21-110: Problem Solving in Recreational Mathematics

Homework assignment 1 solutions

Problem 1. Find my office. Sign your name on the sheet posted outside my office door.

Solution. My office is in the Physical Plant Building (PPB), room 342. One way to get there is to go to Wean Hall, take the elevator down to the first floor, exit the building, follow the sidewalk across the street and around the corner of the FMS Building, and cross the plaza to the southwest corner. My office door is just inside the glass doors. □

Problem 2. "How many children do you have, and what are their ages?" asks the census taker.

The mother answers, "I have three children. The product of their ages is 36, and the sum of their ages is the same as my house number."

The census taker looks at the house number, thinks for a moment, and responds, "I'm sorry, but I need more information."

"My oldest child likes chocolate ice cream," says the mother.

"Thank you," replies the census taker. "I have all the information I require."

How old are the children?

Solution. There are eight possible ways for three children to have ages whose product is 36:

Ages	Sum
1, 1, 36	38
1, 2, 18	21
1, 3, 12	16
1, 4, 9	14
1, 6, 6	13
2, 2, 9	13
2, 3, 6	11
3, 3, 4	10

We don't know the sum of the ages, but we *do* know that the census taker knows the sum (she looked at the house number) and yet she is still unable to determine the three ages. This must be because the house number is 13; otherwise, knowing the sum would be enough information. So the ages of the children are either 1, 6, and 6 or 2, 2, and 9. The mother's statement that her *oldest child* likes chocolate ice cream rules out the possibility that the ages are 1, 6, and 6, because in that case there would not be a single oldest child. Hence we conclude that the ages of the children are 2, 2, and 9. □

Problem 3. ("Ins and Outs," *Thinking Mathematically*, page 175.) Take a strip of paper and fold it in half several times in the same fashion as in *Paper Strip* (page 4). Unfold it and observe that some of the creases are IN and some are OUT. For example, three folds produce the sequence

in in out in in out out

What sequence would arise from 10 folds (if that many were possible)?

Solution. It is helpful to carefully and precisely describe the folding procedure and what is meant by "in" and "out." *Paper Strip* describes the method of folding as follows: "Imagine taking the ends in your hands and placing the right hand end on top of the left. Now press the strip flat so that it is folded in half and has a crease. Repeat the whole operation on the new strip. . . ." After folding the strip in half a number of times, the strip should be unfolded by exactly undoing the folding process (this is important to note, because different unfolding methods can result in different sequences of creases).

Once the strip is unfolded, the creases can be classified as either "in" or "out." An "in" crease is one that, when viewed from the "top" (as this word is used in the description of the folding method), makes an angle of less than 180°, while an "out" crease makes an angle of more than 180°. Equivalently, an "in" crease is one that tends to make the adjacent segments of the paper strip come together by folding toward the viewer, while an "out" crease tends to make them fold away from the viewer.

We discussed *Paper Strip* in class, which asks for the number of creases resulting from a number of folds. We found that after n folds the strip has $(2^n - 1)$ creases. For example, after 3 folds there are $2^3 - 1 = 7$ creases, and after 10 folds there would be $2^{10} - 1 = 1023$ creases.

In reality, it is impossible to fold a paper strip 10 times. (Actually, this is not true—in 2002, Britney Gallivan, a California high school student, succeeded in folding a very long strip of very thin paper in half 12 times, and in the course of studying the problem developed a mathematical theory to describe the limits of paper folding.) In any case, a sequence of 1023 "ins" or "outs" is not a very helpful answer to this question and is easy to get lost in. So we would like to identify a pattern in the creases and describe the pattern instead.

After some experimentation, a pattern emerges. It appears that the sequence of creases after n folds consists of the sequence of creases after $(n-1)$ folds; followed by an "in" crease; followed by the sequence of creases after $(n-1)$ folds, reversed and with all "ins" changed to "outs" and vice versa. For example, after one fold, the sequence is

in;

after two folds,

$$\underline{\text{in}}\quad \text{in}\quad \underline{\text{out}};$$

after three folds,

$$\underline{\underline{\text{in}}\quad \text{in}\quad \underline{\text{out}}}\quad \text{in}\quad \underline{\underline{\text{in}}\quad \text{out}\quad \underline{\text{out}}};$$

and after four folds,

$$\underline{\underline{\underline{\text{in}}\quad \text{in}\quad \underline{\text{out}}}\quad \text{in}\quad \underline{\underline{\text{in}}\quad \text{out}\quad \underline{\text{out}}}}\quad \text{in}\quad \underline{\underline{\underline{\text{in}}\quad \text{in}\quad \underline{\text{out}}}\quad \text{out}\quad \underline{\underline{\text{in}}\quad \text{out}\quad \underline{\text{out}}}}.$$

Because of their symmetry, it turns out that reversing one of these sequences and changing all "ins" to "outs" and vice versa is the same as simply changing the middle "in" to an "out." So this pattern can also be described in a slightly different way. The sequence of creases after n folds consists of the sequence of creases after $(n-1)$ folds; followed by an "in" crease; followed by the sequence of creases after $(n-1)$ folds again, with the middle "in" crease changed to an "out" crease.

Why does this pattern appear? I had to spend quite a bit of time folding and refolding paper to understand what was happening, but in the end I came up with the following explanation.

Suppose we start with the strip all folded. The sequence of creases we see (no creases) is the same as if the strip had not been folded at all. Next we *reveal* the last fold by undoing one fold. What we see now is a single "in" crease, as if a strip had been folded once and then unfolded. Revealing another fold shows the sequence "in, in, out," which is what we would see in a strip that had been folded twice and then unfolded. In general, adding one more fold to a paper strip before unfolding it is like doing one more such "revealing" step to a strip that is initially folded many times.

Now, each "revealing" step mirrors all of the previously existing creases (doubling their number, reversing their order, and also changing all "in" creases to "out" creases), and adds a new "in" crease in the middle. This explains the pattern we observed.

Just for completeness, here is the sequence of creases produced by 10 folds. (This was generated by a computer program, not by hand!)

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It should be clear why this is not the ideal form of the answer to this question. □

Problem 4. (Exercise 1.5 from *Problem Solving Through Recreational Mathematics*, page 27.) Six players—Pietrovich, Cavelli, St. Jacques, Smith, Lord Bottomly, and Fernandez—are competing in a chess tournament over a period of five days. Each player plays each of the others once. Three matches are played simultaneously during each of the five days. The first day, Cavelli beat Pietrovich after 36 moves. The second day, Cavelli was again victorious when St. Jacques failed to complete 40 moves within the required time limit. The third day had the most exciting match of all when St. Jacques declared that he would checkmate Lord Bottomly in 8 moves and succeeded in doing so. On the fourth day, Pietrovich defeated Smith.

Who played against Fernandez on the fifth day?

Solution. The hint on page 381 of the book suggests that we should make a chart showing what happens each day, and we should begin by determining whom St. Jacques can play on the fourth day. Putting the information given in the problem into a chart, we have the following:

Day 1	Day 2	Day 3	Day 4	Day 5
Cav vs. Piet	Cav vs. St. J	St. J vs. Lord B	Piet vs. Smith	

On the fourth day, St. Jacques cannot play Pietrovich or Smith, because they are playing each other. He also cannot play Cavelli or Lord Bottomly, as he has played them already. This means that St. Jacques must play Fernandez on the fourth day. Now that we have two matches determined for the fourth day, the third match is easy—it must be between the two remaining players, Cavelli and Lord Bottomly.

Day 1	Day 2	Day 3	Day 4	Day 5
Cav vs. Piet	Cav vs. St. J	St. J vs. Lord B	Piet vs. Smith St. J vs. Fern Cav vs. Lord B	

We know whom St. Jacques plays on three of the days. Let's consider his opponent on the first day. It cannot be Cavelli, Lord Bottomly, or Fernandez, because these are his opponents on other days. This leaves either Pietrovich or Smith. But Pietrovich is playing Cavelli on the first day, so only Smith is available to play St. Jacques. The remaining match on Day 1 must be Lord Bottomly versus Fernandez. Furthermore, the only player St. Jacques has not yet met is Pietrovich; they must play on the fifth day.

Day 1	Day 2	Day 3	Day 4	Day 5
Cav vs. Piet St. J vs. Smith Lord B vs. Fern	Cav vs. St. J	St. J vs. Lord B	Piet vs. Smith St. J vs. Fern Cav vs. Lord B	St. J vs. Piet

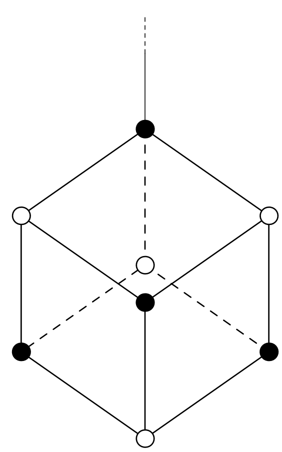
Pietrovich must play Lord Bottomly on one of the days. The only day they are both available is the second day. Once we have filled this in, we see that Smith and Fernandez must play each other on the second day, since they are the only two left. Also, Pietrovich and Lord Bottomly each have just one hole left in their schedules; Pietrovich must play Fernandez on the third day, and Lord Bottomly must play Smith on the fifth day. Finally we see that Cavelli plays Smith on the third day, and Cavelli plays Fernandez on the fifth day. Our completed chart is shown below.

Day 1	Day 2	Day 3	Day 4	Day 5
Cav vs. Piet St. J vs. Smith Lord B vs. Fern	Cav vs. St. J Piet vs. Lord B Smith vs. Fern	St. J vs. Lord B Piet vs. Fern Cav vs. Smith	Piet vs. Smith St. J vs. Fern Cav vs. Lord B	St. J vs. Piet Lord B vs. Smith Cav vs. Fern

So it was Cavelli who played against Fernandez on the fifth day. □

Problem 5. A sculpture in an art gallery consists of a wooden cube suspended from the ceiling by a thin wire. The wire is attached to the cube at one of its corners, so that the cube hangs at an angle. A fly lands on the top corner of the cube (the one at which the wire is attached) at 12:00 noon. Every five minutes thereafter (at 12:05, 12:10, 12:15, and so on) the fly moves along one of the edges of the cube to reach a neighboring corner. The path of the fly around the cube is random (subject to the restriction that it can only move along edges of the cube). What is the probability that the fly will be at the bottom corner of the cube (the corner nearest the floor) at 12:31?

Solution. We can color the corners of the cube with two colors (say, black and white) in such a way that adjacent corners are given opposite colors, as shown below.



With this coloring, every move of the fly takes it from a corner of one color to a corner of the opposite color. Since the fly makes a move every five minutes, it will have made exactly six moves by 12:31. The fly began at a black corner, so after six moves it must also be at a black corner. In particular, it cannot be at the bottom corner of the cube, which is white. So the probability that the fly will be at the bottom corner of the cube at 12:31 is 0. □

Problem 6. Three guests check into a hotel and ask what the nightly rate is. The clerk says a room costs 30 dollars a night, so each guest gives the clerk ten dollars, and they head up to the room.

A while later, the clerk realizes he overcharged the guests; the room they are staying in is only 25 dollars a night. So he takes five one-dollar bills from the cash box and hands them to the bellhop with instructions to return the money.

On the way up to the room, the bellhop realizes that five dollars cannot be split evenly among three guests, so he pockets two dollars and returns three dollars to the guests.

Now, each of the guests initially paid ten dollars for the room, but later received a dollar back, so effectively each guest paid nine dollars. In total, then, the room cost the guests 27 dollars. With the two dollars the bellhop kept, this comes to 29 dollars. What happened to the missing dollar?

Solution. This is a dishonest and misleading question. There is no missing dollar.

If we carefully keep track of where all of the money ends up, we find that the clerk has 25 dollars in the cash box, the bellhop has two dollars in his pocket, and the three guests have one dollar each. This comes to 30 dollars, as it should.

The question is misleading because the addition is done incorrectly in the last paragraph. It is true that in all the three guests paid 27 dollars, but that total includes both the cost of the room and the two-dollar "tip" for the bellhop. It does not make sense to add the bellhop's two dollars to the 27-dollar total, because it is already included. What *should* be added to the 27 dollars in order to get a total of 30 is the three dollars the guests received as change. $\square$

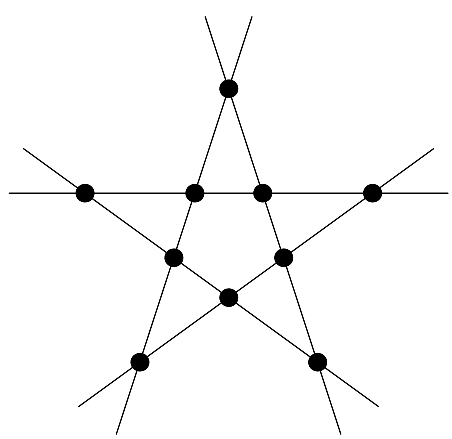
Problem 7. (Note: The two parts of this question are not meant to be related to each other, apart from the fact that they are both about arranging coins.)

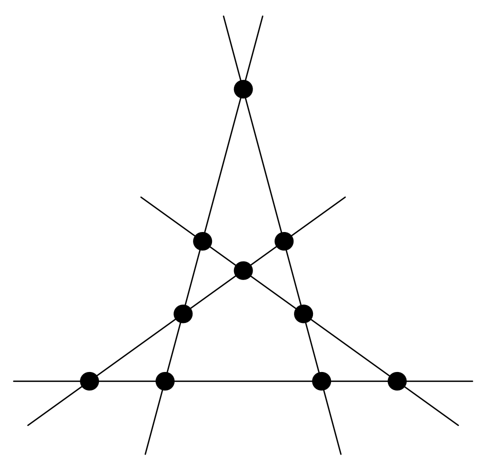
(a) Place 10 coins in five straight lines so that each line contains exactly four coins.

(b) Can you arrange four coins so that if you choose *any* three of them (i.e., no matter which three of the four you pick), the three coins you chose form the corners of an equilateral triangle?

Solution.

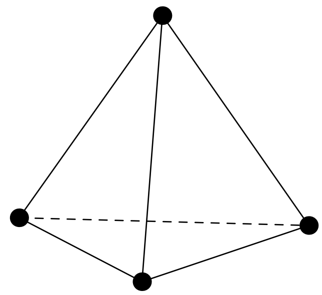
(a) There are several solutions to this puzzle. Here are two:





To make such a figure, it is easier to begin with the five lines than with the 10 coins. Let any five straight lines be drawn in a plane so that no two of the lines are parallel and no three of the lines meet in a single point. Then every line will intersect every other line (so there will be four points of intersection along each line), and in all there will be 10 points of intersection among the five lines. Placing a coin on each point of intersection will give a solution to this puzzle.

(b) It is impossible to solve this puzzle in two dimensions; three dimensions are required. The trick is to place three of the coins flat on a tabletop so that they form an equilateral triangle, and then suspend the fourth coin above the center of the triangle so that it sits at the top of a triangular pyramid (a regular tetrahedron), as shown below.



□

Problem 8. (From the article "Number Games and Other Mathematical Recreations" in the 15th edition of the Encyclopædia Britannica.) Three travelers were aboard a train that had just emerged from a tunnel, leaving a smudge of soot on the forehead of each. While they were laughing at each other, and before they could look into a mirror, a neighboring passenger suggested that although no one of the three knew whether he himself was smudged, there was a way of finding out without using a mirror.

He suggested: "Each of the three of you look at the other two; if you see at least one whose forehead is smudged, raise your hand." Each raised his hand at once. "Now," said the neighbor, "as soon as one of you knows for sure whether his own forehead is smudged or not, he should drop his hand, but not before."

After a moment or two, one of the men dropped his hand with a smile of satisfaction, saying: "I know."

How did that man know that his forehead was smudged?

Solution. Let's give the three travelers names: Augustus, Bertrand, and Kurt (after three famous logicians). Suppose Augustus is the one who figures out that his forehead is smudged. Let's look at the situation from his point of view.

He sees two men with smudged foreheads and raised hands. He cannot conclude anything from this directly. Perhaps Bertrand and Kurt are raising their hands because they see Augustus' smudged forehead, but it is equally possible that they are raising their hands merely because they see each other.

Augustus reasons as follows: "I will assume, for the sake of argument, that my own forehead is clean. Then Bertrand and Kurt must be raising their hands because they see each other. What does the situation look like from Bertrand's point of view? He sees Kurt's smudged forehead and my clean one, and he sees Kurt raising his hand. Kurt is raising his hand because he sees a smudged forehead, and clearly it is not mine (because mine is clean), so Bertrand can easily conclude that his own forehead must be smudged. The situation is exactly the same for Kurt. Therefore, assuming that my own forehead is clean, Bertrand and Kurt can quickly determine that their own foreheads are smudged.

"But wait! Neither Bertrand nor Kurt is dropping his hand. They must not be able to conclude anything about their own foreheads. This must mean that my assumption is wrong, and my own forehead is smudged!" □