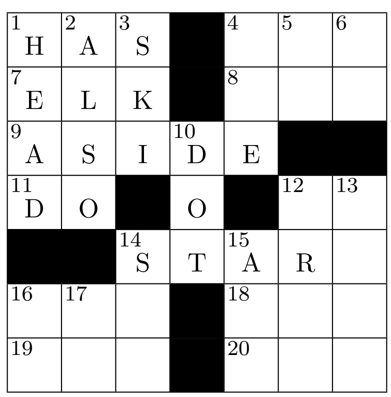
Suppose 9 Across is LEGAL. This forces 3 Down to be TAG, since that is the only __G in the word list, and it forces 1 Down to be HALL, since that is the only __L_. But then 7 Across must be A_A, and there are no words like this in the list. So we conclude that 9 Across cannot be LEGAL.

Suppose 9 Across is VOTED. This forces 1 Down to be HAVE (that's the only __V_), and so 11 Across must be EH (the only E_). But then 2 Down is __OH, which is impossible. So 9 Across cannot be VOTED.

This means that 9 Across must be ASIDE, and therefore 10 Down is DOT and 14 Across is either STARE or STARS. Let's fill in what we know.



Okay, let's attack the northwest corner next. First we see that 3 Down must be SKI (the only __I), so 1 Across must be HAS (the only __S). This means that 1 Down is H_A_, which makes it HEAD, and 2 Down is A_S_, which makes it ALSO. (In doing this, we have also filled in ELK at 7 Across and DO at 11 Across, so we cross out these words from the word list too.) That corner wasn't too bad. The puzzle so far looks like this:



On to the southwest corner. The possibilities for 14 Down are SET and SPA (remember that SKI has already been used). But SPA at 14 Down would require 16 Across to be either ZAP or ZIP, as these are the only words that fit __P, so then 16 Down would need to be Z_, which is a problem. Therefore 14 Down must be SET.

So 19 Across is __T. The possibilities here are ANT, HAT, LOT, NOT, PAT, and VAT (DOT and SET have already been used). We can discard most of these possibilities because they require crossing words that do not exist in the word list: ANT would require 16 Down to be _A; HAT would make 17 Down _A; LOT makes 16 Down _L; PAT requires 16 Down to be _P; and VAT needs _V at 16 Down. The only possibility remaining is NOT, which must be the word at 19 Across.

Hence 16 Down is _N, either AN or IN; and 17 Down is _O, one of LO, NO, SO, or TO (since DO has already been used). Therefore 16 Across must look like [AI][LNST]E. There are two words like this in the word list: ALE and ATE. We don't know for sure which of these is the correct word at 16 Across, so we'll come back later. However, we do know that the first letter is A, making 16 Down AN.

1 H	2 A	3 S	■	4	5	6
7 E	L	K	■	8		
9 A	S	I	10 D	E	■	■
11 D	O	■	O	■	12	13
■	■	14 S	T	15 A	R	
16 A	17	E	■	18		
19 N	O	T	■	20		

Continuing to the southeast corner, we first look at 12 Down. There are three possibilities for _R__, namely, IRIS, IRON, and URNS. But URNS would require 12 Across to be U_, which doesn't work. So the first letter of 12 Down must be I. This means 12 Across is either IN or IT. We already know that 14 Across is either STARE or STARS, so 13 Down looks like [NT][ES]__. Both NS__ and TS__ cause problems, so 14 Across must be STARE. This means 13 Down is [NT]E__; the only word like this is TEND, and so 12 Across is IT.

Now if 12 Down were IRIS we would have __SD at 20 Across, which would be impossible to fill; so 12 Down must be IRON. Then 18 Across looks like _ON, so it's either TON or WON, and 20 Across looks like _ND, making it either AND or END. So 15 Down is of the form A[TW][AE]. Again we find that there are two possibilities, either ATE or AWE, and we can't decide between them at the moment. In any case, we know that 15 Down ends with E, so 20 Across is END.

1 H	2 A	3 S	■	4	5	6
7 E	L	K	■	8		
9 A	S	I	10 D	E	■	■
11 D	O	■	O	■	12 I	13 T
■	■	14 S	T	15 A	R	E
16 A	17	E	■	18	O	N
19 N	O	T	■	20 E	N	D

Finally we come to the northeast corner. This corner is the most difficult of the four, because the only foothold we have is that 4 Down looks like __E, and almost all of the possibilities seem plausible at first. Upon careful examination, however, we will see that only one of them will actually work. The eight possibilities for 4 Down are ALE, ATE, AWE, ICE, LIE, ODE, OWE, and TOE. We'll consider each of these separately. Essentially we will try each possibility until we get stuck, and then backtrack to try the next possibility.

The easiest possibility to eliminate is ICE. Putting ICE at 4 Down would make 8 Across look like C__, but none of the three-letter words in the list begin with C. So 4 Down cannot be ICE.

If 4 Down is ALE, then 8 Across is L__, so it is either LIE or LOT. If it's LIE, then 5 Down must be _I, which is no good. If it's LOT, then 6 Down must be _T, forcing it to be AT (because IT is already used). But then 4 Across is A_A, which is impossible. So 4 Down can't be ALE.

If 4 Down is ATE, then 8 Across is T__, so it must be TAG, TEA, TOE, or TON. If it's TAG, then 5 Down must be _A, which is a problem. If it's TEA, then _A shows up at 6 Down this time—same problem. If 8 Across is TOE, then 6 Down is _E, which means it must be ME. This makes 4 Across look like A_M. The only choice is AIM, but that puts IO at 5 Down, which is not a word. So TOE is no good for 8 Across. The last possibility for 8 Across is TON, which would make 6 Down _N, requiring IN (since AN has already been used). Then 4 Across looks like A_I, which can't be filled. So 4 Down cannot be ATE.

If 4 Down is LIE, then 4 Across looks like L__, so it must be LOT (because we just used LIE). This means 5 Down is O_, which means it must be OH; but then 8 Across looks like IH_, which doesn't work. So 4 Down can't be LIE.

If 4 Down is ODE, then 8 Across is D__, making it DIN (since DOT is already used). Then 5 Down is an impossible _I. So 4 Down cannot be ODE.

If 4 Down is OWE, then 4 Across looks like O__, so it must be ODE (because we just used OWE). Then 5 Down is D_, but we've already used DO, which is the only such word in the list. So 4 Down can't be OWE.

If 4 Down is TOE, then 8 Across is O__, either ODE or OWE. So 5 Down is either _D or _W, both of which are impossible. So 4 Down cannot be TOE.

Therefore, the only possibility for 4 Down is AWE. Phew!

Let's go back to the spaces we left blank earlier. We knew 15 Down was either ATE or AWE, but we didn't know which. Now that we've used AWE, we see that 15 Down must be ATE, giving TON at 18 Across. And now that we've used ATE there, we know 16 Across must be ALE, giving LO for 17 Down.

We have just four squares left. Let's consider 4 Across. It looks like A__, so the possibilities are AGO, AIM, ALL, AND, ANT, and ASH (ALE, ATE, and AWE have already been used). But AGO would make 5 Down look like G_, which is no good; ALL would require both 5 Down and 6 Down to be L_, but LO is the only word like this (and anyway it's already been used); AND would need D_ at 6 Down, but DO has been used; and ASH would make 6 Down look like H_, which can't be filled. So there are really only two possibilities for 4 Across, AIM and ANT.

The word at 8 Across looks like W__, so it's one of WED, WHO, WIN, WON, or WOO. We see that WED needs _D at 6 Down, and WIN needs _I at 5 Down, both of which are impossible. This narrows the possibilities at 8 Across to WHO, WON, and WOO.

Therefore, the second letter of 4 Across (the first letter of 5 Down) is either I or N, and the second letter of 8 Across (the second letter of 5 Down) is either H or O. So 5 Down is IH, IO, NH, or NO; only NO is a word in the list, so we fill in NO at 5 Down. This makes 4 Across ANT, so 6 Down is TO (the only T_ in the list), giving WOO at 8 Across.

The completed puzzle appears below.

1 H	2 A	3 S		4 A	5 N	6 T
7 E	L	K		8 W	O	O
9 A	S	I	10 D	E		
11 D	O		O		12 I	13 T
		14 S	T	15 A	R	E
16 A	17 L	E		18 T	O	N
19 N	O	T		20 E	N	D

□

Problem 7. Indiana Jones has made it through the deadly traps of an ancient temple and has discovered ten treasures inside. Unfortunately, his knapsack is too small to carry them all, so he must choose (wisely). He has made the following estimates of the objects' weights and values. If his knapsack can hold at most 20 pounds of treasure, which objects should he take to maximize the value of his loot?

Treasure	Weight	Value
Crown of Atahualpa	4 lb.	$ 4,000
Itzcoatl's Orb	6 lb.	9,000
Tablet of the Heavens	7 lb.	7,000
Golden Quetzal	13 lb.	17,000
Mask of the Ancients	5 lb.	5,000

Treasure	Weight	Value
Key of Silver Light	2 lb.	$ 2,000
Idol of Inti	10 lb.	15,000
Eternal Quipu	5 lb.	8,000
Goblet of Uxmal	3 lb.	4,000
Sacred Stone of Cuzco	8 lb.	11,000

Solution. We might first try sorting the objects by value, and then choosing the highest-value items one by one, so long as we still have room in the knapsack. First we sort the treasures in descending order by value. (We'll put the Goblet of Uxmal ahead of the Crown of Atahualpa, because they are have the same value but the goblet weighs less.)

Treasure	Weight	Value
Golden Quetzal	13 lb.	$17,000
Idol of Inti	10 lb.	15,000
Sacred Stone of Cuzco	8 lb.	11,000
Itzcoatl's Orb	6 lb.	9,000
Eternal Quipu	5 lb.	8,000
Tablet of the Heavens	7 lb.	7,000
Mask of the Ancients	5 lb.	5,000
Goblet of Uxmal	3 lb.	4,000
Crown of Atahualpa	4 lb.	4,000
Key of Silver Light	2 lb.	2,000

Now let's fill the knapsack. We'll take the Golden Quetzal, since it has the highest value. Since the Golden Quetzal weighs 13 pounds, we have 7 pounds of capacity left. We cannot take either of the next two items in the list, the Idol of Inti or the Sacred Stone of Cuzco, because they weigh too much. So we'll take Itzcoatl's Orb, which weighs 6 pounds. This gives us 19 pounds, worth $26,000. Since there is no 1-pound item, this is the best we can do by this method. (Such a method, in which we simply take the "best" available choice at each step without considering how our choice will affect future options, is called a "greedy algorithm.")

But we can do better if we modify the method slightly. We weren't able to fill our knapsack because there is no 1-pound item to take. So let's try the same method, except that we will skip an item if it would give us a total of exactly 19 pounds. (We'll call this approach the "modified greedy algorithm.") As before we take the Golden Quetzal and skip the Idol of Inti and the Sacred Stone of Cuzco. This time we will also skip Itzcoatl's Orb and take the Eternal Quipu instead, giving us 18 pounds so far. Then we still have room to take the Key of Silver Light. This solution gives us a full 20 pounds of treasure worth $27,000.

However, there is another way to sort the treasure. Instead of sorting by value with no regard to weight, we can sort by value per pound. (As before, if two objects have the same value per pound, let's put the lighter one first.) Sorting the treasures in this way gives the list on the following page.

Treasure	Weight	Value	Value per pound
Eternal Quipu	5 lb.	$ 8,000	$1,600/lb.
Itzcoatl's Orb	6 lb.	9,000	1,500/lb.
Idol of Inti	10 lb.	15,000	1,500/lb.
Sacred Stone of Cuzco	8 lb.	11,000	1,375/lb.
Goblet of Uxmal	3 lb.	4,000	1,333/lb.
Golden Quetzal	13 lb.	17,000	1,308/lb.
Key of Silver Light	2 lb.	2,000	1,000/lb.
Crown of Atahualpa	4 lb.	4,000	1,000/lb.
Mask of the Ancients	5 lb.	5,000	1,000/lb.
Tablet of the Heavens	7 lb.	7,000	1,000/lb.

If we apply the greedy algorithm to this list, we begin by taking the Eternal Quipu and Itzcoatl's Orb, bringing us to a total of 11 pounds. We do not have enough room for the Idol of Inti, but we can take the Sacred Stone of Cuzco. We must stop here, because we have reached 19 pounds; the value of our treasure is $28,000.

This is an improvement upon the solutions we obtained earlier. Perhaps we can improve even further if we try the modified greedy algorithm. We choose the Eternal Quipu and Itzcoatl's Orb and skip the Idol of Inti, as before, but now we skip the Sacred Stone of Cuzco as well, because that would give us a total of 19 pounds. Instead we take the Goblet of Uxmal, the Key of Silver Light, and the Crown of Atahualpa. We have filled our knapsack with 20 pounds, but when we calculate the value it turns out that we have done worse; we have only $27,000.

The greedy algorithm and the modified greedy algorithm are simple methods to use, and they usually work well to get an *approximation* of the best solution, but neither of these methods is *guaranteed* to give the best solution. In this problem there is a better solution than any of the four we have found so far.

Let's go about this in a very systematic way. We have already found a solution worth $28,000. If we are going to improve upon this, we must include in our solution at least one of the five highest-value items (the Golden Quetzal, the Idol of Inti, the Sacred Stone of Cuzco, Itzcoatl's Orb, or the Eternal Quipu) because the other five items are worth only $22,000 in all. So we will consider the five high-value items separately from the five low-value items.

The low-value items are listed below, sorted in ascending order by weight (and, incidentally, also by value).

Treasure	Weight	Value
Key of Silver Light	2 lb.	$2,000
Goblet of Uxmal	3 lb.	4,000
Crown of Atahualpa	4 lb.	4,000
Mask of the Ancients	5 lb.	5,000
Tablet of the Heavens	7 lb.	7,000

We will make a list of ways to form various weights from these low-value treasures. It is easy to see that the only way to make 2 pounds is to take the Key of Silver Light. Similarly, the only way to make 3 pounds is to take the Goblet of Uxmal, and the only way to make 4 pounds is to take the Crown of Atahualpa. To make 5 pounds, however, we have a choice: either we can take the Mask of the Ancients alone, or we can take the Key of Silver Light and the Goblet of Uxmal. In general, when our aim is to form a total weight of n pounds, we must also consider all of the possible ways to write n as a sum of smaller numbers (out of 2, 3, 4, 5, and 7). Following this approach, we construct the table on the next page.

Weight	As a sum	Treasures	Value
2 lb.	2	Key	$ 2,000
3 lb.	3	Goblet	4,000
4 lb.	4	Crown	4,000
5 lb.	5	Mask	5,000
	$2+3$	Key, goblet	6,000
6 lb.	$2+4$	Key, crown	6,000
7 lb.	7	Tablet	7,000
	$2+5$	Key, mask	7,000
	$3+4$	Goblet, crown	8,000
8 lb.	$3+5$	Goblet, mask	9,000
9 lb.	$2+7$	Key, tablet	9,000
	$4+5$	Crown, mask	9,000
	$2+3+4$	Key, goblet, crown	10,000
10 lb.	$3+7$	Goblet, tablet	11,000
	$2+3+5$	Key, goblet, mask	11,000
11 lb.	$4+7$	Crown, tablet	11,000
	$2+4+5$	Key, crown, mask	11,000
12 lb.	$5+7$	Mask, tablet	12,000
	$2+3+7$	Key, goblet, tablet	13,000
	$3+4+5$	Goblet, crown, mask	13,000
13 lb.	$2+4+7$	Key, crown, tablet	13,000
14 lb.	$2+5+7$	Key, mask, tablet	14,000
	$3+4+7$	Goblet, crown, tablet	15,000
	$2+3+4+5$	Key, goblet, crown, mask	15,000
15 lb.	$3+5+7$	Goblet, mask, tablet	16,000
16 lb.	$4+5+7$	Crown, mask, tablet	16,000
	$2+3+4+7$	Key, goblet, crown, tablet	17,000
17 lb.	$2+3+5+7$	Key, goblet, mask, tablet	18,000
18 lb.	$2+4+5+7$	Key, crown, mask, tablet	18,000
19 lb.	$3+4+5+7$	Goblet, crown, mask, tablet	19,000
21 lb.	$2+3+4+5+7$	Key, goblet, crown, mask, tablet	22,000

To use this table, we will consider some combination of the high-value treasures, see how much room is left in the knapsack, and then find the best combination of low-value treasures in the table above that will fit. For example, if we choose the Sacred Stone of Cuzco and Itzcoatl's Orb as our high-value treasures, which together weigh 14 pounds, we will have 6 pounds of capacity remaining. Looking at the "6 lb." row and the rows above it in this table, we see that our best option is to take the Key of Silver Light and the Crown of Atahualpa (or, equivalently, the Key of Silver Light and the Goblet of Uxmal—perhaps this is actually a better choice, since it gives us the same value with less weight).

Proceeding in this way, we can find the best collection of treasures to take by considering all possible combinations of the high-value items. Of course, we can safely ignore those that put us overweight: the Golden Quetzal and the Idol of Inti together weigh 23 pounds, and the Golden Quetzal and the Sacred Stone of Cuzco together weigh 21 pounds. Furthermore, any combination of three or more high-value items will weigh more than 20 pounds (except the three lightest ones, the

Sacred Stone of Cuzco, Itzcoatl's Orb, and the Eternal Quipu, which weigh 19 pounds together). These impossible combinations of high-value items are omitted from the following table.

Quetzal	Idol	Stone	Orb	Quipu	Remaining	Low-value items	Total value
•					7 lb.	Goblet, crown	$25,000
	•				10 lb.	Goblet, tablet	26,000
		•			12 lb.	Key, goblet, tablet	24,000
			•		14 lb.	Goblet, crown, tablet	24,000
				•	15 lb.	Goblet, mask, tablet	24,000
•			•		1 lb.	*(None)*	26,000
•				•	2 lb.	Key	27,000
	•	•			2 lb.	Key	28,000
	•		•		4 lb.	Crown	28,000
	•			•	5 lb.	Key, goblet	29,000
		•	•		6 lb.	Key, crown	26,000
		•		•	7 lb.	Goblet, crown	27,000
			•	•	9 lb.	Key, goblet, crown	27,000
		•	•	•	1 lb.	*(None)*	28,000

From this we see that the best choice is to take the Idol of Inti, the Eternal Quipu, the Key of Silver Light, and the Goblet of Uxmal, which weigh 20 pounds together and have a total value of $29,000. Moreover this solution is *unique*: there is only one best solution.

(With the particular numbers used in this problem, it turns out that we would have found this optimal solution by using the greedy algorithm based on value per pound if we would have broken ties by putting the *heavier* item first rather than the lighter one. But this isn't a general rule that will always give the best possible solution.) □

[This problem is an example of what computer scientists call the *0–1 knapsack problem*, so called because each type of item is to be put into the knapsack either 0 times or 1 time. The general version of the knapsack problem allows several copies of each item to be put in the knapsack (for example, perhaps Indiana Jones is buying donuts to put in his knapsack and must decide how many chocolate donuts, how many glazed donuts, and how many strawberry donuts to buy); sometimes fractional parts of items are allowed. The knapsack problem has many practical applications; it shows up whenever choices have to be made because of a restriction on some kind of capacity. However, it is a hard problem to solve exactly. It is believed, but not known for certain, that there is no efficient method that will always give the best possible solution for the knapsack problem.]

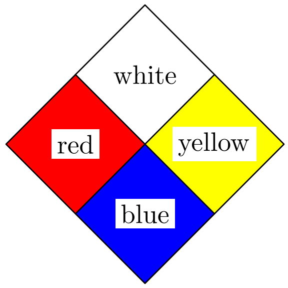
Problem 8. ("Cubes Cubed," *Thinking Mathematically*, page 36.) I have eight cubes. Two of them are painted red, two white, two blue and two yellow but otherwise they are indistinguishable. I wish to assemble them into one large cube with each colour appearing on each face. In how many ways can I assemble the cube?

Solution. The answer to this question depends greatly on the way we choose to decide whether two assemblages of the small cubes are in fact *different*. One possibility is to say that two assemblages are the same if one of them can be rotated somehow (in three dimensions) to get the other, and different otherwise. Under this definition of *different*, we do not get a different arrangement of the small cubes if we simply rotate everything by $90°$, for example. This seems to be a sensible definition, and it is the one we will use here.

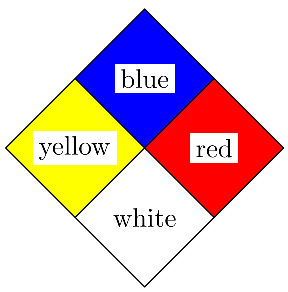
Since there are eight small cubes, the large cube must have dimensions $2 \times 2 \times 2$, that is, two small cubes on each edge. Hence each small cube occupies one of the eight corners of the large cube, and so each small cube appears on three faces of the larger cube. The large cube has six faces, of course, so in order to have each color on each face the two small cubes of any one color must be located at diagonally opposite corners of the large cube.

This means that the arrangement of the small cubes is completely determined once we have arranged four cubes for one of the faces. For instance, if we arrange four cubes, one of each color, to make the bottom layer of the large cube, then we are forced to put the cubes in the top layer so that the blue cube in the top layer is diagonally opposite the blue cube in the bottom layer (and the same for the other colors).

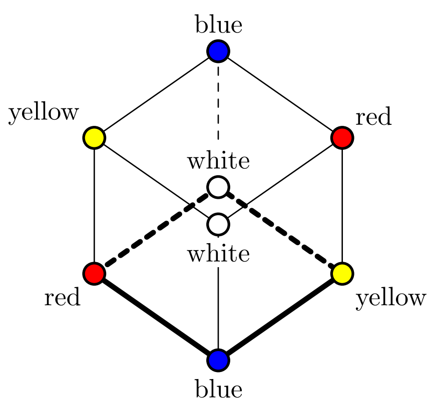
For example, consider the following arrangement for the bottom layer, seen from above.



This arrangement forces the top layer to be as shown below:

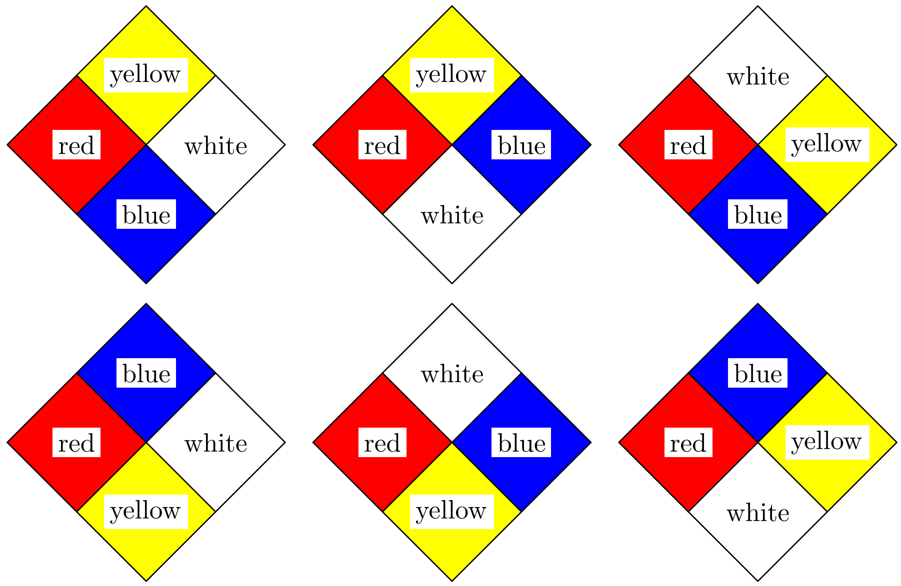


So the large cube as a whole looks like the following. (In this picture the corners of the cube are colored to indicate the colors of the small cubes located there, and the bottom layer is drawn with heavy lines.)

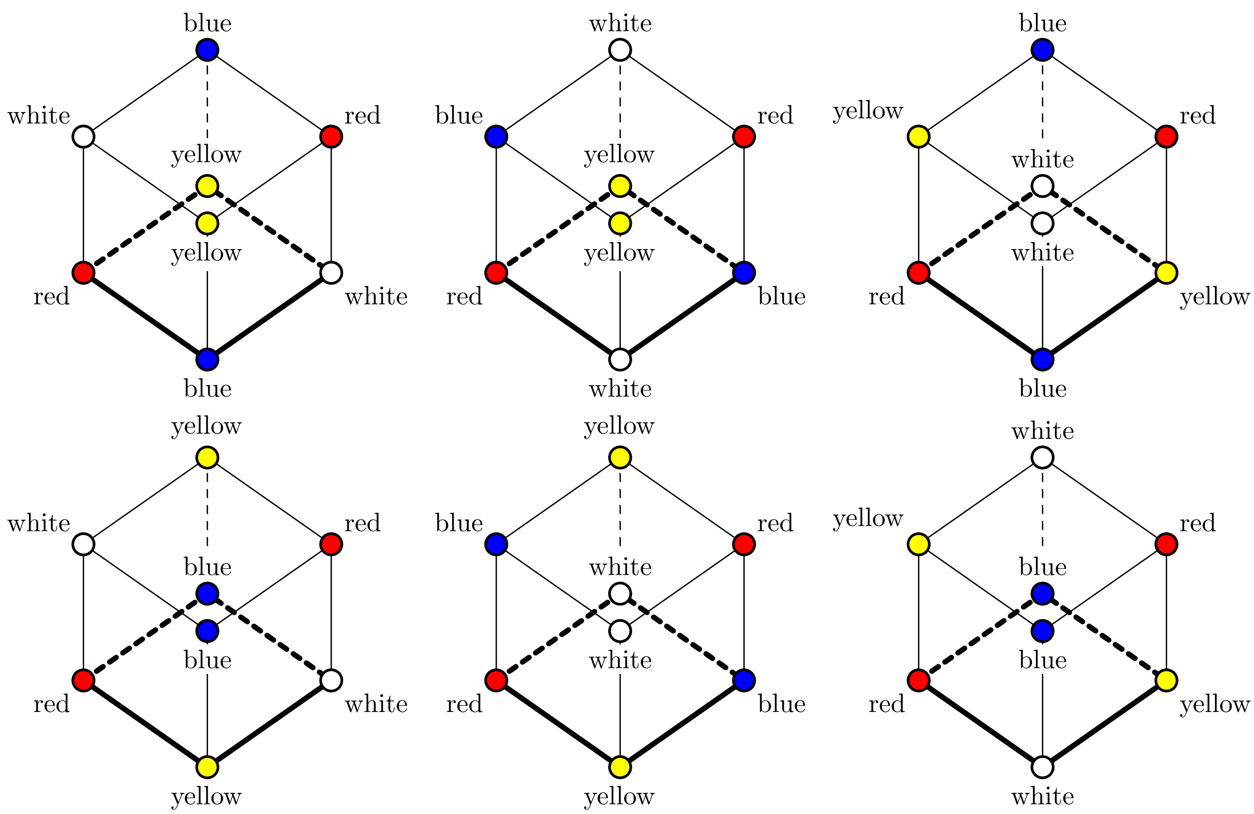


Therefore we can list all of the possible assemblages of the small cubes by simply listing all the ways to arrange four cubes in the bottom layer. Since we have decided (in our definition of *different*) that we can freely rotate a cube without changing it, let's draw the bottom layer so that it always has the red cube on the left side. We can list the possibilities for the bottom layer by noting that the cube opposite the red one can be either white, blue, or yellow, and then the other two positions can be taken by the remaining two cubes in either order. (The order of these is important, because switching their order cannot be accomplished by rotation.)

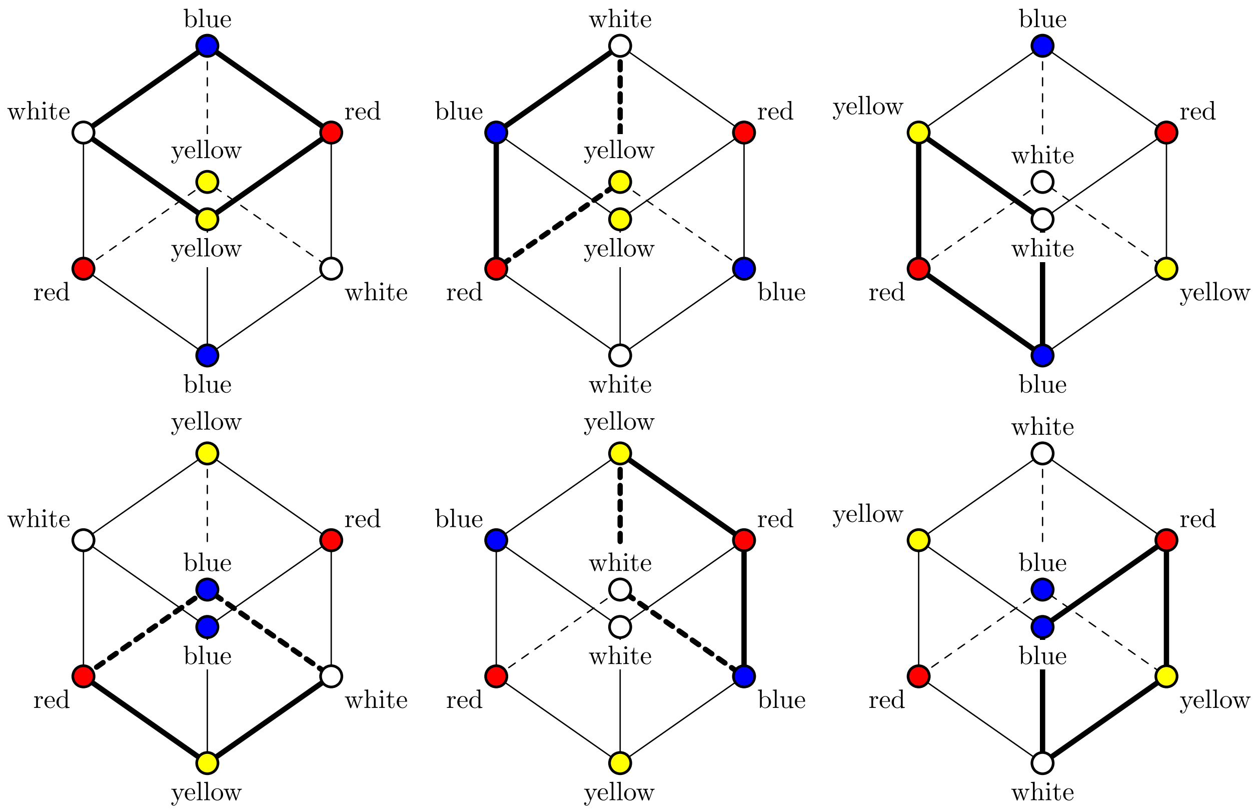
So we have the following six possibilities for the bottom layer:



These correspond to the following arrangements for the large cube (the bottom layer is drawn with heavy lines):



How can we tell when two arrangements of these are the same? Well, if we can identify a face of one large cube that looks the same as a face of another large cube, the two cubes must be the same (because once we have decided what one face looks like, we have determined the entire arrangement). If we look at the top face of the first cube above, we see that the colors go “red–yellow–white–blue” (starting at red and going around counterclockwise). As it turns out, we can find a face with this sequence of colors on all six of the arrangements! This is illustrated below. (Remember that we are going around the face in the clockwise direction, imagining that we are holding the cube so that the face in question is facing toward us. For the bottom face and the two faces on the back side, this will appear to be the counterclockwise direction in these pictures.)



Therefore there is essentially only one way to arrange the four small cubes into one large cube so that each color appears on each face. □